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Jindřich Nečas

Direct Methods in the Theory of Elliptic Equations

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Jindřich Nečas

Direct Methods in the Theory of Elliptic Equations

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Preface

The first edition of Nečas's book *Les méthodes directes en théorie des équations elliptiques* was published in 1967 simultaneously by Academia, the Publishing House of the Czechoslovak Academy of Sciences in Prague, and by Masson in Paris. The publication was unique in that it gave quite a general up-to-date account of Direct Methods (also called Variational Methods) in the context of linear elliptic equations and systems of arbitrary order in Lipschitz domains, together with an excellent general introduction to the theory of Sobolev Spaces. The topics the book addresses were always among the main research interests of J. Nečas (together with non-linear problems). The book was written in French not only because French was J. Nečas's favorite language, but also due to his collaboration with J. L. Lions and the Paris school. After the success of the French edition the translations into English and Russian were discussed at several points, but for various reasons this work was completed only recently.

I am very grateful to J. Nečas's colleagues and friends, particularly to W. Jäger, E. Zeidler, P.G. Ciarlet, C. Simader, V. Solonnikov, H. Sohr, R. Glowinski, and R. Rautmann. The generous support and patience of C. Byrne and M. Reizakis of Springer were also essential for the success of the project. The first version of the translation into English was prepared by G. Tronel of Université Paris VI. Additional modifications of the translation were carried out by A. Kufner in cooperation with O. John, V. Šveràk, and G. Koch. LaTeX2e typesetting was finalized by E. Ritterová and O. Ulrych. As the editorial coordinator of the publication I would like to thank all those mentioned above and also many others who contributed to the final success of the task. In particular, the support of my mother and my sister was invaluable.

The content as well as form of the translated work keep strictly to the author's concept with no attempt to update or restructure it significantly. Such changes would have destroyed the unique historical value of the work. In addition, the huge progress that the theory of PDEs went through since the first edition of the book would require both considerable theoretical extensions and vast additional comments. We thus present Nečas's work essentially in the form in which it was published in 1967, because it is in some sense timeless in that it gives a definitive presentation of the topics it addresses, still very relevant today. Quoting C. Simader, the book is as vital now as it was in 1967.

Prague 2011 *Šárka Nečasová*

About the Translation

When Prof. Jindřich Nečas' book Les méthodes directes en théorie des équations elliptiques in 1967 appeared I was a PhD student at the University of Munich (Ludwigs-Maximilians-Universitaet Muenchen). My advisor, Prof. Dr. Ernst Wienholtz, immediately ordered this book for the library. One has to know that at this time only a few books on Sobolev spaces and their applications to PDE - problems existed: Translations of S. L. Sobolev's book (1964), the books by S. Agmon (1965) and L. Bers, F. John and M. Schechter (1964), where the last two books only treated L^2 - Sobolev spaces. Solely looking at the list of contents of Nečas' book convinced us that this book is of enormous importance for us. But there was a great problem: Prof. Wienholtz and I learned Latin for several years at school, but we never learned French. Therefore trying to understand the book was a permanent fight as well with the language as with the difficulty of the material - even though we found out quickly that the presentation was very well understandable. One great advantage of this book is that it contains all facts on the spaces $W_p^{(k)}(\Omega)$ for all $1 \leq p < \infty$ (nowadays called Sobolev spaces). In addition he regarded the spaces $V_p^{(k)}(\Omega)$ whose definition is more in the spirit of Sobolev's original definition. Here I found in well readable form the spaces $W_p^{(k)}(\Omega)$ with k non integer, all facts on traces and much more. The main part of the book concerns the problem how an elliptic boundary value problem could be translated in a weak form and in the solution of the resulting functional equation in suitable function spaces. Further regularity properties are studied.

In February 1969 after receiving my PhD I had the opportunity to participate in the Oberwolfach meeting on Partial Differential Equations. There I met first time Prof. Nečas. Having the maturity of his book in mind, I was very much surprised by his Juvenileness. He was not yet forty. Unfortunately this was the last time to meet him for more than 20 years. After 1968 he did not behave in the sense of the occupying Soviet forces. Only in May 1990 at a Spring School held in Roudnice nad Labem (Czech Republic) I met Nečas again. He nearly did not change during the past 21 years. On the contrary he see seemed to me more happy as in 1969: He lived now in a democratic, free mother country.

I learned a lot of things from this book and it is until today a standard reference for my students - and I am convinced that many students all over the world will appreciate and use the underlying English edition of the book, which is as vital as in 1967.

C. Simader, Bayreuth 2010

Preface to the French Edition

This book is devoted to the theory of linear partial differential equations of elliptic type; it is based on the work of the Seminar of Partial Differential Equations at the Mathematical Institute of the Czechoslovak Academy of Sciences as well as on a lecture series I gave at the Prague Charles University.

The content of the book concentrates on the solution of boundary value problems; less attention is given to the problem of eigenfunctions and eigenvalues. The solution of the problem is sought in the Sobolev space $W^{k,2}$, i.e. in the Hilbert space of functions with bounded Dirichlet integral. The definition of the boundary value problem, based on the concept of the integral of energy and its first variation, is expressed in the form of an equality between a certain sesquilinear form and some functionals determined by the data. From this equality, existence theorems are deduced. I call this approach the "direct method" which corresponds in general to the use of this notion by other authors. This point of view seems to be very useful and in principle simple, and can be used to solve the other questions considered in this book as well, e.g., the regularity of solutions, the introduction of the concept of very weak solutions, the use of Rellich's identity etc. This book also contains the theory of Sobolev spaces. It is assumed that the reader is familiar with basic notions and results from functional analysis, which can be found in the literature.

I tried to include in the book, in a natural way, a number of recent results, published as well as unpublished, of other authors and of myself, or at least to mention them in the bibliography. The choice of subjects and bibliography is, of course, not exhaustive and depends on the author's taste. In some sense, the book is related to the paper by E.Magenes and G.Stampacchia [1].

The chapters are divided into sections (two digits, the first denoting the corresponding chapter; Sect. 3.2 is the second section in Chap. 3), sections are divided into subsections (three digits, 3.2.5 is the fifth subsection in Sect. 3.2). Formulas are numbered succesively throughout the corresponding chapter; theorems, lemmas, remarks, exercises etc. are numbered succesively throughout the corresponding section. If we refer to a theorem etc from another chapter, we add a new digit at the front, e.g. Proposition 6.2 from Chap. 3 (i.e. the second theorem in Subsect. 3.2) is referred in other chapters as Proposition 3.6.2.

A few words about the content: The first chapter describes, in a simple form, the results of Chaps. 2 and 3 together with some questions concerning numerical methods, etc. This chapter is addressed to a larger group of readers than the other chapters. In the second chapter, we describe the theory of the Sobolev spaces $W^{k,p}$ together with imbedding theorems and the problem of traces and extensions. The third chapter is devoted to the definition of the boundary value problem and its solution and includes the Fredholm alternative. We deal here with apriori estimates of functions satisfying the Lopatinskii conditions at the boundary, and investigate the dependence of the solution on the coefficients of the operator, on the coefficients appearing in the boundary conditions and on the domain. In short, systems of equations are discussed.

In the fourth chapter, the methods of differences and of compensation are used to prove the regularity of solutions. The dual approach leads to the very weak solutions, with the Green kernel as a particular case. The fifth chapter is devoted to Rellich's equality. There we deduce, for instance, the existence of the solution of the Dirichlet problem for a second order differential operator with square integrable boundary data and without assumptions about the regularity of the boundary. In the sixth chapter, weighted spaces are introduced and used for the investigation of boundary value problems with non-degenerated differential operators. The seventh chapter deals with questions of regularity based on the maximum principle and on the mean value theorem.

Besides theorems and their proofs, there are also remarks, examples and exercises in the book. The latter ones can also serve as announcements of new results and approaches, and not only as pure exercises. There is also a list of open problems.

I want to express my gratitude to the fellows of the Department of Partial Differential Equations of the Mathematical Institute, A.Kufner and J.Kadlec, who have read the manuscript and contributed to its improvement, and further to Prof. V. Pták, and Dr. L. Pachta, the reviewer.

Prague 1966 *Jindřich Nečas*

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Chapter 1
Elementary Description of Principal Results

Chapter 1 is intended for those who want to acquire, as quickly as possible, a knowledge of the basic elements and results of the theory of elliptic equations; naturally, it is therefore reduced to the exposition of only the most important and easy results. We also have in mind non-mathematician readers whose interest is oriented more towards applications and numerical methods. Nevertheless, reading Chap. 1 is recommended to those who will study the following chapters to familiarize themselves with the basic notions which theory will meet again in the rest of the book. For the moment, we will limit ourselves to citing only the works of R. Courant and D. Hilbert [1]; S. G. Mikhlin [2]; E. Magenes and G. Stampacchia [1], where further literature can be found, and to S. Agmon [4]; J.L. Lions [4]; V. I. Smirnov [1]; S. L. Sobolev [1]; G. Hellwig [1].

1.1 Beppo Levi Spaces

1.1.1 Definition of $W^{k,2}$

We denote by $\mathbb{R}^N$ the N-dimensional Euclidean space with generic point $x = (x_1, x_2, \dots, x_N)$, and set

$$|x| = (\sum_{i=1}^{N} x_i^2)^{1/2}.$$

We denote by Ω a bounded or unbounded *domain*, i.e. an open and connected subset of $\mathbb{R}^N$.

We denote by $C^\infty(\overline{\Omega})$ the set of infinitely differentiable complex-valued functions on Ω which can be continuously extended with all their derivatives to the closure $\overline{\Omega}$ of Ω. If Ω is unbounded, we suppose that functions from $C^\infty(\overline{\Omega})$ vanish in a neighborhood of infinity; of course, this neighborhood can be different for different functions.

J. Nečas et al., *Direct Methods in the Theory of Elliptic Equations*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-10455-8_1,

We denote by $C_0^\infty(\Omega)$ the subset of all functions from $C^\infty(\overline{\Omega})$ with compact support in Ω; here, the support of ϕ, $\operatorname{supp}\phi$, is defined as the closure (in $\mathbb{R}^N$) of the set of all points x such that $\phi(x) \neq 0$. Consequently, every $\phi \in C_0^\infty(\Omega)$ vanishes identically in a neighborhood of $\partial\Omega$, the boundary of Ω.

We denote by $L^2(\Omega)$ the space of all complex-valued functions which are square integrable over Ω; the set $L^2(\Omega)$ is equipped with the scalar product

$$(v,u) = \int_\Omega v(x)\overline{u(x)}\,\mathrm{d}x \tag{1.1}$$

where $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_N$. The integrals are considered to be Lebesgue integrals.

$L^2(\Omega)$ is a Hilbert space. Wherever there is no ambiguity, we will write simply L^2 instead of $L^2(\Omega)$.

Let $i = (i_1, i_2, \ldots, i_N)$ be a vector whose components are non-negative integers. We call this vector a *multiindex*. Denote

$$|i| = \sum_{n=1}^N i_n.$$

For $u \in C^\infty(\overline{\Omega})$, we denote by $D^i u$ the expression

$$D^i u = \frac{\partial^{|i|} u}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_N^{i_N}}.$$

For $k = 1, 2, 3, \ldots$, we introduce on $C^\infty(\overline{\Omega})$ the scalar product

$$(v,u)_{W^{k,2}(\Omega)} \equiv (v,u)_k \equiv \sum_{|i|\le k} \int_\Omega D^i v D^i \overline{u}\,\mathrm{d}x. \tag{1.2}$$

The space $W^{k,2}(\Omega)$ is defined as the closure of $C^\infty(\overline{\Omega})$ with respect to the norm $(v,v)_k^{1/2}$; we also write simply $W^{k,2}$ instead of $W^{k,2}(\Omega)$. This convention will be used in the sequel for other spaces as well. For the norm $(u,u)_k^{1/2}$, we will use the notation $|u|_k$ or $|u|_{W^{k,2}(\Omega)}$.

If $f \in W^{k,2}(\Omega)$ and $|i| \le k$, $D^i f$ is defined as a function from $L^2(\Omega)$. Indeed, according to the definition of $W^{k,2}(\Omega)$, $f = \lim_{n\to\infty} f_n$ in $W^{k,2}(\Omega)$ where f_n belongs to $C^\infty(\overline{\Omega})$; hence, for $|i| \le k$, $D^i f_n$, $n = 1, 2, \ldots$, is a Cauchy sequence in $L^2(\Omega)$. We denote its limit in $L^2(\Omega)$ by $D^i f$ and call it the *generalized derivative* (distributional derivative, derivative in the sense of distributions). $D^i f$ is uniquely determined by virtue of the following assertion:

Proposition 1.1. *Assume $f \in W^{k,2}(\Omega)$. Then for any $\varphi \in C_0^\infty(\Omega)$ and for any i, $|i| \le k$,*

$$\int_\Omega \varphi D^i f\,\mathrm{d}x = (-1)^{|i|} \int_\Omega D^i \varphi f\,\mathrm{d}x.$$

Indeed, since the identity is satisfied for every f_n from $C^\infty(\Omega)$, we obtain it for f by passing to the limit. The uniqueness of $D^i f$ follows now from the fact that $C_0^\infty(\Omega)$ is dense in $L_2(\Omega)$. For details see S. L. Sobolev [1].

The space $W^{k,2}$ is again a Hilbert space if equipped with the scalar product (1.2).

We denote by $W_0^{k,2}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm of $W^{k,2}(\Omega)$.

The spaces $W^{k,2}(\Omega)$ are called spaces of functions with "finite energy".

1.1.2 Equivalent Norms

For applications, it is important to find different norms which are equivalent to $(v,v)_k^{1/2}$ and to consider boundary values of functions from $W^{k,2}(\Omega)$ as well as the quotient space $W^{k,2}/P$, where P denotes the subspace of polynomials of degree up to $k-1$.

Theorem 1.1. *Let Ω be a bounded domain. The scalar product*

$$(v,u)_{W_0^{k,2}(\Omega)} = \sum_{|i|=k} \int_\Omega D^i v D^j \overline{u}\, \mathrm{d}x, \tag{1.3}$$

defines a norm on $W_0^{k,2}(\Omega)$ *which is equivalent to* $(v,v)_k^{1/2}$.

Proof. It is sufficient to prove the so-called *Friedrichs inequality*

$$\int_\Omega |\varphi|^2\, \mathrm{d}x \le c \sum_{i=1}^N \int_\Omega \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \mathrm{d}x \tag{1.4}$$

for any function φ from $C_0^\infty(\Omega)$.

The domain Ω can be placed inside a cube $|x_i| < a$, $i = 1,2,\ldots,N$. Extending φ by zero outside of Ω, we have that

$$\varphi(x_1,x_2,\ldots,x_N) = \int_{-a}^{x_1} \frac{\partial \varphi}{\partial x_1}(\xi_1,x_2,\ldots,x_N)\, \mathrm{d}\xi_1,$$

and the Schwarz inequality leads to

$$|\varphi(x_1,x_2,\ldots,x_N)|^2 \le 2a \int_{-a}^{a} \left| \frac{\partial \varphi}{\partial x_1}(\xi_1,x_2,\ldots,x_N) \right|^2 \mathrm{d}\xi_1. \tag{1.5}$$

By integration with respect to x_1 over the interval $[-a,a]$ it follows that

$$\int_{-a}^{a} |\varphi(x_1,x_2,\ldots,x_N)|^2\, \mathrm{d}x_1 \le 4a^2 \int_{-a}^{a} \left| \frac{\partial \varphi}{\partial x_1}(\xi_1,x_2,\ldots,x_N) \right|^2 \mathrm{d}\xi_1. \tag{1.6}$$

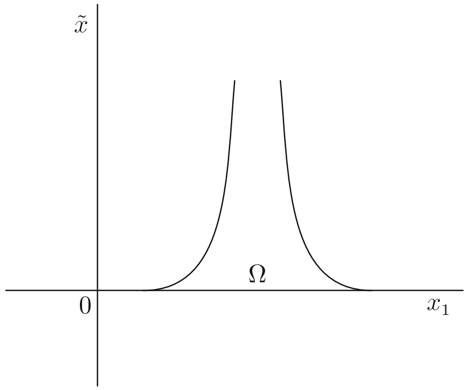

Fig. 1.1

By repeated integrations with respect to $x_2, x_3, \ldots, x_N$ over the interval $[-a, a]$ we get (1.4) with the constant c equal to $4a^2$.

Remark 1.1. The previous proof holds even in the case that Ω is bounded only in one direction, for instance in the direction of the x_1 axis (cf. Fig. 1.1 where $\tilde{x} = (x_2, \ldots, x_N)$).

Exercise 1.1. Let Ω be bounded or unbounded, $\Omega' \subset \Omega$, Ω' bounded, $\mathbb{R}^N - \overline{\Omega} \neq \emptyset$. Prove that for $u \in W_0^{k,2}(\Omega)$, the following inequality holds:

$$|u|_{W^{k,2}(\Omega')} \leq \mathrm{const}(\Omega') \left(\sum_{i=k} \int_\Omega |D^i u|^2 \, \mathrm{d}x \right)^{1/2} . \tag{1.7}$$

1.1.3 Concept of a Trace

In what follows, we will mainly work with bounded domains whose boundaries are continuous. The boundary $\partial\Omega$ of Ω is called *continuous* if the following conditions are satisfied:
– there exist positive numbers $\alpha > 0$ and $\beta > 0$, systems of local charts $(x_{r1}, x_{r2}, \ldots, x_{rN}) = (x_r', x_{rN})$, $r = 1, 2, \ldots, M$, and continuous functions a_r defined on closed $(N-1)$-dimensional cubes $|x_{ri}| \leq \alpha$, $i = 1, 2, \ldots, N-1$, and such that every point x at the boundary $\partial\Omega$ can be represented at least in one of the charts in the form $x = (x_r', a_r(x_r'))$;
– the points (x_r', x_{rN}) such that $|x_{ri}| \leq \alpha$, $i = 1, \ldots, N-1$ (we denote this set by $\overline{\Delta}_r$) and $a_r(x_r') < x_{rN} < a_r(x_r') + \beta$ belong to Ω while the points (x_r', x_{rN}) such that $x_r' \in \overline{\Delta}_r$, $a_r(x_r') - \beta < x_{rN} < a_r(x_r')$ are outside of $\overline{\Omega}$. (Cf. Fig. 1.2.).

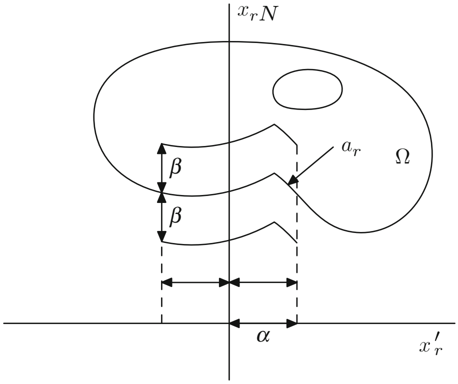

Fig. 1.2

The boundary $\partial\Omega$ is called *Lipschitz* (or *lipschitzian*) if the functions a_r above satisfy the Lipschitz condition on $\overline{\Delta}_r$, i.e.

$$x'_r, y'_r \in \overline{\Delta}_r \Longrightarrow |a_r(x'_r) - a_r(y'_r)| \leq \text{const}|x'_r - y'_r|.$$

We will show that, in a certain sense (which will be specified), the functions from $W^{k,2}(\Omega)$ have boundary values on $\partial\Omega$ together with their derivatives of order $\leq k-1$; these values generalize in a natural way the boundary values of a function continuous on $\overline{\Omega}$. We will call them *traces*.

We denote by $L^2(\partial\Omega)$ the space of functions which are square integrable over $\partial\Omega$.

Theorem 1.2. *Let Ω be a bounded domain with lipschitzian boundary. Then there exists a uniquely defined, linear and continuous mapping $T : W^{k,2}(\Omega) \to L^2(\partial\Omega)$ such that for $x \in \partial\Omega$ and $v \in C^\infty(\overline{\Omega})$, it is defined by $T(v)(x) = v(x)$.*[1]

Proof. Let V_r be the subset of Ω of points $x = (x'_r, x_{rN})$ for which $x'_r \in \Delta_r$, $a_r(x'_r) < x_{rN} < a_r(x'_r) + \beta$.

During the proof, we will omit the index r. Let $v \in C^\infty(\overline{\Omega})$; then we have

$$v(x', a(x')) = -\int_{a(x')}^{\tau} \frac{\partial v}{\partial x_N}(x', \xi_N)\,\mathrm{d}\xi_N + v(x', \tau), \quad a(x') < \tau < a(x') + \beta.$$

Using the Schwarz inequality, we obtain

$$|v(x', a(x'))|^2 \leq 2\left(\beta \int_{a(x')}^{a(x')+\beta} \left|\frac{\partial v}{\partial x_N}(x', \xi_N)\right|^2 \mathrm{d}\xi_N + 2|v(x', \tau)|^2\right). \tag{1.8}$$

[1] In the sequel, we will write simply $T(v) = v$.

Integrating this inequality with respect to τ over the interval $[a(x'), a(x')+\beta]$, we get

$$\beta|v(x',a(x'))|^2 \le 2\left(\beta^2 \int_{a(x')}^{a(x')+\beta} \left|\frac{\partial v}{\partial x_N}(x',\xi_N)\right|^2 \mathrm{d}\xi_N\right) + 2\int_{a(x')}^{a(x')+\beta} |v(x',\tau)|^2 \mathrm{d}\tau. \tag{1.9}$$

Now integration over the cube Δ leads to

$$\beta \int_\Delta |v(x',a(x'))|^2 \,\mathrm{d}x' \le 2\beta^2 \int_\Delta \mathrm{d}x' \int_{a(x')}^{a(x')+\beta} \left|\frac{\partial v}{\partial x_N}(x',x_N)\right|^2 \mathrm{d}x_N + 2\int_\Delta \mathrm{d}x' \int_{a(x')}^{a(x')+\beta} |v(x',x_N)|^2 \,\mathrm{d}x_N. \tag{1.10}$$

The surface element of $\partial\Omega$ can be expressed as

$$\mathrm{d}S = \left(1 + \sum_{i=1}^{N-1} \left(\frac{\partial a}{\partial x_i}\right)^2\right)^{1/2} \mathrm{d}x'.$$

for $i \le N-1$, the derivative $\frac{\partial a}{\partial x_i}$ is bounded since the function $a(x')$ satisfies the Lipschitz condition and its first derivatives exist almost everywhere in Δ. Hence the assertion of our theorem follows from (1.10). □

In Chap. 2, the reader will find several generalizations of Theorem 1.2; the space $L^2(\partial\Omega)$ will be replaced there by other Banach spaces equipped with a more fine structure.

Corollary 1.1. *Let Ω be a bounded domain with lipschitzian boundary, $|i| \le k-1$. Then there exists a uniquely defined, linear and continuous mapping $T_i : W^{k,2}(\Omega) \to L^2(\partial\Omega)$ such that for $u \in C^\infty\overline{\Omega}$,*

$$T_i u = D^i u.$$

In the sequel, we will denote for simplicity the trace of $u \in W^{1,2}(\Omega)$ on $\partial\Omega$ again by u.

1.1.4 The Poincaré Inequality

We shall prove the Poincaré inequality for the cube $|x_i| < a$, $i = 1,2,\dots,N$.

Theorem 1.3. *Let Ω be the cube $|x_i| < a$, $i = 1,2,\dots,N$. For $u \in W^{1,2}(\Omega)$, the following inequality holds:*

$$\int_\Omega |u|^2\,\mathrm{d}x \le 2a^2N\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2 \mathrm{d}x + \frac{1}{(2a)^N}\left|\int_\Omega u(x)\,\mathrm{d}x\right|^2. \tag{1.11}$$

Proof. For $u \in C^\infty(\overline{\Omega})$ and $x, y \in \Omega$, we can write

$$\begin{aligned} u(x) - u(y) = &\int_{y_1}^{x_1} \frac{\partial u}{\partial x_1}(\xi_1, y_2, \ldots, y_N)\,\mathrm{d}\xi_1 \\ &+ \int_{y_2}^{x_2} \frac{\partial u}{\partial x_2}(x_1, \xi_2, y_3 \ldots, y_N)\,\mathrm{d}\xi_2 + \ldots \\ &+ \int_{y_N}^{x_N} \frac{\partial u}{\partial x_N}(x_1, x_2, \ldots, x_{N-1}, \xi_N)\,\mathrm{d}\xi_N, \end{aligned} \tag{1.12}$$

and consequently,

$$\begin{aligned} |u(x) - u(y)|^2 &= |u(x)|^2 + |u(y)|^2 - u(x)\overline{u(y)} - u(y)\overline{u(x)} \\ &\le 2aN\left(\int_{-a}^{a}\left|\frac{\partial u}{\partial x_1}(\xi_1, y_2, \ldots, y_N)\right|^2 \mathrm{d}\xi_1 + \ldots \right. \\ &\left. + \int_{-a}^{a}\left|\frac{\partial u}{\partial x_N}(x_1, x_2, \ldots, \xi_N)\right|^2 \mathrm{d}\xi_N\right). \end{aligned} \tag{1.13}$$

Integration over the product $\Omega \times \Omega$ yields

$$\begin{aligned} &2(2a)^N \int_\Omega |u(x)|^2\,\mathrm{d}x \\ &\le 2aN(2a)^{(N+1)}\int_\Omega \left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}(x)\right|^2\right)\mathrm{d}x + 2\left|\int_\Omega u(x)\,\mathrm{d}x\right|^2, \end{aligned} \tag{1.14}$$

and the result follows. □

1.1.5 Rellich's Theorem

The identity mapping $W^{1,2}(\Omega) \to L^2(\Omega)$ is linear and continuous, but we have a more precise result, namely Rellich's theorem:

Theorem 1.4. *Let Ω be a bounded domain with continuous boundary. The identity mapping $W^{1,2}(\Omega) \to L^2(\Omega)$ is compact (cf. F. Riesz, B. Sz. Nagy [1]).*

Proof. A bounded subset M of some Banach space is *precompact* if and only if for every $\varepsilon > 0$ there exists a finite ε-net in M, i.e. some elements $u_1, u_2, \ldots, u_l \in M$

such that for every $v \in M$, one has

$$v \in M \Longrightarrow \min_{i=1,2,\dots,l} |v - u_i| < \varepsilon.$$

Let $\varepsilon > 0$. We can find a set $\Omega_1, \overline{\Omega}_1 \subset \Omega$, such that for all functions u from $W^{1,2}(\Omega)$, for which $|u|_{W^{1,2}(\Omega)} \leq 1$, we have the estimate

$$\left(\int_{\Omega - \Omega_1} |u|^2 \, \mathrm{d}x \right)^{1/2} < \varepsilon/6. \tag{1.15}$$

To prove it, let us consider the set V_r (cf. 1.1.3), in the proof we again omit the index r. Let $v \in C_0^\infty(\Omega)$. If $a(x') < y_N < a(x') + \beta/2$, we then have:

$$v(x', y_N) = -\int_{y_N}^{\tau} \frac{\partial v}{\partial x_N}(x', \xi_N) \, \mathrm{d}\xi_N + v(x', \tau), \quad a(x') + \beta/2 < \tau < a(x') + \beta.$$

We can deduce that

$$|v(x', y_N)|^2 \leq 2\beta \int_{a(x')}^{a(x')+\beta} \left| \frac{\partial v}{\partial x_N}(x', \xi_N) \right|^2 \mathrm{d}\xi_N + 2|v(x', \tau)|^2. \tag{1.16}$$

Integrating the last inequality with respect to τ over the interval $[a(x'), a(x') + \beta]$ we get

$$\beta |v(x', y_N)|^2 \leq 2\beta^2 \int_{a(x')}^{a(x')+\beta} \left| \frac{\partial v}{\partial x_N}(x', \xi_N) \right|^2 \mathrm{d}\xi_N + 2 \int_{a(x')}^{a(x')+\beta} |v(x', \tau)|^2 \, \mathrm{d}\tau. \tag{1.17}$$

Now we integrate (1.17) first over Δ and then with respect to y_N over the interval $[a(x'), a(x') + \delta]$, $\delta \leq \beta$. We get

$$\begin{aligned} \int_\Delta \mathrm{d}x' \int_{a(x')}^{a(x')+\delta} |v(x', y_N)|^2 dy_N &\leq 2\beta\delta \int_\Delta \mathrm{d}x' \int_{a(x')}^{a(x')+\beta} \left| \frac{\partial v}{\partial x_N}(x', x_N) \right|^2 \mathrm{d}x_N \\ &\quad + 2\delta/\beta \int_\Delta \mathrm{d}x' \int_{a(x')}^{a(x')+\delta} |v(x', x_N)|^2 \, \mathrm{d}x_N. \end{aligned} \tag{1.18}$$

This inequality is satisfied also for $u \in W^{1,2}(\Omega)$. If δ is small enough, using (1.18) for all r (omitted in (1.18)) we obtain (1.15); for Ω_1 we can take

$$\Omega_\lambda = \{x \in \Omega, \ \mathrm{dist}(x, \partial\Omega) > \lambda\},$$

λ small enough.

Let us cover Ω_1 by cubes $C_i \subset \Omega, i = 1, 2, \dots, l$, with edges less than or equal to $\varepsilon/(6\sqrt{2N})$. We deduce that

$$\begin{aligned}\left(\int_{\Omega_1} |u|^2 \,\mathrm{d}x\right)^{1/2} &\le \left(\int_{\cup_{i=1}^l C_i} |u|^2 \,\mathrm{d}x\right)^{1/2} \\ &\le \frac{\varepsilon}{6}\left(\int_{\cup_{i=1}^l C_i} \sum_{j=1}^N \left|\frac{\partial u}{\partial x_j}\right|^2 \mathrm{d}x\right)^{1/2} + \frac{1}{(2a)^N}\sum_{i=1}^l \left|\int_{C_i} u(x)\,\mathrm{d}x\right|.\end{aligned} \tag{1.19}$$

Let us consider the linear continuous mapping of $W^{1,2}(\Omega)$ to the l-dimensional complex space A defined by $y_i = \int_{C_i} u(x)\,\mathrm{d}x$, $i = 1,2,\ldots,l$. Since A is finite-dimensional, this mapping is compact. Hence for the unit ball of $W^{1,2}(\Omega)$, $|u|_{W^{1,2}(\Omega)} \le 1$, it is possible to find $u_1, u_2, \ldots, u_s$ such that for every u in this ball there exists some u_t, satisfying

$$\frac{1}{(2a)^N}\sum_{i=1}^l \left|\int_{C_i} |u - u_t|\,\mathrm{d}x\right| < \varepsilon/3. \tag{1.20}$$

For this function u we have that

$$\begin{aligned}\left(\int_{\Omega} |u-u_t|\,\mathrm{d}x\right)^{1/2} &\le \left(\int_{\Omega-\Omega_1} |u-u_t|\,\mathrm{d}x\right)^{1/2} + \left(\int_{\Omega_1} |u-u_t|\,\mathrm{d}x\right)^{1/2} \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

as a consequence of (1.15) and (1.20). Hence $u_1, u_2, \ldots, u_s$ form an ε-net in $L^2(\Omega)$ for the ball $|u|_{W^{1,2}(\Omega)} \le 1$. □

1.1.6 The Generalized Poincaré Inequality

Now, we generalize Theorem 1.3:

Theorem 1.5. *Let Ω be a bounded domain with continuous boundary. Then for $u \in W^{k,2}(\Omega)$, the following inequality holds:*

$$|u|^2_{W^{k,2}(\Omega)} \le \mathrm{const}\left(\sum_{|i|=k}\int_\Omega |D^i u|^2\,\mathrm{d}x + \sum_{|i|<k}\left|\int_\Omega D^i u\,\mathrm{d}x\right|^2\right). \tag{1.21}$$

Proof. To the contrary, let us assume that there does not exist any constant for which inequality (1.21) holds. Then we can find a sequence of functions $u_s \in W^{k,2}(\Omega)$, $s = 1,2,\ldots$, such that $|u_s|_{W^{k,2}(\Omega)} = 1$ and

$$|u_s|^2_{W^{k,2}(\Omega)} > s\left(\sum_{|i|=k}\int_\Omega |D^i u_s|^2\,\mathrm{d}x + \sum_{|i|<k}\left|\int_\Omega D^i u_s\,\mathrm{d}x\right|^2\right).$$

Hence, it follows that for $|i| = k$, $\lim_{s\to\infty} D^i u_s = 0$ in $L^2(\Omega)$. On the other hand, due to Theorem 1.4 we can find a subsequence u_{s_i} which strongly converges in $W^{k-1,2}(\Omega)$. Let $u = \lim_{i\to\infty} u_{s_i}$ in $W^{k-1,2}(\Omega)$. We have $\lim_{i\to\infty} u_{s_i} = u$ in $W^{k,2}(\Omega)$; but $D^i u = 0$ for $|i| = k$, and hence u is a polynomial of degree $\le k-1$. Obviously, we have

$$\lim_{j\to\infty} \int_\Omega D^i u_{s_j}\,\mathrm{d}x = \int_\Omega D^i u\,\mathrm{d}x = 0, \qquad |i| < k$$

and so $u \equiv 0$. But this contradicts the fact that $1 = \lim_{i\to\infty} \|u_{s_i}\|_{W^{k,2}(\Omega)} = \|u\|_{W^{k,2}(\Omega)}$. □

1.1.7 The Quotient Spaces

Let $P_{(k-1)}$ be the space of polynomials of degree $\le (k-1)$ and $P \subset P_{(k-1)}$ a linear subspace. We denote by $W^{k,2}(\Omega)/P$ the *quotient space* of classes $\tilde{u}$ of functions $u \in W^{k,2}(\Omega)$; $u, v \in \tilde{u} \Leftrightarrow u - v \in P$. The norm in the space $W^{k,2}(\Omega)/P$ is defined in the usual way:

$$|\tilde{u}|_{W^{k,2}(\Omega)/P} = \inf_{u\in\tilde{u}} |u|_{W^{k,2}(\Omega)}. \tag{1.22}$$

The quotient space of a Banach space is a Banach space. (Cf. F. Riesz, B.Sz. Nagy [1], L.A. Ljusternik, V.I. Sobolev [1].)

Example 1.1. Let $k = 2$; P can be the set of constants or the set of first order polynomials $a_0 + a_1x_1 + a_2x_2 + \ldots + a_Nx_N$ with $\sum_{i=1}^N a_i = 0$; etc.

If $P = P_{(k-1)}$, we obviously have for $u \in \tilde{u}$:

$$\left(\int_\Omega \sum_{|i|=k} |D^i u|^2\,\mathrm{d}x\right)^{1/2} \le |\tilde{u}|_{W^{k,2}(\Omega)/P_{(k-1)}}, \tag{1.23}$$

and the left hand side of (1.23) depends only on the class $\tilde{u}$. Moreover we have the following theorem:

Theorem 1.6. *Let Ω be a bounded domain with continuous boundary. The following inequality*

$$|\tilde{u}|_{W^{k,2}(\Omega)/P_{(k-1)}} \le \mathrm{const}\left(\int_\Omega \sum_{|i|=k} |D^i u|^2\,\mathrm{d}x\right)^{1/2} \tag{1.24}$$

holds for $u \in \tilde{u}$.

The space $W^{k,2}(\Omega)/P_{(k-1)}$ is a Hilbert space with the scalar product

$$\int_\Omega \sum_{|i|=k} D^i v D^i \overline{u}\,\mathrm{d}x \quad \text{with} \quad v \in \tilde{v}, u \in \tilde{u}.$$

Proof. If $\tilde{u} \in W^{k,2}(\Omega)/P_{(k-1)}$, then it is possible to find a function $u \in \tilde{u}$ such that $\int_\Omega D^i u \, dx = 0$ for $|i| < k$. Hence (1.24) is a consequence of (1.21). □

The quotient spaces $W^{k,2}(\Omega)/P$ are Hilbert spaces too:

Theorem 1.7. *The quotient space $W^{k,2}(\Omega)/P$ is a Hilbert space with the norm* (1.22).

Indeed: $W^{k,2}(\Omega) = P \dot{+} K$ i.e. the direct sum of P and its orthogonal complement in the hilbertian structure (1.2). If $\tilde{u} \in W^{k,2}(\Omega)/P$, then there exists exactly one element $u_K \in \tilde{u}, u_K \in K$; let us put $(\tilde{v},\tilde{u})_{W^{k,2}(\Omega)/P} = (v_K, u_K)$. Then we have:

$$
\begin{aligned}
|u|^2_{W^{k,2}(\Omega)/P} = \inf_{u\in\tilde{u}} |u|^2_{W^{k,2}(\Omega)} &= \inf_{u\in\tilde{u}} (|u_K|^2_{W^{k,2}(\Omega)} + |u_P|^2_{W^{k,2}(\Omega)}) \\
&= |u_K|^2_{W^{k,2}(\Omega)}.
\end{aligned}
$$

□

1.1.8 Other Equivalent Norms

In the previous subsections we have constructed equivalent norms in the space $W^{k,2}(\Omega)$ or in some subspaces. Now, we give three theorems with some other typical examples.

Theorem 1.8. *Let Ω be a bounded domain with continuous boundary. Then we have for $u \in W^{k,2}(\Omega)$ that*

$$
|u|_{W^{k,2}(\Omega)} \leq \text{const} \left(\int_\Omega |u|^2 \, dx + \int_\Omega \sum_{i=k} |D^i u|^2 \, dx \right)^{1/2}. \tag{1.26}
$$

Proof. We shall prove that $W^{k,2}(\Omega)$ is a Banach space (indeed a Hilbert space) with respect to the norm given by the right hand side of (1.26). Then we use the Banach theorem on isomorphism (cf. L.A. Ljusternik, V.I. Sobolev [1]).

Hence let u_s be a Cauchy sequence with respect to the "new" norm. According to Theorem 1.6 there exist polynomials p_s of degree $\leq (k-1)$ such that $\lim_{s\to\infty}(u_s + p_s) = u$ in $W^{k,2}(\Omega)$ for the "natural" norm. Since u_s is a Cauchy sequence in $L^2(\Omega)$, the sequence p_s is a Cauchy sequence too, but in a finite-dimensional space. Because all norms in a finite-dimensional space are equivalent, $p_s \to p$ in $W^{k,2}(\Omega)$. □

By the same way we obtain the *Friedrichs* inequality:

Theorem 1.9. *Let Ω be a bounded domain with lipschitzian boundary. Let $\Gamma \subset \partial\Omega$, meas$(\Gamma) \neq 0$. Then for $u \in W^{1,2}(\Omega)$ we have*

$$|u|_{W^{1,2}(\Omega)} \leq \text{const} \left(\int_{\Gamma} |u|^2 \, \mathrm{d}S + \int_{\Omega} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^2 \mathrm{d}x \right)^{1/2}. \tag{1.27}$$

Finally we prove

Theorem 1.10. *Let Ω be a bounded domain with lipschitzian boundary. Then for $u \in W^{2,2}(\Omega)$ we have*

$$|u|_{W^{2,2}(\Omega)} \leq \text{const} \left(\int_{\partial\Omega} |u|^2 \, \mathrm{d}S + \int_{\Omega} \sum_{|i|=2} |D^i u|^2 \, \mathrm{d}x \right)^{1/2}. \tag{1.28}$$

Proof. We proceed as in the proof of Theorem 1.8 taking into account that $(\int_{\partial\Omega} |u|^2 \, \mathrm{d}S)^{1/2}$ is a norm in $P_{(1)}$. □

In Chap. 2, we shall consider Sobolev spaces $W^{k,p}(\Omega)$, $1 \leq p < \infty$, which are L^p analogs of $W^{k,2}(\Omega)$. We shall prove various imbedding theorems for which the theorems of this section are model cases. For $p > 1$ and for sufficiently smooth domains, we will define trace spaces. Cf. S.L. Sobolev [1], J. Deny, J.L. Lions [1], E. Gagliardo [1, 2], J. Nečas [11].

1.1.9 An Imbedding Theorem

The inclusion $W^{k,2}(\Omega) \subset L^2(\Omega)$ has the following sense: the imbedding (i.e. the identity mapping) of $W^{k,2}(\Omega)$ into $L^2(\Omega)$ is continuous; we call this inclusion an *algebraic and topological inclusion*. Another type of inclusion, which is a particular case of an imbedding theorem, is given by the following:

Theorem 1.11. *Let Ω be a bounded domain in $\mathbb{R}^2$. We have $W_0^{2,2}(\Omega) \subset C(\overline{\Omega})$ algebraically and topologically; $C(\overline{\Omega})$ is the space of continuous functions on $\overline{\Omega}$, which is a Banach space with respect to the norm*

$$|u|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |u(x)|.$$

Proof. We can put Ω into a big square, say into $(-a,a) \times (-a,a)$. For $\varphi \in C_0^\infty(\Omega)$ we have that

$$\varphi(x) = \int_{-a}^{x_1} \int_{-a}^{x_2} \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(\xi_1, \xi_2) \, \mathrm{d}\xi_1 \, \mathrm{d}\xi_2,$$

hence

$$|\varphi(x)| \leq \int_{-a}^{a} \int_{-a}^{a} \left| \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(\xi) \right| \mathrm{d}\xi \leq 2a \left(\int_{-a}^{a} \int_{-a}^{a} \left| \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(\xi) \, \mathrm{d}\xi \right|^2 \right)^{1/2}.$$

□

Remark 1.2. For domains with lipschitzian boundary and for $u \in W^{k,2}(\Omega)$, it is possible to define, in terms of traces, $D^i u$ on $\partial\Omega$, $|i| \le k-1$, also exterior normal derivatives, $\partial^i u/\partial n^i$, $i = 0,1,\dots,k-1$; $\partial^i u/\partial n^i \in L^2(\partial\Omega)$. If $\partial\Omega$ is smooth, for instance a $(k+1)$-times continuously differentiable hypersurface, then using local charts we can define the spaces $W^{l,2}(\partial\Omega)$, and prove without difficulty the inequalities

$$\left|\frac{\partial^i u}{\partial n^i}\right|_{W^{(k-i-1),2}(\partial\Omega)} \le \text{const}|u|_{W^{k,2}(\Omega)}, \quad i = 1,2,\dots,k-1.$$

We can formulate the converse question, i.e.: Given $\varphi_i \in W^{(k-i-1),2}(\partial\Omega)$, $i = 0,1,\dots,k-1$, can one find $u \in W^{k,2}(\Omega)$ such that $\partial^i u/\partial n^i = \varphi_i$ on $\partial\Omega$? The answer is negative. For $k = 1$, we can construct a continuous function φ_0 on $\partial\Omega$ such that there is no function $u \in W^{1,2}(\Omega)$ with φ_0 as trace. An example of such a function (cf. for instance S.G. Mikhlin [2]) was given by J. Hadamard.

But if $\varphi_i \in W^{k-i,2}(\partial\Omega)$, the answer is positive. For details see Chap. 2.

Exercise 1.2. The mapping T defined in Theorem 1.2 is compact. Hint: Use (1.10) with β small enough.

Exercise 1.3. Let Ω be a bounded domain with continuous boundary. Then we have

$$|u|_{W^{k,2}(\Omega)} \le \text{const}\left(\sum_{|i|<k}\left|\int_\Omega x_1^{i_1}x_2^{i_2}\dots x_N^{i_N}u(x)\,\mathrm{d}x\right|^2 + \sum_{|i|=k}\int_\Omega |D^i u|^2\,\mathrm{d}x\right)^{1/2}.$$

(Cf. Theorem 1.5.)

Exercise 1.4. Let Ω be a bounded domain with continuous boundary. Let P be the set of constants; then the norm $|u|_{W^{k,2}(\Omega)/P}$ is equivalent to:

$$\left(\sum_{1\le|i|\le k}\int_\Omega |D^i u|^2\,\mathrm{d}x\right)^{1/2}.$$

Exercise 1.5. Let Ω be a bounded domain with continuous boundary and $C \subset \Omega$ a cube; then

$$|u|_{W^{k,2}(\Omega)} \le \text{const}\left(\int_C |u|^2\,\mathrm{d}x + \int_\Omega \sum_{|i|=k}|D^i u|^2\,\mathrm{d}x\right)^{1/2}.$$

Exercise 1.6. Let Ω be an ellipsoid. The analogue of (1.28), i.e. the inequality

$$|u|_{W^{k,2}(\Omega)} \le \text{const}\left(\int_{\partial\Omega} |u|^2\,\mathrm{d}S + \int_\Omega \sum_{|i|=k}|D^i u|^2\,\mathrm{d}x\right)^{1/2}$$

is not true if $k = 3$; if Ω is not an ellipsoid, and if Ω is bounded with lipschitizan boundary, then the inequality is true for $k = 3$.

Exercise 1.7. Theorem 1.2 is not true in case that the boundary $\partial\Omega$ is not lipschitzian. Hint: Consider a domain in $\mathbb{R}^2$ with a cusp.

1.2 Boundary Value Problems for Elliptic Operators

1.2.1 Elliptic Operators

Let k be a nonnegative integer, let a_{ij} be measurable and bounded complex-valued functions defined on Ω, $|i|, |j| \le k$; $i = (i_1, i_2, \dots, i_N)$, $j = (j_1, j_2, \dots, j_N)$.

A *differential operator* given by a matrix a_{ij} will be written in the following form:

$$A = \sum_{|i|,|j| \le k} (-1)^{|i|} D^i (a_{ij} D^j). \tag{1.29}$$

If the coefficients a_{ij} are $|i|$-times continuously differentiable in Ω, then we can define for u $2k$-times continuously differentiable in Ω:

$$Au = \sum_{|i|,|j| \le k} (-1)^{|i|} D^i (a_{ij} D^j u).$$

The operator (1.29) is *elliptic at the point* $x \in \Omega$, if for $\xi \in \mathbb{R}^N$, $\xi \ne 0$, we have:

$$\sum_{|i|,|j| = k} \overline{a}_{ij}(x) \xi^i \xi^j \ne 0, \quad \xi^i = \xi_1^{i_1} \xi_2^{i_2} \dots \xi_N^{i_N}. \tag{1.30}$$

The operator is *elliptic in* Ω if (1.30) is satisfied almost everywhere in Ω, and *uniformly elliptic in* Ω, if there exists a constant $c > 0$ such that almost everywhere in Ω the following inequality holds:

$$\left| \sum_{|i|,|j| = k} \overline{a}_{ij}(x) \xi^i \xi^j \right| \ge c |\xi|^{2k}. \tag{1.31}$$

Let u be in $W^{k,2}(\Omega)$ and let f be in $L^2(\Omega)$; we shall say that $Au = f$ *weakly in* Ω or that u is a *weak solution* of the equation $Au = f$ in Ω if for all $\varphi \in C_0^\infty(\Omega)$ the following identity is satisfied:

$$\sum_{|i|,|j| \le k} \int_\Omega \overline{a}_{ij} D^i \varphi D^j \overline{u} \, \mathrm{d}x = \int_\Omega \varphi \overline{f} \, \mathrm{d}x. \tag{1.32}$$

Example 2.1. We take $a_{ij} = 1$ for $i = j = 1$, $a_{ij} = 0$ otherwise (i, j are usual indices, not multiindices, $i, j = 1, 2, \ldots, N$.) As a result, we obtain the Laplace operator:

$$\triangle = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(1 \frac{\partial}{\partial x_i}\right).$$

Moreover

$$\sum_{i,j=1}^{N} \overline{a}_{ij} \xi_i \xi_j = |\xi|^2,$$

which implies that (1.31) is satisfied.

Example 2.2. Let a_{ij} (again with usual indices i, j), $i, j = 1, 2, \ldots N$, be real constants, and assume that the quadratic form

$$\sum_{i,j=1}^{N} \overline{a}_{ij} \xi_i \xi_j$$

is positive definite. Then the operator

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial}{\partial x_j}\right)$$

is uniformly elliptic. The same property holds for the operator

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(b_i) + \sum_{j=1}^{N} c_j \frac{\partial}{\partial x_j} + d,$$

where a_{ij}, b_i, c_j, d are arbitrary real constants.

Example 2.3. Consider $|i| = |j| = 2$, $a_{ij} = 1$ if $i = j$ and only one index i_τ is non zero, $a_{ij} = 2$ if $i = j$ and two indices i_μ, i_τ are non zero; $a_{ij} = 0$ if $i \neq j$. As a result, we obtain the biharmonic operator $\triangle^2$, which is uniformly elliptic since the following identity is satisfied:

$$\sum_{|i|,|j|=2} \overline{a}_{ij} \xi^i \xi^j = |\xi|^4.$$

The same is true for the operator

$$\triangle^2 + \frac{\partial}{\partial x_1}(\triangle) - \triangle + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \ldots + \frac{\partial}{\partial x_N} - 2.$$

Example 2.4. For $N = 2$, we take $a_{11} = 1$, $a_{12} = a$, $a_{21} = -a$, $a_{22} = 1$, where a is a constant. The operator

$$-\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

is formaly equal to $-\triangle$!

Example 2.5. For $N=2$ and σ a constant, we define

$$A=\frac{\partial^2}{\partial x_1^2}\left(1\frac{\partial^2}{\partial x_1^2}\right)+\frac{\partial^2}{\partial x_1\partial x_2}\left(2(1-\sigma)\frac{\partial^2}{\partial x_1\partial x_2}\right)$$
$$+\frac{\partial^2}{\partial x_1^2}\left(\sigma\frac{\partial^2}{\partial x_2^2}\right)+\frac{\partial^2}{\partial x_2^2}\left(\sigma\frac{\partial^2}{\partial x_1^2}\right)+\frac{\partial^2}{\partial x_2^2}\left(1\frac{\partial^2}{\partial x_2^2}\right),$$

which is formally equal to $\triangle^2$!

Example 2.6. For $l=1,2,3,\dots$ we define

$$\triangle^i=(-1)^l l!\sum_{|i|=l}D^i\left(\frac{1}{i_1!i_2!\dots i_N!}D^i\right).$$

1.2.2 Decomposition of Operators

Using integration by parts, we get:

Proposition 2.1. *Let a_{ij} be $|i|$-times continuously differentiable functions in Ω, $u\in W^{k,2}(\Omega)$. Let us assume that u is $2k$-times continuously differentiable in Ω, and $Au\in L^2(\Omega)$. Set $f=Au$. Then $Au=f$ is satisfied in the weak sense in Ω.*

Examples 2.4 and 2.5 show that a particular operator allows several *decompositions*. We will say that $A_1=A_2$ in Ω, if for all $u\in C^\infty(\overline{\Omega})$, $\varphi\in C_0^\infty(\Omega)$ we have:

$$\int_\Omega\sum_{|i|,|j|\le k}\overline{a}_{ij,1}D^i\varphi D^j\overline{u}\,\mathrm{d}x=\int_\Omega\sum_{|i|,|j|\le k}\overline{a}_{ij,2}D^i\varphi D^j\overline{u}\,\mathrm{d}x. \tag{1.33}$$

Proposition 2.2. *Let A_1,A_2 be two second order operators,*

$$A_l=-\sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}\left(a_{ij,l}\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{N}b_{i,l}\frac{\partial}{\partial x_i}+c_l\quad l=1,2.$$

Assume that $a_{ij,l}=\overline{a}_{ji,l}$, $l=1,2$, $a_{ij,l}$ are continuously differentiable in $\overline{\Omega}$, and that $b_{i,l},c_l$ are continuous in $\overline{\Omega}$, $l=1,2$. If $A_1=A_2$ in Ω, then we have Re $a_{ij,1}=$ Re $a_{ij,2}$, *$b_{i,1}=b_{i,2}$, $c_1=c_2$.*

Indeed: choosing $u = 1$ in (1.33) we have:

$$\int_\Omega \overline{c_1}\varphi\,\mathrm{d}x = \int_\Omega \overline{c_2}\varphi\,\mathrm{d}x \Longrightarrow c_1 = c_2.$$

For $y \in \Omega$ and i_0, j_0 two indices set $u(x) = (x_{i_0} - y_{i_0})(x_{j_0} - y_{j_0})$. Integrating by parts in (1.33) we get for $\varphi \in C_0^\infty(\Omega)$:

$$\int_\Omega \overline{A_1 u}\varphi\,\mathrm{d}x = \int_\Omega \overline{A_2 u}\varphi\,\mathrm{d}x \Longrightarrow A_1 u = A_2 u \text{ in } \Omega.$$

But at the point y we have:

$$A_1 u = a_{i_0 j_0,1}(y) + a_{j_0 i_0,1}(y) = a_{i_0 j_0,2}(y) + a_{j_0 i_0,2}(y) \Longrightarrow \operatorname{Re} a_{ij,1} = \operatorname{Re} a_{ij,2},$$

and if we choose $u(x) = x_{i_0}$, we get $b_{i_0,1} = b_{i_0,2}$. □

Remark 2.1. Under the same hypotheses as in Proposition 2.2, with real functions a_{ij} such that $a_{ij} = a_{ji}$, we can show that the decomposition of the operator is uniquely determined. This is not true if $k \geq 2$, cf. Example 2.5.

1.2.3 The Boundary Operators

Let Ω be a bounded domain with lipschitzian boundary. We say that the boundary $\partial\Omega$ is *smooth in a neighborhood of* $y \in \partial\Omega$ if for an atlas of charts (x', x_N) (cf. the definition in 1.1.3), y belongs to a subset Λ of $\partial\Omega$ defined by $x' \in \overline{\Delta}$, $x_N = a(x')$, where $a \in C^\infty(\overline{\Delta})$.

If $\partial\Omega$ is smooth in a neighborhood of each point $y \in \partial\Omega$ we say that the boundary is *smooth*.

Let $k = 1, 2, 3, \ldots$ be a fixed integer. We divide the set $\{0, 1, 2, \ldots, k-1\}$ into two subsets $\{j_1, j_2, \ldots, j_\mu\}$, $\{i_1, i_2, \ldots, i_{k-\mu}\}$.

If $\mu = 0$, the j subset is empty, if $\mu = k$, the i subset is empty.

Let Ω be a bounded domain with boundary $\partial\Omega$ almost everywhere smooth, and let Γ be an open set of $\partial\Omega$. We define on Γ the *boundary operators* B_s, $s = 1, 2, \ldots, \mu$ in the following form:

$$B_s = \frac{\partial^{j_s}}{\partial n^{j_s}} - F_s, \tag{1.34a}$$

where $\partial^{j_s}/\partial n^{j_s}$ is the j_s-th derivative in the direction of the exterior normal n; with respect to the surface measure, the tangent plane exists almost everywhere on $\partial\Omega$. The operator F_s is defined by:

$$F_s = \sum_{|i| \leq k-1} c_{si} D^i, \tag{1.34b}$$

where the functions c_{si} are measurable and bounded on $\partial\Omega$. In a neighborhood of a smooth point y, the boundary is given in a chart by $x = (\sigma, a(\sigma))$, $\sigma \in \overline{\Delta}$, and we assume that F_s is given by:

$$F_s = \sum_{|i| \le k-1} d_{si} \frac{\partial^{|i|}}{\partial\sigma_1^{i_1}\partial\sigma_2^{i_2}\dots\partial\sigma_{N-1}^{i_{N-1}}\partial n^{i_N}}, \tag{1.34c}$$

where $i_N = i_t$ and i_t is one of indices $i_1, i_2, \dots, i_{k-\mu}$. For simplicity we can define B_s first on $C^\infty(\overline{\Omega})$ and then on $W^{k,2}(\Omega)$ using the trace operators.

Now we can define *stable boundary conditions* in the following setting: we decompose the almost everywhere smooth boundary $\partial\Omega$ into disjoined open sets Γ_i, $i = 1, 2, \dots, \kappa$, such that

$$\text{meas}\left(\partial\Omega \setminus \bigcup_{i=1}^{\kappa} \Gamma_i\right) = 0.$$

For every i, we introduce a non-negative integer μ_i, $\mu_i \le k-1$, and operators B_{is}, $s = 1, 2, \dots, \mu_i$; a function $u \in W^{k,2}(\Omega)$ satisfies the boundary conditions $B_{is}u = 0$ on $\partial\Omega$, $i = 1, 2, \dots, \kappa$ and $s = 1, 2, \dots, \mu_i$, if $B_{is}u = 0$ on $\partial\Omega$ in the sense of traces. We denote by V the subspace of functions u from $W^{k,2}(\Omega)$ such that $B_{is}u = 0$ on $\partial\Omega$.

Let us remark that for some particular cases the definition of V can be generalized for domains with weaker smoothness.

1.2.4 Green's Formula

Let us give a particular case of *the partition of unity* (cf. L. Schwartz [1]):

Proposition 2.3. *Let F be a compact set in $\mathbb{R}^N$, and $G_1, G_2, \dots, G_M$ be open sets covering F. We can find $\varphi_i \in C_0^\infty(G_i)$, $i = 1, 2, \dots, M$, $0 \le \varphi_i \le 1$, such that*

$$x \in F \Longrightarrow \sum_{i=1}^{M} \varphi_i(x) = 1.$$

Proof. We construct open sets $G'_1, G'_2, \dots, G'_M$ such that $G'_i \subset G_i$, $i = 1, 2, \dots, M$, which also form a covering of F. There exists an open set G_{M+1} such that $\cup_{i=1}^{M+1} G_i = \mathbb{R}^N$ and that $\text{dist}(G_{M+1}, F) > h > 0$. Let h be small enough with $\text{dist}(G'_i, \partial G_i) > h$, $i = 1, 2, \dots, M$. Let us consider $\omega(h,x) = \exp(|x|^2/(|x^2 - h^2|))$ for $|x| < h$, and $\omega(h,x) = 0$ for $|x| \ge h$. The function $\omega(h,x)$ belongs to $C_0^\infty(\mathbb{R}^N)$. Each $\overline{G'_i}$, $i = 1, 2, \dots, M+1$, can be covered by a set of balls with radius equal to h and center $y_{ij}, j = 1, 2, \dots$. Let us assume that the covering is locally finite, which is possible. Denote

$$\psi_i(x) = \sum_j \omega(h, x - y_{ij}).$$

Then we have that

$$\psi_i \in C_0^\infty(G_i),\ x \in F \Longrightarrow \sum_{i=1}^{M} \psi_i(x) \neq 0,\ \psi_{M+1}(x) = 0.$$

Let us take

$$\varphi_i(x) = \frac{\psi_i(x)}{\sum_{i=1}^{M+1} \psi_i(x)}.$$

The functions φ_i satisfy the hypotheses of Proposition 2.3. □

Let Ω be a domain with continuous boundary, cf. 1.1.3. Hereafter we shall use the following notations:

$$U_r = \{x \in \mathbb{R}^N, x = (x'_r, x_{rN}), x'_r \in \Delta_r, a_r(x'_r) < x_{rN} < a_r(x') + \beta\},$$
$$r = 1, 2, \ldots, M.$$

Let U_{M+1} be a subdomain of Ω such that

$$\overline{U}_{M+1} \subset \Omega, \quad \cup_{i=1}^{M+1} U_r \supset \overline{\Omega}.$$

Let $\varphi_r \in C_0^\infty(U_r)$, $r = 1, 2, \ldots, M+1$, be functions which form a partition of unity for $\overline{\Omega}$ and such that φ_r, $r = 1, 2, \ldots, M$, form a partition of unity for $\partial\Omega$.

Let us recall the definition from 1.1.3:

$$\Delta_r = \{x'_r \in \mathbb{R}^{N-1}, |x_{ri}| < \alpha,\ i = 1, 2, \ldots, N-1\}.$$

We denote

$$V_r^+ = \{x \in \mathbb{R}^N, x = (x'_r, x_{rN}), x'_r \in \Delta_r, a_r(x'_r) < x_{rN} < a_r(x') + \beta\},$$
$$V_r^- = \{x \in \mathbb{R}^N, x = (x'_r, x_{rN}), x'_r \in \Delta_r, a_r(x'_r) - \beta < x_{rN} < a_r(x')\},$$
$$\Lambda_r = \{x \in \mathbb{R}^N, x = (x'_r, x_{rN})\}, x'_r \in \Delta_r, x_{rN} = a_r(x'_r)\}.$$

We denote by $n = (n_1, n_2, \ldots, n_N)$ the exterior normal at a regular (smooth) point y of $\partial\Omega$. Let us assume that $y \in \Lambda$ where Λ is described by $(\sigma, a(\sigma))$, $\sigma \in \Delta$. Let us denote:

$$G = \{x,\ x = (x', x_N),\ x_i = \sigma_i + t n_i(\sigma),\ i = 1, 2, \ldots, N-1,$$
$$x_N = a(\sigma) + t n_N(\sigma),\ |t| < \delta,\ \delta > 0\}.$$

We assume that δ is small enough and such that the transformation defined by

$$x = (x', x_N), \quad x_i = \sigma_i + t n_i(\sigma), \quad i = 1, 2, \ldots, N-1,$$
$$x_N = a(\sigma) + t n_N(\sigma), \quad \sigma \in \Delta, |t| < \delta \tag{1.35}$$

is one-to-one and infinitely differentiable together with its inverse transformation from $\overline{\Delta} \times [-\delta, \delta]$ to $\overline{G}$.

If $\partial\Omega$ is smooth, and if it is covered by the open sets $G_1, G_2, \ldots, G_M$ defined above, let $\psi_i \in C_0^\infty(G_i)$, $i = 1, 2, \ldots, M$, be a partition of unity on $\partial\Omega$. Let

$$G_{M+1} \subset \overline{G}_{M+1} \subset \Omega, \quad \cup_{r=1}^{M+1} G_r \supset \Omega$$

and, moreover, let $\psi_{M+1} \in C_0^\infty(G_{M+1})$ be such that

$$x \in \Omega \Longrightarrow \sum_{r=1}^{M+1} \psi_r(x) = 1.$$

Now we give *Green's formula*, a formula which will be very useful later:

Proposition 2.4. *Let Ω be a bounded domain with smooth boundary. Let A be the differential operator (1.29) with coefficients in $C^\infty(\overline{\Omega})$. Let $u, v \in C^\infty(\overline{\Omega})$. There exist boundary operators:*

$$L_s = \sum_{|i| \le 2k-1-s} d_{si} D^i,$$

with d_{si} infinitely differentiable on $\partial\Omega$, $s = 1, 2, \ldots, k-1$, such that

$$\int_\Omega \sum_{|i|,|j| \le k} \overline{a}_{ij} D^i v D^j \overline{u} \, \mathrm{d}x = \int_\Omega v \overline{Au} \, \mathrm{d}x + \int_{\partial\Omega} \sum_{s=1}^{k-1} \frac{\partial^s v}{\partial n^s} \overline{L_s(u)} \, \mathrm{d}S. \tag{1.36}$$

Proof. According to the definition from 1.2.4, we put $v_r = v\psi_r$, and compute (1.36) for v_r. For $r = M+1$ we obtain (1.36) immediately since $v_{M+1} \in C_0^\infty(\Omega)$. For $r \le M$, we use Green's formula: for $w, \omega \in C_0^\infty(\overline{\Omega})$ it is

$$\int_\Omega w \frac{\partial \omega}{\partial x_i} \, \mathrm{d}x = -\int_\Omega \frac{\partial w}{\partial x_i} \omega \, \mathrm{d}x + \int_{\partial\Omega} w \omega n_i \, \mathrm{d}S. \tag{1.37}$$

We denote by n_i the component of the exterior normal in a local chart.

Starting with $\int_\Omega v \overline{Au} \, \mathrm{d}x$ we obtain the left hand side of (1.36), and using the local charts (σ, t), the sum of integrals can be written

$$\int_\Delta b \frac{\partial^{|i|} v_r}{\partial \sigma_1^{i_1} \ldots \partial \sigma_{N-1}^{i_{N-1}} \partial n^{i_N}} \frac{\partial^{|j|} \overline{u}}{\partial \sigma_1^{j_1} \ldots \partial \sigma_{N-1}^{j_{N-1}} \partial n^{j_N}} \, \mathrm{d}S,$$

where $b \in C_0^\infty(\overline{\Omega})$, $|i| \le k-1$, $|j| \le 2k-1-|i|$. Integration by parts with respect to σ gives (1.36) for v_r, $r = 1, 2, \ldots, M$; the summation of formulae (1.36) for v_r, $r = 1, 2, \ldots, M+1$ gives the result. □

1.2.5 Sesquilinear Forms

We define for the operator (1.29) a *corresponding sesquilinear form:*

$$A(v,u) = \int_{\Omega} \sum_{|i|,|j| \le k} \overline{a}_{ij} D^i v D^j \overline{u} \, \mathrm{d}x, \tag{1.38}$$

i.e. a form linear in v, antilinear in u and continuous on $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$.

Besides (1.38) we introduce the *boundary forms* $a(v,u)$:

Proposition 2.5. *Let $\partial\Omega$ be almost everywhere smooth, and*

$$a(v,u) = \int_{\partial\Omega} \sum_{r=1}^{k-1} \sum_{|i| \le k-1} \overline{b}_{ri} \frac{\partial^r v}{\partial n^r} D^i \overline{u} \, \mathrm{d}S, \tag{1.39}$$

where b_{ri} are measurable and bounded on $\partial\Omega$, the derivatives being considered in the trace sense. Then $a(v,u)$ is a sesquilinear form on $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$ which vanishes if at least one of the elements v,u is in $W_0^{k,2}(\Omega)$.

Indeed, it is sufficient to observe that

$$\frac{\partial^r v}{\partial n^r} = \sum_{|i|=r} \frac{r!}{i_1! i_2! \dots i_N!} D^i v n^i,$$

where $n^i = n_1^{i_1} n_2^{i_2} \dots n_N^{i_N}$.

If now $\partial\Omega$ is smooth, under some conditions, it is possible to take $|i| \le 2k-1-r$ in (1.39). Details can be found in J.L. Lions [3].

Let us consider the boundary operator

$$\sum_{|i| \le 2k-1-r} b_{ri} D^i, \quad r = 0,1,2,\dots,k-1,$$

where the coefficients b_{ri} are $(|i|-k+1)$-times continuously differentiable on $\partial\Omega$ for $|i| \ge k$, and measurable and bounded otherwise. We say that the operator is at the most $(k-1)$-*transversal* if, in local charts, it can be written as

$$\sum_{|i| \le 2k-1-r} b_{ri}^* \frac{\partial^{|i|}}{\partial \sigma_1^{i_1} \dots \partial \sigma_{N-1}^{i_{N-1}} \partial t^{i_N}}, \quad i_N \le k-1.$$

Theorem 2.1. *Let $\partial\Omega$ be smooth.*[2] *Let v,u be in $C_0^\infty(\overline{\Omega})$ and b_{ri} measurable and bounded for $|i| < k$ and $(|i|-k+1)$-times continuously differentiable on $\partial\Omega$ for*

[2]It is possible to weaken the hypotheses on $\partial\Omega$; it suffices that $\partial\Omega$ is smooth enough.

$|i| \geq k$. Define

$$a(v,u) = \int_{\partial\Omega} \sum_{r=1}^{k-1} \sum_{|i|\leq 2k-1-r} \overline{b}_{ri} \frac{\partial^r v}{\partial n^r} D^i \overline{u} \, \mathrm{d}S, \tag{1.40}$$

and assume that the operators

$$\sum_{|i|\leq 2k-1-r} b_{ri} D^i$$

are at most $(k-1)$-transversal. Then the form $a(v,u)$ can be extended by continuity to a sesquilinear form on $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$.

Proof. Using the partition of unity from 1.2.4, we consider only the case $v = w\psi$, $w \in C_0^\infty(\overline{\Omega})$. $a(v,u)$ is a sum of integrals of the following type (we omit the index r):

$$\int_\Delta \overline{b} \frac{\partial^s v}{\partial t^s} \frac{\partial^{|i|} \overline{u}}{\partial \sigma_1^{i_1} \ldots \partial \sigma_{N-1}^{i_{N-1}} \partial t^{i_N}} \, \mathrm{d}S,$$

where b are as smooth as b_{si}. If $|i| < k$, we are in the setting of the assumptions of Proposition 2.5. Let us consider $|i| \geq k$; we have $|i| - k \leq |i| - i_N - 1$, and by integration by parts in (1.40) we obtain integrals of the following type:

$$\int_\Delta \overline{c} \frac{\partial^{|i|+s-k} v}{\partial \sigma_1^{i_1} \ldots \partial \sigma_{N-1}^{i_{N-1}} \partial t^s} \frac{\partial^k \overline{u}}{\partial \sigma_1^{l_1} \ldots \partial \sigma_{N-1}^{l_{N-1}} \partial t^{i_N}} \, \mathrm{d}S,$$

where c is continuously differentiable on $\partial\Omega$. We have $s + |i| - k \leq k-1$, $k - i_N \geq 1$; without loss of generality we can assume $l_1 \geq 1$, and denote

$$\frac{\partial^{|i|+s-k} v}{\partial \sigma_1^{i_1} \ldots \partial \sigma_{N-1}^{i_{N-1}} \partial t^s} = w, \quad \frac{\partial^{k-1} \overline{u}}{\partial \sigma_1^{l_1} \ldots \partial \sigma_{N-1}^{l_{N-1}} \partial t^{i_N}} = \omega.$$

We investigate the following integral:

$$\int_\Delta \overline{c} w \frac{\partial \omega}{\partial \sigma_1} \, \mathrm{d}S.$$

Then we have:

$$\begin{aligned} \int_\Delta \overline{c} w \frac{\partial \omega}{\partial \sigma_1} \, \mathrm{d}S &= -\int_0^\delta dt \int_\Delta \frac{\partial}{\partial t} \left(\overline{c} w \frac{\partial \overline{\omega}}{\partial \sigma_1} \right) \mathrm{d}S = \\ & - \int_0^\delta dt \int_\Delta \left(\frac{\partial \overline{c}}{\partial t} w \frac{\partial \overline{\omega}}{\partial \sigma_1} + \overline{c} \frac{\partial w}{\partial t} \frac{\partial \overline{\omega}}{\partial \sigma_1} + \overline{c} w \frac{\partial^2 \omega}{\partial \sigma_1 \partial t} \right) \mathrm{d}S. \end{aligned} \tag{1.41}$$

Setting $c(\sigma,t) = c(\sigma,0)$ yields

$$\int_0^\delta dt \int_\Delta \overline{c}w \frac{\partial^2 \omega}{\partial \sigma_1 \partial t} \, \mathrm{d}S = -\int_0^\delta dt \int_\Delta \frac{\partial}{\partial \sigma_1} (\overline{c}w) \frac{\partial \overline{\omega}}{\partial t} \, \mathrm{d}S. \tag{1.42}$$

Taking into account the regularity of the transformation (1.35), we get from (1.41) and (1.42)

$$|a(v,u)| \leq \mathrm{const}|v|_{W^{k,2}(\Omega)}|u|_{W^{k,2}(\Omega)}.$$

□

Remark 2.2. If $\partial\Omega$ is smooth in the neighborhood of $y \in \partial\Omega$, we can define (1.40) for b_{ri} with support in Λ (cf. definition in 1.2.4). We obtain again a sesquilinear form on $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$.

1.2.6 Boundary Value Problems

The aim of this section is the definition of a *boundary value problem* for an elliptic operator with general data:

Let Ω be a domain with almost everywhere smooth and lipschitzian boundary. (1.43*a*)

Let $\Gamma_1, \dots, \Gamma_\kappa$ be a partition of the boundary $\partial\Omega$ into disjoint open sets. (1.43*b*)

Let A be the differential operator (1.29) with the corresponding sesquilinear form (1.38). (1.43*c*)

Let $a(v,u)$ be the sesquilinear boundary form (1.39) or (1.40), the choice depending on the smoothness of the boundary $\partial\Omega$. (1.43*d*)

Let B_{is}, $i = 1,2,\dots,\kappa$, $s = 1,2,\dots,\mu_i$, be boundary operators. (1.43*e*)

Let $f \in L^2(\Omega)$, $u_0 \in W_0^{k,2}(\Omega)$, $g_{i_t} \in L^2(\Gamma_i)$, $i = 1,2,\dots,\kappa$, $t = 1,2,\dots,k-\mu_i$. (1.43*f*)

A function $u \in W^{k,2}(\Omega)$ is called *a weak solution of the boundary value problem* if the following conditions are satisfied:

$$u - u_0 \in V \tag{1.44a}$$

and for every $v \in V$ (cf. 1.2.3)

$$A(v,u) + a(v,u) = \int_\Omega v\overline{f} \, \mathrm{d}x + \sum_{i=1}^{\kappa} \sum_{t=1}^{k-\mu_i} \int_{\Gamma_i} \frac{\partial^{i_t} v}{\partial n^{i_t}} \overline{g}_{i_t} \, \mathrm{d}S. \tag{1.44b}$$

We shall give a formal interpretation of the properties of the solution; a precise interpretation will be given in Chap. 4.

The sense of (1.44a) is well defined: on Γ_i, $B_{is}u = B_{is}u_0$ in the trace sense. For convenience we shall denote $B_{is}u_0$ by h_{is} and we shall write $B_{is}u = h_{is}$ on $\partial\Omega$.

Let us consider (1.44b): first we have $C_0^\infty(\Omega) \subset V$, hence $Au = f$ in the weak sense (cf. Proposition 2.1). To give a sense of functions g_{i_t} let us for simplicity assume $\kappa = 1$, $\partial\Omega$ regular (smooth), a_{ij} from (1.29), b_{ri} from (1.39) or (1.40), c_{si} from (1.34) infinitely continuously differentiable. Let us assume $u \in C^\infty(\overline{\Omega})$ (in this case $f \in C^\infty(\overline{\Omega})$), and $v \in V$. By Proposition 2.4 we have:

$$A(v,u) + a(v,u) = \int_\Omega v\overline{Au}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{i=1}^{k-1} \frac{\partial^i v}{\partial n^i} \overline{L_i u}\,\mathrm{d}S + \int_{\partial\Omega} \sum_{i=1}^{k-1} \frac{\partial^i v}{\partial n^i} \overline{M_i u}\,\mathrm{d}S,$$

with

$$M_i u = \sum_{|j| \le 2k-1-i} b_{ij} D^i u.$$

Now applying (1.34), $\partial^{j_s}/\partial n^{j_s} = F_s$ we get:

$$A(v,u) + a(v,u) = \int_\Omega v\overline{Au}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{t=1}^{k-\mu} \frac{\partial^{i_t} v}{\partial n^{i_t}} \left(\overline{L_{i_t} u + M_{i_t} u}\right) \mathrm{d}S$$
$$+ \int_{\partial\Omega} \sum_{s=1}^{\mu} F_s v \left(\overline{L_{j_s} u + M_{ij_s} u}\right) \mathrm{d}S.$$

Using the partition of unity ψ_r, $r = 1,2,\ldots,M$ from 2.4, we integrate by parts and taking into account (1.34c) we finally obtain

$$A(v,u) + a(v,u) = \int_\Omega v\overline{Au}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{t=1}^{k-\mu} \frac{\partial^{i_t} v}{\partial n^{i_t}} \overline{C_t u}\,\mathrm{d}S = \int_\Omega v\overline{f}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{t=1}^{k-\mu} \frac{\partial^{i_t} v}{\partial n^{i_t}} \overline{g_t}\,\mathrm{d}S,$$

where C_t is the operator generated by $L_{i_t} + M_{i_t}$ and by the corresponding derivatives of F_s, $s = 1,2,\ldots,\mu$. Here $C_t u = g_t$ on $\partial\Omega$. If $\kappa > 1$, we "obtain" $C_{i_t} u = g_{i_t}$ on Γ_i, $i = 1,2,\ldots,\kappa$, $t = 1,2,\ldots,k-\mu_i$. For simplicity we shall write $C_{i_t} u = g_{i_t}$ on $\partial\Omega$.

The conditions on $C_{i_t} u$ depend on B_{is}, on the decomposition of A, and on the form $a(v,u)$.

We shall give a justification of these considerations in Chap. 4.

1.2.7 Examples

For a given operator A with the associated sesquilinear form $A(v,u)$, and for the given boundary sesquilinear form $a(v,u)$ and the space V, *we define on* $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$:

$$((v,u)) = A(v,u) + a(v,u). \tag{1.45}$$

We can define also the *adjoint operator:*

$$A^* = \sum_{|i|,|j|\le k} (-1)^{|i|} D^i(\overline{a}_{ji} D^j),$$

and the *sesquilinear form* $A^*(v,u) = \overline{A(u,v)}$. If we put $a^*(v,u) = \overline{a(u,v)}$, we define by (1.43a)–(1.43f) the *adjoint boundary value problem.*

Example 2.7. Let us consider $A = -\triangle$ with the associated sesquilinear form

$$A(v,u) = \int_\Omega \sum_{i=1}^N \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \,\mathrm{d}x.$$

Let us take $\Gamma_1 = \partial\Omega$, $a(v,u) \equiv 0$, $Bu = u$, $V = \{v \in W^{1,2}(\Omega), v = 0 \text{ on } \partial\Omega\}$. For the solution of $-\triangle u = f$ in Ω, we get $u = u_0 = g$ on $\partial\Omega$. This problem is called the *Dirichlet problem.*

Example 2.8. Let $A = -\triangle$, with the same decomposition as in previous example, $a(v,u) \equiv 0$. We don't put any conditions on B on $\partial\Omega$, so $k-\mu = 1$ (it is not necessary to prescribe u_0), $V = W^{1,2}(\Omega)$. We have:

$$v \in V: \quad \int_\Omega \sum_{i=1}^N \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \,\mathrm{d}x = \int_\Omega v\overline{f} \,\mathrm{d}x + \int_{\partial\Omega} v\overline{g} \,\mathrm{d}S.$$

Then $-\triangle u = f$ weakly in Ω and formally

$$\int_\Omega \sum_{i=1}^N \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \,\mathrm{d}x = \int_{\partial\Omega} v \sum_{i=1}^N \frac{\partial \overline{u}}{\partial x_i} n_i \,\mathrm{d}S - \int_\Omega v\overline{\triangle u} \,\mathrm{d}x$$

and the solution u takes on $\partial\Omega$ the value

$$g = \sum_{i=1}^N \frac{\partial u}{\partial x_i} n_i = \frac{\partial u}{\partial n},$$

where $\partial u/\partial n$ is the exterior normal derivative. This problem is called the *Neumann problem.*

Example 2.9. Let $N = 2$, $A = -\triangle$, with the decomposition given in Example 2.4, a a real number, $a(v,u) \equiv 0$. We choose again $V = W^{1,2}(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. We have for $v \in W^{1,2}(\Omega)$

$$\int_\Omega \left(\frac{\partial v}{\partial x_1} \frac{\partial \overline{u}}{\partial x_1} + a \frac{\partial v}{\partial x_1} \frac{\partial \overline{u}}{\partial x_2} - a \frac{\partial v}{\partial x_2} \frac{\partial \overline{u}}{\partial x_1} + \frac{\partial v}{\partial x_2} \frac{\partial \overline{u}}{\partial x_2} \right) \mathrm{d}x = \int_\Omega v\overline{f} + \int_{\partial\Omega} v\overline{g} \,\mathrm{d}S,$$

then $-\triangle u = f$ weakly in Ω and by formal integration by parts

$$A(v,u) = -\int_{\Omega} v \triangle \overline{u} \, \mathrm{d}x + \int_{\partial\Omega} v \left(\frac{\partial \overline{u}}{\partial x_1} n_1 + a \frac{\partial \overline{u}}{\partial x_2} n_1 - a \frac{\partial \overline{u}}{\partial x_1} n_2 + \frac{\partial \overline{u}}{\partial x_2} n_2 \right) \mathrm{d}S$$

and

$$g = \frac{\partial u}{\partial x_1}(n_1 - a n_2) + \frac{\partial u}{\partial x_2}(a n_1 + n_2).$$

The vector $(n_1 - a n_2, a n_1 + n_2)$ is directed into the exterior of Ω and is never tangent to the boundary. This problem is called *the oblique derivative problem.*

Example 2.10. We can obtain the same problem in a different way: Let us put $A = -\triangle$, with the usual decomposition. We assume $\partial\Omega$ is smooth enough and for simplicity let Ω be simply connected; let s be the length of the arc of the curve $\partial\Omega$ positively oriented. Set

$$a(v,u) = \int_{\partial\Omega} a v \frac{\partial \overline{u}}{\partial s} ds.$$

Choosing $V = W^{1,2}(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, we get $-\triangle u = f$ in Ω, $\partial u/\partial n + a\partial u/\partial s = g$ on $\partial\Omega$. This is the previous boundary condition.

Example 2.11. Let $\partial\Omega = \Gamma_1 + \Gamma_2 + \Lambda$, $\mathrm{meas}\Lambda = 0$, $A = -\triangle$ with the classical decomposition, $a(v,u) = 0$. We take $B_1 u = u$ on Γ_1, no condition prescribed on Γ_2, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_2)$, $V = \{v \in W^{1,2}(\Omega), v = 0 \text{ on } \Gamma_1\}$; let $u_0 \in W^{1,2}(\Omega)$. Then the solution of the problem is as follows: $-\triangle u = f$ in $\Omega, u = u_0 = g_0$ on Γ_1, $\partial u/\partial n = g$ on Γ_2. We call this the *mixed problem.*

Example 2.12. Denote $A = -\triangle$ with the classical decomposition. Let h be a measurable and bounded function on $\partial\Omega$, $a(v,u) = \int_{\partial\Omega} \overline{h} v \overline{u} \, \mathrm{d}S$. Let $V = W^{1,2}(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. For the solution of the problem we have:

$$v \in W^{1,2}(\Omega) \Longrightarrow \int_{\Omega} \sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \mathrm{d}x + \int_{\partial\Omega} \overline{h} v \overline{u} \, \mathrm{d}S = \int_{\Omega} v \overline{f} \, \mathrm{d}x + \int_{\partial\Omega} v \overline{g} \, \mathrm{d}S.$$

Then formally $-\triangle u = f$ in Ω, $\partial u/\partial n + hu = g$ on $\partial\Omega$. This problem is called *the Newton problem.*

Example 2.13. Let us decompose the domain Ω into two subdomains $\Omega_1 + \Omega_2$ and a set of measure zero (cf. Fig. 1.3). On Ω_1 we put $A = -a\triangle$, on Ω_2 $A = -b\triangle$, where a, b are positive constants, $a \neq b$. Set

$$A(v,u) = \int_{\Omega_1} a \sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \mathrm{d}x + \int_{\Omega_2} b \sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \mathrm{d}x, \quad a(v,u) \equiv 0$$

and let $Bu = u$ on $\partial\Omega$. Let u_0 be in $W^{1,2}(\Omega)$, $f \in L^2(\Omega)$. We denote $u_i = u$ in Ω_i. Formally the problem corresponds to $-a\triangle u_1 = f$ in Ω_1, $-b\triangle u_2 = f$ in Ω_2, $u_1 = u_0$ on Λ_1, $u_2 = u_0$ on Λ_2, where $\Lambda_1 = \partial\Omega \cap \partial\Omega_1$, $\Lambda_2 = \partial\Omega \cap \partial\Omega_2$, and let us denote $\Lambda = \partial\Omega_1 \cap \partial\Omega_2$. On Λ we have $u_1 = u_2$ (in the trace sense) and formally, denoting

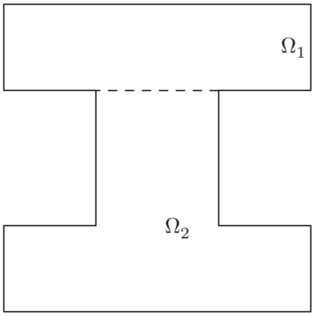

Fig. 1.3

by n the exterior normal to $\partial\Omega_1$, $a\partial u_1/\partial n = b\partial u_2/\partial n$; this relation is an interface condition on Λ. This problem is called the *transmission problem.*

Example 2.14. We take $N = 2$, $k = 2$, $A = \triangle^2$ with the sesquilinear form

$$A(v,u) = \int_\Omega \left(\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 \overline{u}}{\partial x_1^2} + 2\frac{\partial^2 v}{\partial x_1 \partial x_2}\frac{\partial^2 \overline{u}}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial x_2^2}\frac{\partial^2 \overline{u}}{\partial x_2^2} \right) \mathrm{d}x, \quad a(v,u) \equiv 0;$$

we consider $\Gamma_1 = \partial\Omega$; $B_1 u = u$, $B_2 u = \partial u/\partial n$; $V = \{v \in W^{2,2}(\Omega),\ v = \partial v/\partial n = 0 \text{ on } \partial\Omega\}$, $f \in L^2(\Omega)$, $u_0 \in W^{2,2}(\Omega)$. The solution corresponds to the problem $\triangle^2 u = f$ in Ω, $u = u_0 = g_1$ on $\partial\Omega$, $\partial u/\partial n = \partial u_0/\partial n = g_2$ on $\partial\Omega$. This is the *Dirichlet problem* for the biharmonic operator $\triangle^2$.

Example 2.15. We take $N = 2$, $k = 2$, $A = \triangle^2$, the decomposition of the operator as in Example 2.5, σ real, $a(v,u) \equiv 0$, $V = W^{2,2}(\Omega)$, $f \in L^2(\Omega)$, $g_1 \in L^2(\partial\Omega)$, $g_2 \in L^2(\partial\Omega)$. For the solution of the problem we consider:

$$v \in W^{2,2}(\Omega) \Longrightarrow$$

$$\int_\Omega \left(\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 \overline{u}}{\partial x_1^2} + 2(1-\sigma)\frac{\partial^2 v}{\partial x_1 \partial x_2}\frac{\partial^2 \overline{u}}{\partial x_1 \partial x_2} + \sigma\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 \overline{u}}{\partial x_2^2} + \sigma\frac{\partial^2 v}{\partial x_2^2}\frac{\partial^2 \overline{u}}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2}\frac{\partial^2 \overline{u}}{\partial x_2^2} \right) \mathrm{d}x$$
$$= \int_\Omega v\overline{f}\,\mathrm{d}x + \int_{\partial\Omega} v\overline{g_1}\,\mathrm{d}S + \int_{\partial\Omega} \frac{\partial v}{\partial n}\overline{g_2}\,\mathrm{d}S.$$

If the curve $\partial\Omega$ has a positive orientation, formally we obtain:

$$A(v,u) = \int_\Omega v\overline{Au}\,\mathrm{d}x + \int_{\partial\Omega} v\overline{Tu}ds + \int_{\partial\Omega} \frac{\partial v}{\partial n}\overline{Mu}ds$$

where

$$Mu = \sigma\triangle u + (1-\sigma)\left(\frac{\partial^2 u}{\partial x_1^2}n_1^2 + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}n_1 n_2 + \frac{\partial^2 u}{\partial x_2^2}n_2^2 \right),$$

$$Tu = -\frac{\partial}{\partial n}\triangle u + (1-\sigma)\frac{\partial}{\partial s}\left(\frac{\partial^2 u}{\partial x_1^2}n_1 n_2 - \frac{\partial^2 u}{\partial x_1 \partial x_2}(n_1^2 - n_2^2) - \frac{\partial^2 u}{\partial x_2^2}n_1 n_2\right).$$

Then the solution u satisfies $\triangle^2 u = f$ in $\Omega, Tu = g_1$ on $\partial\Omega$ and $Mu = g_2$ on $\partial\Omega$. This is called the *Neumann problem* for the biharmonic operator.

Example 2.16. Take the same hypotheses as in the previous example, but with $Bu = u$ on $\partial\Omega$, $V = \{v \in W^{2,2}(\Omega),\ v = 0 \text{ on } \partial\Omega\}$. Let $u_0 \in W^{2,2}(\Omega)$, $f \in L^2(\Omega)$, $g_2 \in L^2(\partial\Omega)$. The solution u satisfies $\triangle^2 u = f$ in Ω, $u = u_0 = g_1$ on $\partial\Omega$, $Mu = g_2$ on $\partial\Omega$. This problem is the *intermediary problem* for the biharmonic operator.

Example 2.17. Under the same hypotheses as in Examples 2.15 and 2.16, we define for a a real constant :

$$a(v,u) = a\int_{\partial\Omega}\frac{\partial v}{\partial n}\frac{\partial \overline{u}}{\partial n}ds.$$

The solution satisfies $\triangle^2 u = f$ in Ω, $u = u_0 = g_1$, $Mu + a\,\partial u/\partial n = g_2$ on $\partial\Omega$.

Example 2.18. Let us solve the problem given formally by $\triangle^2 u = f$ in Ω, $u = g_1$ on $\partial\Omega$, $\triangle u = g_2$ on $\partial\Omega$. We cannot set simply $\sigma = 1$ in Example 2.16 for reasons to be explained later. But we can proceed as follows: we find $\omega \in W^{1,2}(\Omega)$ such that $\triangle\omega = f$ in Ω, $\omega = g_2$ on $\partial\Omega$. Then we solve the problem to find u such that $\triangle u = \omega$ in Ω, $u = g_1$ on $\partial\Omega$. The initially posed problem leads to a system of simple equations that we can solve successively as a unique equation.

Example 2.19. Let $A = \triangle^2 + 1$. We want to solve (formally) the following problems:

$$\triangle^2 u + u = f \text{ in } \Omega, \quad \triangle u = g_1, \quad (\partial/\partial n)\triangle u = g_2 \text{ on } \partial\Omega; \qquad (a)$$

$$\triangle^2 u + u = f \text{ in } \Omega, \quad u = g_1, \quad \triangle u = g_2 \text{ on } \partial\Omega; \qquad (b)$$

$$\triangle^2 u + u = f \text{ in } \Omega, \quad \partial u/\partial n = g_1, \quad (\partial/\partial n)\triangle u = g_2 \text{ on } \partial\Omega. \qquad (c)$$

For (a) we put $\triangle u = \omega$ and for ω we have $\triangle^2\omega + \omega = \triangle f$ in Ω, $\omega = g_1$, $\partial\omega/\partial n = g_2$ on $\partial\Omega$; this is the Dirichlet problem for the operator $\triangle^2 + 1$: i.e. a problem of our type is considered. For ω known, we find u by $\triangle^2 u = \triangle\omega$, and then $u = f - \triangle^2 u$.

Concerning (b), we put $\triangle^2 + 1 = (\triangle + i)(\triangle - i)$, and $(\triangle - i)u = \omega$; we have $(\triangle + i)\omega = f$, $\omega = g_2 - ig_1$ on $\partial\Omega$; this is again a problem of the considered type. If ω is found, we want to find u such that $(\triangle - i)u = \omega$ in Ω, $u = g_1$ on $\partial\Omega$ and we are in the setting of the definition from 1.2.6. For (c) we can proceed as in the case (b).

There exist problems which don't enter in the setting of the definition of 1.2.6, or which cannot be transformed into problems of that type. For instance if we want to solve the problem $(\triangle^2 + 1)u = f$ in Ω, (a) $u = g_1$ on $\partial\Omega$, $(\partial/\partial n)\Delta u = g_2$ on $\partial\Omega$, or (b) $(\partial u/\partial n) = g_1$, $\triangle u = g_2$ on $\partial\Omega$.

If we put $\triangle u = \omega$, the problem is to solve the system $\triangle u - \omega = 0$, $\Delta\omega + u = f$ in Ω and (a) $u = g_1$, $\partial\omega/\partial n = g_2$ on $\partial\Omega$; (b) $\partial u/\partial n = g_1$, $\omega = g_2$ on $\partial\Omega$.

Such systems will be solved in Chap. 3.

Generally to find a weak solution is more natural than try to find a classical solution; this is connected with physical considerations. Nevertheless the problem of regularity of weak solutions which will be discussed in Chap. 4 is very important, in particular in applications.

Other possible formulations of boundary value problems, even in the classical sense or in a more general form than in the definition of 1.2.6, will be discussed in Chaps. 4 through 7. For references see M. Schechter [4, 5], F. E. Browder [3–5], S. Agmon, A. Douglis, L. Nirenberg [1, 2], J. Nečas [1, 2, 8], etc.

The terminology concerning the boundary value problems as in the definition of 1.2.6 is not unified: if $B_1 u = u$, $B_2 u = \partial u/\partial n, \dots, B_k u = \partial^{k-1} u/\partial n^{k-1}$, we call it a *Dirichlet problem*; on the contrary, if no condition is given, we call it a *Neumann problem.* For other problems with $a(v,u) \equiv 0$, $\Gamma_1 = \partial\Omega$ we call it an *intermediary problems.* If $a(v,u) \neq 0$ with condition (1.40), and $V = W^{k,2}(\Omega)$, we call it an *oblique derivative problem.* If $\partial\Omega$ is decomposed into more pieces $\Gamma_1, \Gamma_2, \dots, \Gamma_\kappa$ we call it a *mixed problem.* Cf. also E. Magenes, G. Stampacchia [1].

Exercise 2.1. Let us take for $\eta \in C_0^\infty(\overline{\Omega})$,

$$-\triangle = -\frac{\partial}{\partial x_1}\left(1\frac{\partial}{\partial x_1}\right) - \frac{\partial}{\partial x_1}\left(\eta\frac{\partial}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\eta\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(1\frac{\partial}{\partial x_2}\right) + \frac{\partial \eta}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial \eta}{\partial x_2}\frac{\partial}{\partial x_1}.$$

Formulate the Neumann problem for this operator.

Exercise 2.2. Later in Chap. 2, we shall see that if the boundary of Ω is almost everywhere smooth, then $V = \{v \in W^{k,2}(\Omega),\ v = \partial v/\partial n = \ldots = \partial^{k-1} v/\partial n^{k-1} = 0 \text{ on } \partial\Omega\} = W_0^{k,2}(\Omega)$. Prove that the formulation of the Dirichlet problem does not depend on the decomposition of the operator A.

Exercise 2.3. Let be $N = 2$, $\partial\Omega$ smooth. Prove that Example 2.16 is formally equivalent, for $g_1 = g_2 = 0$, to the problem $\triangle^2 u = f$ in Ω, $u = 0$, $\triangle u - [(1-\sigma)/\rho](\partial u/\partial n) = 0$ on $\partial\Omega$, where ρ is the curvature radius on $\partial\Omega$.

1.3 The V-ellipticity, Existence and Uniqueness of the Solution

1.3.1 The Lax-Milgram Lemma

The proof of the existence of a solution to a boundary value problem is based on a simple generalization of the F. Riesz theorem due to P.D. Lax and A. Milgram:

Lemma 3.1. *Let H be a Hilbert space, and $((v,u))$ a sesquilinear form in $H \times H$ satisfying:*

$$|((v,u))| \le c|v||u|, \quad c \ge 0, \tag{1.46}$$

$$|((v,v))| \ge \alpha|v|^2, \quad \alpha > 0. \tag{1.47}$$

Then every functional F on H can be uniquely represented in the form $((v,u)) = Fv$ with $u \in H$. Moreover, $|u| \le (c/\alpha)|F|$.

Proof. Let (v,u) be the scalar product on H. The sesquilinear form $((v,u))$ defines for $u \in H$ a functional on H which can be uniquely represented, according to the F. Riesz theorem, by $(v,\omega), \omega \in H$. This defines a one-to-one bounded linear mapping $Z : H \to H$: if $Zu = 0$, $((v,u)) \equiv 0$, then $((u,u)) = 0$ and (1.47) implies $u \equiv 0$. The mapping Z is open; indeed, $\alpha|u|^2 \le |((u,u))| = |(u,Zu)| \le c|u||Zu|$, then $\alpha|u| \le c|Zu|$; $Z(H)$ is a closed subset of H. Let us assume $Z(H) \ne H$; in this case there would exist a $w \ne 0$ such that $(w,Zu) = 0$, then for $u \in H$, $\alpha|w|^2 \le |(w,Zw)| = 0 \Longrightarrow w = 0$; hence we have a contradiction. □

1.3.2 Solving the Boundary Value Problem

Let us assume that the space V and the sesquilinear form (1.45) are given. The form (1.45) is called *V-elliptic* if there exists a constant $\alpha > 0$ such that

$$v \in V \Longrightarrow |((v,v))| > \alpha|v|^2. \tag{1.48}$$

We have

Theorem 3.1. *A boundary value problem with a V-elliptic sesquilinear form $((v,u))$ has a unique solution u, satisfying*

$$|u|_{W^{k,2}(\Omega)} \le \text{const}(|f|_{L^2(\Omega)} + |u_0|_{W^{k,2}(\Omega)} + \sum_{i=1}^{\kappa}\sum_{t=1}^{k-\mu_i} |g_{it}|_{L^2(\partial\Omega)}). \tag{1.49}$$

Proof. Let us put

$$Fv = \int_\Omega v\overline{f}\,\mathrm{d}x + \sum_{i=1}^{\kappa}\sum_{t=1}^{k-\mu_i} \int_{\partial\Omega} \frac{\partial^t v}{\partial n^t}\overline{g}_{it}\,\mathrm{d}S - ((v,u_0)).$$

According to Theorem 1.2, the expression

$$\int_{\partial\Omega} \frac{\partial^t v}{\partial n^t}\overline{g}_{it}\,\mathrm{d}S$$

is a functional on V, which is a Hilbert space with scalar product (1.2). From Lemma 3.1, we deduce the existence of a unique $\omega \in V$ such that for all $v \in V$, $((v,\omega)) = Fv$, and

$$|\omega|_{W^{k,2}(\Omega)} \leq \text{const}(|f|_{L^2(\Omega)} + |u_0|_{W^{k,2}(\Omega)} + \sum_{i=1}^{\kappa} \sum_{t=1}^{k-\mu_i} |g_{it}|_{L^2(\partial\Omega)}). \tag{1.50}$$

If we put $u = \omega + u_0,$ we obtain a solution such that (1.49) is true due to (1.50). Let u_1, u_2 be two solutions of the problem. Then $u_1 - u_2 \in V$, $((u_1 - u_2, u_1 - u_2)) = 0$, and consequently $u_1 = u_2$. □

Example 3.1. Let us consider

$$((v,u)) = \int_\Omega \left(\sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} + v\overline{u} \right) \mathrm{d}x.$$

Clearly $((v,u))$ is $W^{1,2}(\Omega)$-elliptic, hence V-elliptic if $V \subset W^{1,2}(\Omega)$. The considered sesquilinear form corresponds to the operator $-\triangle + 1$.

Example 3.2. Let us consider

$$((v,u)) = \int_\Omega \left(\sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \right) \mathrm{d}x.$$

Let $V = \{v = W^{1,2}(\Omega), v = 0 \text{ on } \partial\Omega\}$. The form $((v,u))$ is V-elliptic according to (1.27) where we choose $\Gamma = \partial\Omega$.

Example 3.3. The sesquilinear form,

$$((v,u)) = \int_\Omega \left(\sum_{i=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \right) \mathrm{d}x$$

is not $W^{1,2}(\Omega)$-elliptic. Indeed: for $u = \text{const}$, $((u,u)) = 0$.

Remark 3.1. If $\partial\Omega$ is almost everywhere smooth, then

$$V = \{v \in W^{k,2}(\Omega),\ v = \partial v/\partial n = \ldots = \partial^{k-1} v/\partial n^{k-1} = 0 \text{ on } \partial\Omega\} = W_0^{k,2}(\Omega).$$

The Dirichlet problem can be defined for any bounded domain if $V = W_0^{k,2}(\Omega)$ (and under some restrictions also for Ω unbounded, $\mathbb{R}^N - \Omega \neq \emptyset$.)

1.3.3 The Case of Quotient Spaces

To overcome the difficulties which appeared in Example 3.3 we proceed as follows: Let $P \subset P_{(k-1)}$ (cf. 1.1.7) and for a given V, let $P \subset V$. We assume that $((\tilde{v}, \tilde{u}))$ is a bounded sesquilinear form on $W^{k,2}(\Omega)/P \times W^{k,2}(\Omega)/P$ defined by $((\tilde{v}, \tilde{u})) =$

$((v,u))$, $v \in \tilde{v}$, $u \in \tilde{u}$, $|((\tilde{v},\tilde{u}))| \le \text{const}|\tilde{v}|_{W^{k,2}(\Omega)/P}|\tilde{u}|_{W^{k,2}(\Omega)/P}$; we say that the sesquilinear form $((\tilde{v},\tilde{u}))$ is *V/P-elliptic* if:

$$\tilde{v} \in V/P \Longrightarrow |((\tilde{v},\tilde{v}))| \ge \alpha|\tilde{v}|^2_{W^{k,2}(\Omega)/P}\,, \quad \alpha > 0. \tag{1.51}$$

After a simple adaptation of the proof of Theorem 1.7, and if we put $V = P\dot{+}K$, we get:

Proposition 3.1. *The space V/P is an Hilbert space with the norm:*

$$|\tilde{v}|_{V/P} = \inf_{v\in\tilde{v}} |v|_{W^{k,2}(\Omega)}. \tag{1.52}$$

Theorem 3.2. *Let a boundary value problem be given for the sesquilinear form $((\tilde{v},\tilde{u}))$ which is V/P-elliptic. A necessary and sufficient condition for the existence of a solution of the problem is the so-called compatibility condition, i.e.*

$$p \in P \Longrightarrow \int_\Omega p\overline{f}\,\mathrm{d}x + \sum_{i=1}^{\kappa}\sum_{t=1}^{k-\mu_i}\int_{\partial\Omega}\frac{\partial^t p}{\partial n^t}\overline{g}_{it}\,\mathrm{d}S = 0. \tag{1.53}$$

In this case the solution is determined uniquely modulo a polynomial $p \in P$; we have for a appropriately chosen function u

$$|u|_{W^{k,2}(\Omega)} \le \text{const}(|f|_{L^2(\Omega)} + |u_0|_{W^{k,2}(\Omega)} + \sum_{i=1}^{\kappa}\sum_{t=1}^{k-\mu_i}|g_{it}|_{L^2(\partial\Omega)}). \tag{1.54}$$

A possible and uniquely determined choice for $p \in P$ is :

$$\int_\Omega p\overline{u}\,\mathrm{d}x = 0. \tag{1.55}$$

Proof. Let us set

$$F\tilde{v} = \int_\Omega v\overline{f}\,\mathrm{d}x + \sum_{i=1}^{\kappa}\sum_{t=1}^{k-\mu_i}\int_{\partial\Omega}\frac{\partial^t v}{\partial n^t}\overline{g}_{it}\,\mathrm{d}S - ((v,u_0)).$$

According to (1.53), and since $((\tilde{v},\tilde{u_0})) = ((v,u_0))$, $F\tilde{v}$ is a functional on V/P. We apply again Lemma 3.1, taking into account Proposition 3.1. Then there exists a uniquely determined $\overline{\varpi} \in V/P$ such that $\tilde{v} \in V/P \Longrightarrow ((\tilde{v},\tilde{\varpi})) = F\tilde{v}$, and the following inequality holds

$$\sup_{|\tilde{v}|_{V/P}\le 1} |F\tilde{v}| \le \text{const}(|f|_{L^2(\Omega)} + |u_0|_{W^{k,2}(\Omega)} + \sum_{i=1}^{\kappa}\sum_{t=1}^{k-\mu_i}|g_{it}|_{L^2(\partial\Omega)}). \tag{1.56}$$

Let us put $u = \omega + u_0$, $\omega \in \tilde{\omega}$. Clearly, u is a solution of the problem. If u_1, u_2 are two solutions, we have $((\tilde{u_1} - \tilde{u_2}, \tilde{u_1} - \tilde{u_2})) = 0$, so $u_1 - u_2 \in P$. If we decompose $W^{k,2}(\Omega) = P \dot{+} K$ using the scalar product

$$\int_\Omega (v\overline{u} + \sum_{|i|=k} D^i v D^i \overline{u})\,\mathrm{d}x,$$

we get a solution $u \in \tilde{u}$, $u \in W^{k,2}(\Omega)$, $u = u_P + u_K$, $u_P \in P$, $u_K \in K$; $|u_K|_{W^{k,2}(\Omega)} = |\tilde{u}|_{W^{k,2}(\Omega)/P} \le |\tilde{\omega}|_{W^{k,2}(\Omega)/P} + |u_0|_{W^{k,2}(\Omega)}$. Then by (1.56) and Lemma 3.1, we get for u_K the estimate (1.54); clearly (1.55) is true for u_K. □

Let us observe that V-ellipticity is a particular case of V/P-ellipticity for $P = \{0\}$.

Exercise 3.1. Let Ω be a domain with continuous boundary, $P \subset P_{(k-1)}$. In $L^2(\Omega)$, let $P_{(k-1)} = P \dot{+} Q$ and $p_1, p_2, \ldots, p_l$ be an orthonormal basis of Q. Then $W^{k,2}(\Omega)/P$ is a Hilbert space with the scalar product

$$\int_\Omega \sum_{|i|=k} D^i v D^i \overline{u}\,\mathrm{d}x + \sum_{i=1}^{l} \int_\Omega v\overline{u} p_i\,\mathrm{d}x.$$

1.3.4 Conditions of V-ellipticity

The aim of this chapter is not to consider very general conditions, usually of algebraic type, implying V-ellipticity and V/P-ellipticity. We restrict the problem to a simple theorem with hypotheses often satisfied in practical cases.

Theorem 3.3. *Let Ω be a bounded domain with continuous boundary. Let for $i = (i_1, i_2, \ldots, i_N)$, $|i| = k$, ζ_i be arbitrary complex numbers. Let us assume:*

$$\sum_{|i|,|j|=k} \frac{\overline{a}_{ij} + a_{ji}}{2} \zeta_i \overline{\zeta}_j \ge c \sum_{|i|=k} |\zeta_i|^2, \quad c > 0. \tag{1.57}$$

Suppose that the sesquilinear form $((v,u)) = A(v,u) + a(v,u)$ satisfies

$$\mathrm{Re}\left(((v,v)) - \sum_{|i|,|j|=k} \int_\Omega \overline{a}_{ij} D^i v D^j \overline{v}\,\mathrm{d}x \right) \ge 0. \tag{1.58}$$

Let V be a subspace of $W^{k,2}(\Omega)$, $P \subset V \cap P_{(k-1)}$, $((\tilde{v}, \tilde{u})) = ((v,u))$ a sesquilinear form on $W^{k,2}(\Omega)/P \times W^{k,2}(\Omega)/P$; moreover we assume:

$$p \in V \cap P_{(k-1)} \Longrightarrow \{\mathrm{Re}\,((p,p)) = 0 \Leftrightarrow p \in P\}. \tag{1.59}$$

Then the sesquilinear form $((\tilde{v}, \tilde{u}))$ is V/P-elliptic.

Proof. It is easy to see that the sesquilinear form

$$\int_\Omega \sum_{i=k} D^i v D^i \overline{u} \, \mathrm{d}x \tag{1.60}$$

is a scalar product on $V/Q, Q = V \cap P_{(k-1)}$. Indeed, according to Theorem 1.7, $W^{k,2}(\Omega)/Q$ is a Hilbert space which can be decomposed into a direct sum: $W^{k,2}(\Omega)/Q = V/Q \dot{+} H/Q$; let $\tilde{u}_n \in V/Q$ be a Cauchy sequence for the norm generated by (1.60). Theorem 1.6 implies the existence of $p_n \in P_{(k-1)}$ such that $u_n + p_n$ is a Cauchy sequence in $W^{k,2}(\Omega)$ and *a fortiori* $\tilde{u}_n + \tilde{p}_n$ is a Cauchy sequence in $W^{k,2}(\Omega)/Q$; but $\tilde{u}_n + \tilde{p}_n = \tilde{u}_n + \tilde{q}_n$, $\tilde{q}_n \in H/Q$, so $\tilde{u}_n$ is a Cauchy sequence in $W^{k,2}(\Omega)/Q$. Now we prove that

$$((\tilde{v},\tilde{u})) + \overline{((\tilde{u},\tilde{v}))} \tag{1.61}$$

is a scalar product on V/P. Let $\tilde{v}_s$ be a Cauchy sequence with respect to the norm $(\mathrm{Re}\, ((\tilde{v},\tilde{v})))^{1/2}$. Using (1.57), (1.58) there exists $p_s \in Q$ such that $\lim_{s\to\infty}(v_s + p_s) = v$ in $W^{k,2}(\Omega)$.

Obviously (1.59) implies that (1.61) defines a scalar product on Q/P. Since $\tilde{v}_s + \tilde{p}_s$ is a Cauchy sequence in $W^{k,2}(\Omega)/P$, it is also Cauchy with respect to the scalar product (1.61). Hence $\tilde{p}_s$ is a Cauchy sequence in $W^{k,2}(\Omega)/P$, and also $\tilde{v}_s$ is a Cauchy sequence, and the result follows from the Banach isomorphism theorem. □

Remark 3.2. The condition (1.57) implies the uniform ellipticity, i.e. (1.31); the converse is in general not true.

Let us observe that very often, if (1.57) is true, there exists a constant $\lambda > 0$, such that $((v,u)) + \lambda(v,u)$ is $W^{k,2}(\Omega)$-elliptic. The case $\lambda = 0$ is treated by the Fredholm alternative (cf. Sect. 1.5 and Chap. 3).

Example 3.4. We consider Example 2.7; the form $((v,u))$ is V-elliptic; cf. Example 3.2.

Example 3.5. Let us consider Example 2.8; according to Theorem 1.6 or Theorem 3.3 we get the $W^{1,2}(\Omega)/P_{(0)}$-ellipticity. The necessary and sufficient condition for the existence of a solution (the compatibility condition) reads

$$\int_\Omega f \, \mathrm{d}x + \int_{\partial\Omega} g \, \mathrm{d}S = 0.$$

The solution is unique modulo a constant.

Example 3.6. Let us consider Example 2.9. Obviously, (1.57) is satisfied. We have the $W^{1,2}(\Omega)/P_{(0)}$-ellipticity, and we have to add

$$\int_\Omega f \, \mathrm{d}x + \int_{\partial\Omega} g \, \mathrm{d}S = 0.$$

Example 3.7. Let us consider Example 2.9 with $a = \mathrm{i}$. If $v(x) = x_1 - \mathrm{i}x_2$, we have $((v,v)) = 0$, hence the $W^{1,2}(\Omega)/P_{(0)}$-ellipticity does not hold.

Example 3.8. Let us consider Example 2.10, a real. We have

$$\mathrm{Re}\, a \int_{\partial\Omega} u \frac{\partial \overline{u}}{\partial s}\, \mathrm{d}s = 0,$$

which implies that (1.58) is satisfied. (1.57) is obvious and we have the $W^{1,2}(\Omega)/P_{(0)}$-ellipticity. We have to add the compatibility condition

$$\int_{\Omega} f\, \mathrm{d}x + \int_{\partial\Omega} g\, \mathrm{d}S = 0.$$

Example 3.9. Considering Example 2.12 with $h \geq 0, h \neq 0$, from (1.27) or Theorem 3.3 we get the $W^{1,2}(\Omega)$-ellipticity.

Example 3.10. We consider Example 2.11. Then the $W^{1,2}(\Omega)$-ellipticity follows as in Example 3.9.

Example 3.11. We consider Example 2.13. The V-ellipticity is a consequence of Theorem 1.6 or Theorem 3.3.

Example 3.12. We consider Example 2.14. The hypotheses (1.57) and (1.58) are satisfied. Concerning (1.59), let $p \in V \cap P_{(1)}$. We have $p = \partial p/\partial n = 0$ on $\partial\Omega$, but since $p = a + bx_1 + cx_2$, we have $bn_1 + cn_2 = 0$ on $\partial\Omega$; as Ω is a bounded set, $b = c = 0$, then $a = 0$ because $p = 0$ on $\partial\Omega$. Then Theorem 3.3 can be applied. We can also prove the V-ellipticity directly from (1.28).

Example 3.13. Let us consider Example 2.15. We have

$$\begin{aligned} & |\zeta_{11}|^2 + 2(1-\sigma)|\zeta_{12}|^2 + \sigma\zeta_{11}\overline{\zeta}_{22} + \sigma\zeta_{22}\overline{\zeta}_{11} + |\zeta_{22}|^2 \\ = {} & (1-\sigma)(|\zeta_{11}|^2 + |\zeta_{22}|^2) + \sigma(|\zeta_{11}|^2 + |\zeta_{22}|^2) + 2(1-\sigma)|\zeta_{12}|^2 + \sigma(\zeta_{11}\overline{\zeta}_{22} \\ & + \zeta_{22}\overline{\zeta}_{11}) \geq (1-\sigma)(|\zeta_{11}|^2 + |\zeta_{22}|^2 + 2|\zeta_{12}|^2). \end{aligned}$$

Then if $0 \leq \sigma < 1$, (1.57) is satisfied and according to Theorem 3.3 we get the $W^{k,2}(\Omega)/P_{(1)}$-ellipticity. A necessary and sufficient condition for the existence of a solution can be written as

$$\int_{\Omega} f\, \mathrm{d}x + \int_{\partial\Omega} g_1\, \mathrm{d}S = 0, \quad \int_{\Omega} x_1 f\, \mathrm{d}x + \int_{\partial\Omega} x_1 g_1\, \mathrm{d}S + \int_{\partial\Omega} x_1 g_2\, \mathrm{d}S = 0,$$
$$\int_{\Omega} x_2 f\, \mathrm{d}x + \int_{\partial\Omega} x_1 g_1\, \mathrm{d}S + \int_{\partial\Omega} x_1 g_2\, \mathrm{d}S = 0.$$

Example 3.14. We consider Example 2.19 (b). For the sesquilinear form

$$\int_\Omega \left(\sum_{i=1}^N \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \right) \mathrm{d}x \pm \mathrm{i} \int_{\partial\Omega} v\overline{u}\, \mathrm{d}x$$

the hypotheses of Theorem 3.3 are satisfied. If $V = \{v \in W^{1,2}(\Omega), v = 0 \text{ on } \partial\Omega\}$, we get the V-ellipticity.

Inequalities (1.49), (1.54) imply that the solution depends continuously on the data.

In Chap. 4 it will be proved that under conditions of regularity of f and of the coefficients a_{ij}, the regularity of the solution in the interior of Ω is given by Theorems 3.1, 3.2. If f and a_{ij} are smooth enough, then the solution u is a classical solution of the equation $Au = f$ in Ω.

Moreover, if Λ is a smooth open subset of $\partial\Omega$ and if u_0, g_{it} are smooth enough, we obtain a justification of the formal interpretation for the boundary conditions $C_{it}u = g_{it}$ in the trace sense or in the classical sense; we have also the same interpretation for the conditions $B_{is}u = h_{is}$.

Example 3.15. Let us define the operator

$$A = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j} \right) + \sum_{i=1}^N b_i \frac{\partial}{\partial x_i} + c,$$

where b_i are continuously differentiable in $\overline{\Omega}$, a_{ij}, b_i, c are real functions. We assume (the so-called Picard condition):

$$\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2, \quad \alpha > 0, \quad c(x) - \frac{1}{2}\sum_{i=1}^N \frac{\partial b}{\partial x_i} \geq 0.$$

If $V = W_0^{1,2}(\Omega)$, then the form

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_j} \mathrm{d}x + \int_\Omega \sum_{i=1}^N b_i \frac{\partial v}{\partial x_i} \overline{u}\, \mathrm{d}x + \int_\Omega c v \overline{u}\, \mathrm{d}x$$

is V- elliptic. Indeed, we have:

$$\int_\Omega \sum_{i=1}^N b_i \frac{\partial v}{\partial x_i} \overline{v}\, \mathrm{d}x = -\frac{1}{2} \int_\Omega \sum_{i=1}^N \frac{\partial b_i}{\partial x_i} |v|^2\, \mathrm{d}x,$$

and on the other hand, for complex numbers ζ_i,

$$\sum_{i,j=1}^{N} \frac{a_{ij}+a_{ji}}{2}\zeta_i\overline{\zeta}_j \geq \alpha|\zeta|^2;$$

we are in the hypotheses of Theorem 3.3 or we simply use Theorem 1.1.

Exercise 3.2. Consider Example 3.13 and prove that if $\sigma = 1$, then the $W^{2,2}(\Omega)/P_{(1)}$-ellipticity does not hold.

1.3.5 Nonstable Boundary Conditions

The conditions $B_{is}u = 0$ in the definition of 2.3 are called *stable*; clearly if $v_n \in V$, and $\lim_{n\to\infty} v_n = v$, then $v \in V \Leftrightarrow B_{is}v = 0$ on $\partial\Omega$. In contrast to this, the conditions $C_{is}u = 0$ on $\partial\Omega$ are called *nonstable* (cf. 1.2.6), this can be justified by the following:

Theorem 3.4. *Let the sesquilinear form $((v,u))$ be V-elliptic, let $N \subset V$ be the subspace of all solutions of the problem $Au = f$ in Ω as $f \in L^2(\Omega)$ changes, with boundary conditions $B_{is}u = C_{it}u = 0$ on $\partial\Omega$. Then $\overline{N} = V$ (with respect to the norm (1.2)).*

Proof. Let Z be the mapping defined in Lemma 3.1. Let us assume $\overline{N} \neq V$, then $Z(\overline{N}) \neq V$, hence there exists $v \in V, v \neq 0$, such that for all $u \in N$, $0 = (v, Zu)_k = ((v,u)) = (v,f)$. Then $v \equiv 0$, and this is a contradiction. □

1.3.6 Orthogonal Projections

The solution of the Dirichlet problem can be obtained by the method of orthogonal projections (cf. S. Zaremba [1], H. Weyl [1], J. Deny, J.L. Lions [1], etc.):

Theorem 3.5. *Let $A = A^*$ be a selfadjoint operator and the sesquilinear form $A(v,u)$ be $W^{k,2}(\Omega)/P$-elliptic; moreover we assume $A(v,v) \geq 0$.*[3] *Let $V = \{v \in W^{k,2}(\Omega), v = \partial v/\partial n = \ldots = \partial^{k-1}v/\partial n^{k-1} = 0$ on $\partial\Omega\}$. We denote by $Q = V \dot{+} P$ the direct sum; Q is a closed subspace of $W^{k,2}(\Omega)$, H/P is the orthogonal complement of Q/P in $W^{k,2}(\Omega)/P$ obtained using $A(\tilde{v},\tilde{u})$. Let u_0 be in $W^{k,2}(\Omega)$, $\tilde{u}_0$ be the class generated by u_0, $\tilde{u}_0 = \tilde{u} + \tilde{v}$, $\tilde{u} \in H/P$, $\tilde{v} \in Q/P$. Then there exists precisely one $u \in \tilde{u}$, $v \in \tilde{v}$, $v \in V$ such that $u_0 = u + v$; u is the solution of the Dirichlet problem $Au = 0$ in Ω, $u = u_0$, $\frac{\partial u}{\partial n} = \frac{\partial u_0}{\partial n}, \ldots, \frac{\partial^{k-1}u}{\partial n^{k-1}} = \frac{\partial^{k-1}u_0}{\partial n^{k-1}}$ on $\partial\Omega$.*

Proof. The sesquilinear form $A(\tilde{v},\tilde{u})$ is a scalar product on $W^{k,2}(\Omega)/P$. But $V \cap P_{(k-1)} = \emptyset$, and so there exists exactly one $v \in \tilde{v}$ such that $v \in V$. Let us set $u = u_0 - v$. We have $u \in \tilde{u}$, and we also have $((v,u)) = 0$ for $v \in V$ which implies $Au = 0$ in Ω. □

[3] The expression $A(v,v)$ does not change sign; if necessary, we take $-A(v,u)$ instead of $A(v,u)$.

1.4 The Ritz, Galerkin, and Least Squares Methods

1.4.1 The Variational Method

For simplicity, in this section we consider only the case $P = \{0\}$ and homogeneous boundary conditions ($B_{is}u = C_{it}u = 0$ on $\partial\Omega$ – see 1.2.6). These considerations can be easily extended to the general case, and this extension is left to the reader.

Proposition 4.1. *Let* $((v,u)) = \overline{((u,v))}$ *for* $u, v \in V$, $((v,u))$ *V-elliptic,* $((v,v)) \geq 0$ *(cf. footnote in Theorem 3.5);* $((v,u))$ *is a scalar product on* V*. Let* v_s*,* $s = 1,2,\ldots,$ *be an orthonormal basis in* V *with respect to* $((v,u))$*,* u *the solution of the problem* $Au = f$ *in* Ω*,* $B_{is}u = C_{it}u = 0$ *on* $\partial\Omega$*. Then*

$$u = \sum_{s=1}^{\infty} c_s v_s,$$

where c_s *are the Fourier coefficients of* u*:*

$$c_s = \int_\Omega f \overline{v}_s \,\mathrm{d}x. \tag{1.62}$$

Indeed, $((u,v)) = \int_\Omega f\overline{v}\,\mathrm{d}x$ *for* $v \in V$. □

Now let us assume that the functions v_s are elements of a basis in V, that they are linearly independent, but in general not orthogonal. Then we have:

Proposition 4.2. *We preserve the hypotheses given in Proposition 4.1 without the assumption that* v_s *are orthonormal, assuming only that they are linearly independent. For every* n *we compute the Fourier coefficients* c_{ni}*,* $i = 1,2,\ldots,n$ *as a solution of the linear system:*

$$\sum_{i=1}^{n} ((v_i, v_j)) c_{ni} = \int_\Omega f \overline{v_j}\,\mathrm{d}x, \quad j = 1,2,\ldots,n; \tag{1.63}$$

the determinant of (1.63) is not equal zero.

We have $\lim_{n\to\infty} \sum_{i=1}^{n} c_{ni} v_i = u$ *in* $W^{k,2}(\Omega)$*, where* u *is the solution of the problem.*

Proof. The determinant of (1.63) is a Gram determinant, and in our case, it is $\neq 0$. The condition (1.63) corresponds to the minimal value of

$$((u - \sum_{i=1}^{n} c_{ni} v_i, u - \sum_{i=1}^{n} c_{ni} v_i)). \tag{1.64}$$

However, since v_s form a basis, we have that

$$\lim_{n\to\infty}((u-\sum_{i=1}^{n}c_{ni}v_i, u-\sum_{i=1}^{n}c_{ni}v_i))=0. \qquad \square$$

We now introduce the original formulation of the *variational method*; for details cf. S.G. Mikhlin [2, 3]:

Proposition 4.3. *With the previous hypotheses, $u \in V$ is a solution of the problem mentioned if and only if it realizes the minimal value of the quadratic functional*

$$((v,v)) - 2\mathrm{Re}\int_\Omega v\overline{f}\,\mathrm{d}x,\ v \in V. \tag{1.65}$$

Proof. Let u be the solution of the problem; then we have for all $h \in V$:

$$\begin{aligned}((u+h,u+h)) - 2\mathrm{Re}\,(u+h,f) &= ((u,u)) + ((u,h)) + ((h,u)) + ((h,h)) \\ &- (u+h,f) - (f,u+h) = -((u,u)) + ((h,h)).\end{aligned} \tag{1.66}$$

If u is a solution, it follows from (1.66) that (1.65) is minimal for u. If for $v = u+h$ the functional (1.65) attains its minimum, then (1.66) implies that $v = u$. $\square$

In problems from physics, (1.65) expresses the energy.

Remark 4.1. It follows from (1.66) that (1.64) attains its minimum for $v = \sum_{i=1}^{n} c_{ni}v_i$ if and only if it is the case for (1.65): this is the *method of Ritz*.

The choice of v_s is crucial for the numerical stability of Ritz' method as $n \to \infty$; cf. S.G. Mikhlin [1], I. Babuška, M. Práger, E. Vitásek [1].

Remark 4.2. The system (1.63) can be obtained by the *method of Galerkin*: we look for the solution in the form $\sum_{i=1}^{n} c_{ni}v_i$ and we impose $((\sum_{i=1}^{n} c_{ni}v_i, v_j)) = (f, v_j)$, $j = 1,2,\ldots,n$.

1.4.2 The Galerkin Method

We now *describe a generalization of Galerkin's method* giving also a theoretical tool for the proof of the convergence of the finite differences method. Cf. later Example 4.1.

Let V_h be a finite-dimensional subspace of V defined for all $h \in (0,1)$. We say that $\lim_{h\to 0} V_h = V$ if for all $v \in V : \lim_{h\to 0}(\mathrm{dist}\,(v,V_h)) = 0$.

Theorem 4.1. *Let a boundary value problem, with homogeneous boundary conditions and the corresponding V-elliptic sesquilinear form (not necessarily hermitian) $((v,u))$ be given. Let u be the solution of the problem. Then there exists a uniquely determined $u_h \in V_h$ such that for all $v \in V_h$:*

$$((v,u_h)) = (v,f), \tag{1.67}$$

and $\lim_{h\to 0} u_h = u$ *in* $W^{k,2}(\Omega)$.

Proof. Let v_i, $i=1,2,\dots,n_h$ be a basis of linearly independent functions in V_h. We set $u_h=\sum_{i=1}^{n_h} c_i v_i$. Using (1.67) we obtain a system of linear equations with nonzero determinant, hence u_h is uniquely defined. We have $\alpha|u_h|_k^2 \le |((u_h,u_h))| = |(u_h,f)| \le |u_h|_k|f|_0$, and so,

$$\alpha|u_h|_k \le |f|_0. \tag{1.68}$$

We claim that $\lim_{h\to 0} u_h = u$ weakly in $W^{k,2}(\Omega)$.

Indeed: if this is not the case, then due to (1.68) the $|u_h|_k$ are bounded by a constant, and we can extract a subsequence u_{h_i}, $\lim_{i\to\infty} u_{h_i} = u^*$ weakly, $u^* \neq u$. Let us consider $v\in V$. We can find $v_{h_i}\in V_{h_i}$, $\lim_{i\to\infty} v_{h_i} = v$ strongly. We have $((v_{h_i},u_{h_i})) = (v_{h_i},f)$, hence $\lim_{i\to\infty}((v_{h_i},u_{h_i})) = ((v,u^*)) = (v,f)$ for $v\in V$ and then $u^*=u$. This is a contradiction to our assumption.

Let us chose $v_h\in V_h$ such that $\lim_{h\to 0} v_h = u$; we have:

$$\begin{aligned}
&\lim_{h\to 0}([((v_h-u_h,v_h-u_h))]\\
&\qquad = \lim_{h\to 0}[((v_h,v_h)) - ((u_h,v_h)) - ((v_h,u_h)) + ((u_h,u_h))]\\
&\qquad = \lim_{h\to 0}[((v_h,v_h)) - ((u_h,v_h)) - ((v_h,u_h)) + (u_h,f)] = 0.
\end{aligned}$$

□

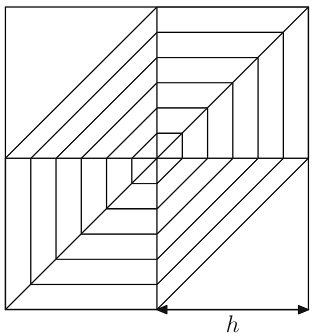

Fig. 1.4

Example 4.1. We consider in the plane $\mathbb{R}^2$ the square $\Omega=(0,1)\times(0,1)$, $A=-\triangle$. We solve the mixed problem $-\triangle u = f$ in Ω, $f\in L^2(\Omega)$, $u=0$ for $x_1=0$, $x_1=1$, $0<x_2<1$; $x_2=0$, $0<x_1<1$; $\partial u/\partial n = 0$ for $x_2=1$, $0<x_1<1$. Let $x_{ij}=(ih,jh)$, $i,j=1,2,\dots,(n+1)$, $(n+1)h=1$ be the points of the network mesh h. The space V_h is a space of functions of height 1, cf. Fig. 1.4; for $j=n+1$ we take the restriction of a pyramidal function on Ω. We denote by v_{ij} the pyramid with center x_{ij}; $V=\{v\in W^{1,2}(\Omega), v=0 \text{ for } x_1=0,1,\ 0<x_2<1; x_2=0,\ 0<x_1<1\}$. It is easy to see that the restrictions of functions in $C_0^\infty(\mathbb{R}^2)$ on Ω are equal to zero on the part of boundary introduced in the definition of V, which are dense in V; it follows that $\lim_{h\to 0} V_h = V$. If we are looking for a solution of the form $u_h = \sum_{i,j} c_{ij} v_{ij}$, an approximating solution, we obtain (1.67) by setting $c_{ij} = 0$, $i = 1, n+1$, $j = 0$, $d_{ij} = (v_{ij}, f)$, and we have the classical linear system in finite differences method:

$$i,j \le n, \quad 4c_{ij} - c_{i+1,j} - c_{i-1,j} - c_{i,j+1} - c_{i,j-1} = d_{ij},$$

$$j = n+1, \quad 2c_{ij} - \frac{1}{2}c_{i+1,j} - \frac{1}{2}c_{i-1,j} - c_{i,j+1} - c_{i,j-1} = d_{ij}.$$

More details on this example can be found in J.L. Lions [6], J. Céa [1–3], I. Babuška, M. Práger, E. Vitásek [1], S.G. Mikhlin [2].

1.4.3 The Least Squares Method

For the least squares method, starting from inequality (1.49), we obtain:

Proposition 4.4. *Let a boundary value problem be given with homogeneous conditions and with a V-elliptic form $((v,u))$. Let v_i be a sequence of approximating solutions of our problem such that $Av_i = f_i$ is a basis in $L^2(\Omega)$. Let $u_n = \sum_{i=1}^{n} c_{ni} v_i$ be defined by the minimum of*

$$\int_\Omega \left| \sum_{i=1}^{n} c_{ni} A v_i - f \right|^2 \mathrm{d}x. \tag{1.69}$$

Then $\lim_{n\to\infty} u_n = u$ *in* $W^{k,2}(\Omega)$.

If $f \in W^{l,2}(\Omega)$, then the solution of the boundary value problem can be found as the minimum of the functional (for $((v,u))$ hermitian, $((v,v)) \ge 0$):

$$((v,v)) - 2\mathrm{Re}\,(v,f) + \sum_{|j|\le l} (D^j Av - D^j f, D^j Av - D^j f);$$

this is the *Courant method.*

The sequence u_n in Proposition 4.4 can converge to the solution in stronger norms, for instance in $W^{2k,2}(\Omega')$ for every $\overline{\Omega'} \subset \Omega$. These questions will be considered in Chap. 4.

We can again generalize Proposition 4.4: instead of taking the minimum of $\sum_{i=1}^{n} c_{ni} A v_i - f$ in $L^2(\Omega)$ we can study this minimization problem in $W^{l,2}(\Omega)$ if $f \in W^{l,2}(\Omega)$. The convergence of the sequence u_n to the solution can be stronger, for instance in $W^{2k+l,2}(\Omega')$ for every $\overline{\Omega}' \subset \Omega$; for l sufficiently high, $\lim_{n\to\infty} u_n = u$ in the classical sense. Cf. Chap. 4 for these questions.

For other numerical processes, like the gradient method, the Treftz method, the Schwarz method, see S.G. Mikhlin [2, 3], I. Babuška [1], M. Práger, [1],

F. E. Browder [2], I. Babuška, M. Prager, E. Vitásek [1], Z. Ch. Rafalson [1], M. Sh. Birman [1], etc.

Exercise 4.1. Prove that u is the solution of the Dirichlet problem $-\triangle u = 0$ in Ω, $u = u_0$ on $\partial\Omega$ if and only if the functional

$$\int_\Omega \left(\sum_{i=1}^{N} \left| \frac{\partial v}{\partial x_i} \right|^2 \right) \mathrm{d}x$$

attains at u its minimum among functions from $W^{1,2}(\Omega)$ for which $u = u_0$ on $\partial\Omega$.

1.5 Basic Notions from the Spectral Theory

1.5.1 Eigenvalues and Eigenfunctions, the Fredholm Alternative

In this section we introduce the basic material and properties of spectral theory and related questions. We restrict ourselves to the case of a hermitian V-elliptic form $((v,u))$, $((v,v)) \geq 0$ and endowe V with the scalar product $((v,u))$. The general case will be considered in Chap. 3.

Let a boundary value problem be given, and let us consider the homogeneous case, i.e. the data on $\partial\Omega$ are equal to zero. If $f \in L^2(\Omega)$, there exists a uniquely determined solution u of this problem. We define the *Green operator*, a continuous linear operator $G : L^2(\Omega) \to W^{k,2}(\Omega)$, by $Gf = u$.

Hereafter the following proposition will be fundamental:

Proposition 5.1. *The Green operator is compact from V to V.*

Proof. Let f_n be a bounded sequence in V. According to Theorem 1.4 it is possible to extract a subsequence f_{n_i} which converges in $L^2(\Omega)$, and then by Theorem 3.1 the sequence Gf_{n_i} converges in V. □

A complex number λ is called the *eigenvalue* of the operator A and of the considered boundary value problem if there exists a function $u \in V, u \not\equiv 0$, such that $((v,u)) - \overline{\lambda}(v,u) = 0$ for $v \in V$. This function u is called the *eigenfunction corresponding to* λ.

Let us observe that we can consider the so called generalized spectral problem: to find the eigenvalues associated to the form $((v,u)) - \overline{\lambda}(v,Bu) = 0$ where B is a continuous linear operator $B : V \to L^2(\Omega)$; cf. S.G. Mikhlin [2].

From the definition of G, we can obtain the following:

Proposition 5.2. *The number λ is an eigenvalue and the corresponding function u is an eigenfunction if and only if*

$$u - \lambda G u = 0. \tag{1.70}$$

If $f \in L^2(\Omega)$, then the function u is a weak solution of the problem $Au - \lambda u = f$ in Ω with homogeneous boundary conditions $B_{is}u = 0, C_{it}u = 0$ on $\partial\Omega$ if and only if

$$u - \lambda Gu = Gf. \tag{1.71}$$

We immediately get

Proposition 5.3. *The Green operator is selfadjoint (on V with the scalar product $((v,u))$) and positive.*

Proof. We have $((Gv,u)) = (v,u) = \overline{(u,v)} = \overline{((Gu,v))} = ((v,Gu))$. □

According to the Riesz-Fredholm and Hibert-Schmidt theory, cf. F. Riesz, B.Sz. Nagy [1], using Propositions 5.1–5.3 we have

Proposition 5.4. *The set of eigenvalues of the operator G is a countable set $\lambda_n, n = 1,2,\ldots$ The eigenvalues are real and positive, the sequence λ_n is non-decreasing and tends to infinity. There exists a basis of orthonormal eigenfunctions in V, say v_n, where v_n corresponds to the eigenvalue λ_n. We have:*

$$\frac{1}{\lambda_n} = \max((Gf,f)) = ((Gv_n,v_n)) \tag{1.72}$$

the maximum being taken over $f \in V$ such that $((f,f)) = 1$ and $((f,v_i)) = 0$ for $i = 1,2,\ldots,n-1$.

If $\lambda \neq \lambda_i$, then the equation

$$u - \lambda Gu = F \tag{1.73}$$

has a unique solution for every $F \in V$; if $\lambda = \lambda_i$ then the previous equation has a solution if and only if

$$((v_i,F)) = 0 \tag{1.74}$$

for all eigenfunctions v_i corresponding to the eigenvalue $\lambda = \lambda_i$. If the condition (1.74) is satisfied, the equation (1.73) has a unique solution modulo a linear combination of eigenfunctions corresponding to λ_i.

1.5.2 Eigenvalues and Eigenfunctions, the Fredholm Alternative (Continuation)

We now give another interpretation of Proposition 5.4.

Theorem 5.1. *For the boundary value problem considered above the set of eigenvalues is countable, the eigenvalues are real, positive, non-decreasing and tend to infinity. There exists an orthogonal basis of eigenfunctions, say v_n, with v_n corresponding to λ_n.*

We have

$$\lambda_n = \min \frac{((v,v))}{(v,v)}, \tag{1.75}$$

for $v \in V$, $(v, v_i) = 0$, $i = 1, 2, \ldots, n-1$,

$$\lambda_n = \frac{((v_n, v_n))}{(v_n, v_n)}. \tag{1.76}$$

If $\lambda \neq \lambda_i$, $i = 1, 2, \ldots$, *then for every* $f \in L^2(\Omega)$, *there exists exactly one solution of the problem* $Au - \lambda u = f$ *in* Ω, $B_{is}u = C_{it}u = 0$ *on* $\partial\Omega$; *we get*

$$|u|_{W^{k,2}(\Omega)} \leq \mathrm{const}|f|_{L^2(\Omega)}. \tag{1.77}$$

If $\lambda = \lambda_i$, *the problem considered has a solution if and only if*

$$(v_i, f) = 0 \tag{1.78}$$

for all eigenfunctions v_i *corresponding to* λ_i. *If condition (1.78) is satisfied, the solution is unique modulo a linear combination of eigenfunctions corresponding to* λ_i. *In this case we can find a unique u satisfying*

$$(v_i, u) = 0, \tag{1.79}$$

where the v_i *are the eigenfunctions mentioned, and we get:*

$$|u|_{W^{k,2}(\Omega)} \leq \mathrm{const}|f|_{L^2(\Omega)}. \tag{1.80}$$

Proof. Using the fact that $((Gv, u)) = ((v, Gu)) = (v, u)$ and $v_i - \lambda_i G v_i = 0$, the result follows from Proposition 5.4. □

Proposition 5.5. *The set of functions* $\lambda_i^{1/2} v_i$ *is an orthonormal basis in* $L^2(\Omega)$.

Indeed

$$\sqrt{\lambda_i \lambda_j}(\varphi_i, \varphi_j) = \sqrt{\lambda_i \lambda_j}((\varphi_i, G\varphi_j)) = \frac{\sqrt{\lambda_i \lambda_j}}{\lambda_j}((\varphi_i, \varphi_j)) = \delta_{ij};$$

on the other hand, V is dense in $L^2(\Omega)$, hence v_n, which is a basis of V, is also a basis in $L^2(\Omega)$.

Proposition 5.6. *For* $f \in L^2(\Omega)$, *let*

$$f = \sum_{i=1}^{\infty} (f, \sqrt{\lambda_n} v_n) \sqrt{\lambda_n} v_n$$

be its Fourier series. The solution u of the boundary value problem with homogeneous boundary conditions can be written as

$$u = \sum_{i=1}^{\infty} (f, v_n) v_n.$$

Indeed, we have

$$Gf = \sum_{n=1}^{\infty} ((Gf, v_n)) v_n = \sum_{n=1}^{\infty} (f, v_n) v_n. \tag{1.81}$$

Remark 5.1. The eigenfunctions are useful if we want to develop the solutions of boundary value problem into a series. From (1.81) we get in $L^2(\Omega)$:

$$\lim_{n\to\infty} A \sum_{i=1}^{N} (f, v_n) v_n = \lim_{n\to\infty} \sum_{i=1}^{N} \lambda_n (f, v_n) v_n = \lim_{n\to\infty} [\sum_{i=1}^{N} (f, \sqrt{\lambda_n} v_n) \sqrt{\lambda_n} v_n] = f;$$

this is not the general case. The details can be found in S.G. Mikhlin [1].

In the numerical computation of eigenvalues we can use (1.75) and try to solve the problem of minimization in subspaces generated by functions $h_1, h_2, \ldots, h_N$, where h_n, $n = 1, 2, \ldots$ form a basis in V. This is the *Ritz method*, cf. S.G. Mikhlin [2] . There are plenty of references about the computation of eigenvalues and eigenfunctions: the Courant principle (cf. R. Courant, D. Hilbert [1]) or the comparison method (cf. L. Collatz [1]). Cf. also G. Polya, G. Szegö [1], L.E. Payne, H.F. Weinberger [3], A. Weinstein [1], Y. Dejean [1], etc.

1.5.3 The Gårding Inequality

The spectral theory provides us with a general tool to solve boundary value problems. We replace the V-ellipticity by the Gårding inequality (cf. Chap. 3 and E. Magenes, G. Stampacchia [1], L. Gårding [1]). Now we assume $((v,u))$ to be hermitian in V.

We say that the sesquilinear form satisfies the *Gårding inequality* if there exists a $\lambda_0 > 0$ such that $((v,u)) + \lambda_0 (v,u)$ is V-elliptic.

From Theorem 5.1 we deduce immediately:

Corollary 5.1. *Let a boundary value problem be given with homogeneous boundary conditions. Assume that the Gårding inequality is satisfied for $((v,u))$, i.e., for $v \in V$ we have*

$$((v,v)) + \lambda_0 (v,v) \geq \text{const}|v|_k^2. \tag{1.81 bis}$$

If 0 is not an eigenvalue, then there exists a unique solution of the problem $Au = f$ in Ω, $f \in L^2(\Omega)$, $B_{is}u = 0$, $C_{it}u = 0$ on $\partial\Omega$, and the following inequality holds:

$$|u|_{W^{k,2}(\Omega)} \leq \text{const}|f|_{L^2(\Omega)}. \tag{1.82}$$

If 0 *is an eigenvalue, the solution exists if and only* $(v_i, f) = 0$ *for all eigenfunctions corresponding to* 0*. In this case the solution is unique modulo a linear combination of these eigenfunctions.*

Remark 5.2. Corollary 5.1 gives us again, under the above hypotheses, Theorem 3.2: if $((v,u))$ is a V/P-elliptic form (and hermitian with $((v,v)) \geq 0$) $(1.81bis)'$ holds for all $\lambda_0 > 0$. The space of eigenfunctions corresponding to 0 is exactly P.

Here we give a lemma which is a particular case of a lemma by J.L. Lions:

Lemma 5.1. *Let* Ω *be a bounded domain with continuous boundary,* $k \geq 2$*. Then for every* $\varepsilon > 0$*, there exists* $\lambda(\varepsilon)$ *such that for* $u \in W^{k,2}(\Omega)$

$$|u|_{W^{k-1,2}(\Omega)} \leq \varepsilon |u|_{W^{k,2}(\Omega)} + \lambda(\varepsilon)|u|_{L^2(\Omega)}. \tag{1.83}$$

Proof. We proceed by contradiction. There exists $\varepsilon > 0$ and a sequence u_n such that

$$u_n \in W^{k,2}(\Omega), \quad |u_n|_{W^{k-1,2}(\Omega)} > \varepsilon |u_n|_{W^{k,2}(\Omega)} + n|u_n|_{L^2(\Omega)}. \tag{1.84}$$

Without loss of generality we can assume $|u_n|_{W^{k,2}(\Omega)} = 1$. It follows from Theorem 1.4 that we can extract a subsequence u_{n_i} converging to some u in $W^{k-1,2}(\Omega)$. Then (1.84) implies that $\lim_{i\to\infty} u_{n_i} = 0$ in $L^2(\Omega)$ and hence $u \equiv 0$. But this is not possible because by (1.84), $|u|_{W^{k-1,2}(\Omega)} \geq \varepsilon$. □

We can prove a theorem giving conditions sufficient for the validity of the Gårding inequality:

Theorem 5.2. *Let the following conditions be satisfied:* $A = A^*$, $((v,u)) = A(v,u)$, $a(v,u) \equiv 0$, $\partial\Omega$ *continuous. We assume (1.57). Then if* λ_0 *is big enough, (1.81 bis) holds with* $V = W^{k,2}(\Omega)$.

Proof. We have

$$((v,v)) + \lambda_0(v,v) = \int_\Omega \sum_{|i|,|j|=k} \bar{a}_{ij} D^i v D^j \bar{v} \,\mathrm{d}x + \int_\Omega \sum_{\substack{|i|,|j|\leq k \\ |i|+|j|\leq 2k-1}} \bar{a}_{ij} D^i v D^j \bar{v} \,\mathrm{d}x + \lambda_0 \int_\Omega |v|^2 \,\mathrm{d}x.$$

For $\delta > 0$, and two numbers a, b, we have:

$$|ab| \leq \frac{\delta}{2} a^2 + \frac{b^2}{2\delta}. \tag{1.85}$$

From (1.57), (1.85), and Lemma 5.1 it follows the existence of c_1 such that

$$v \in W^{k,2}(\Omega), \qquad ((v,v)) \geq c_1 \int_\Omega \sum_{|i|=k} |D^i v|^2 \,\mathrm{d}x - \kappa \int_\Omega |v|^2 \,\mathrm{d}x. \tag{1.86}$$

Then $(1.81\,\mathrm{bis})$ follows with $\lambda_0 = 2\kappa$.

Exercise 5.1. Prove Theorem 5.1 for the general boundary value problem. Hint: Given u_0, g_{it}, f, let u_1 be the solution for the operator A. Finally consider the problem with homogeneous boundary conditions: $Au_2 - \lambda u_2 = \lambda u_1$ in Ω, $B_{is}u_2 = 0$, $C_{it}u_2 = 0$ on $\partial\Omega$.

Chapter 2
The Spaces $W^{k,p}$

The theory of the spaces $W^{k,p}$ for $p = 2$, outlined in Chap. 1, has been substantially developed; now there exist plenty of spaces of analogous type. The fundamental sources are: S.L. Sobolev [1], S.M. Nikolskii [2], J. Deny, J.L. Lions [1], E. Gagliardo [1, 2]; see also N. Aronszajn [3], N. Aronszajn, K.T. Smith [1–3], N. Aronszajn, F. Mulla, P. Szeptycki [1], V.M. Babich [1], O.V. Besov [1, 2], S. Campanato [3–7], E. Gagliardo [3], V.P. Ilyin [1, 2], G.N. Jakovlev [1], W. Kondrashov [1], L.D. Kudriavcev [2], J.L. Lions [5], E. Magenes [4], K. Maurin [1], N.G. Meyers, J. Serrin [1], J. Nečas [11], S.M. Nikolskii [3–8], G. Prodi [1], L.N. Slobodetskii [1, 2], V.I. Smirnov [1], S.V. Uspenskii [1–4], L. De Vito [1].

2.1 Definitions and Auxiliary Theorems

2.1.1 Classification of Domains, Pseudotopology in $C_0^\infty(\Omega)$

In Sect. 1.1.3, we introduced domains with continuous or lipschitzian boundaries; it was a particular case of a more general definition:

A bounded domain Ω is of type $\mathfrak{N}^{k,\mu}$, where k is a non-negative integer or infinity, $0 \le \mu \le 1$, if there exist functions a_r as in 1.1.3 defined in the closures of cubes $\overline{\Delta}_r = \{x' \in \mathbb{R}^{N-1},\ |x_{ri}| \le \alpha,\ i = 1,2,\dots,N-1\}$, μ-hölderian together with their derivatives of order $\le k$, which means that for $x'_r, y'_r \in \overline{\Delta}_r$, there is $|D^i a_r(x'_r) - D^i a_r(y'_r)| \le c|x'_r - y'_r|^\mu$, $|i| \le k$.[1] If $\mu = 0$, the functions a_r and their derivatives of order $\le k$ are only continuous in $\overline{\Delta}_r$, and for simplicity we shall write $\mathfrak{N}^{k,0} = \mathfrak{N}^k$.

Let Ω be a domain in $\mathbb{R}^N$, k a non-negative integer or $k = \infty$, $0 \le \mu \le 1$. We denote by $C^{k,\mu}(\overline{\Omega})$ the space of complex-valued functions whose derivatives of

[1]Hereafter various constants will be mostly denoted by the same letter c. If necessary, we shall use indices or another appropriate notation.

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order $\leq k$ are μ-hölderian on the closure of Ω. If $\mu = 0$, the functions and their derivatives of order $\leq k$ are only continuous on $\overline{\Omega}$; we then write simply $C^k(\overline{\Omega})$ instead of $C^{k,0}(\overline{\Omega})$. If $k < \infty$, we endow $C^k(\overline{\Omega})$ with the following norm:

$$|u|_{C^k(\overline{\Omega})} = \sum_{|\alpha| \leq k} \max_{x \in \overline{\Omega}} |D^\alpha u(x)|, \tag{2.1}$$

and $C^{k,\mu}(\overline{\Omega})$ with the norm defined by:

$$|u|_{C^{k,\mu}(\overline{\Omega})} = |u|_{C^k(\overline{\Omega})} + \sum_{|\alpha|=k} \sup_{x,y \in \Omega, x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\mu}. \tag{2.2}$$

The spaces $C^k(\overline{\Omega})$ and $C^{k,\mu}(\overline{\Omega})$ are Banach spaces. We denote also by $C^k(\Omega)$ (resp. $C^{k,\mu}(\Omega)$) the spaces of functions continuous (resp. $\mu-$hölderian) together with derivatives of order $\leq k$ in Ω.

For $C_0^\infty(\Omega)$, see 1.1.1.

On $C_0^\infty(\Omega)$ we introduce a *pseudotopology* (cf. L. Schwartz [1]): Let φ_n be a sequence in $C_0^\infty(\Omega)$. Then $\lim_{n\to\infty} \varphi_n = \varphi$ in $C_0^\infty(\Omega)$, if there exists $\Omega' \subset \overline{\Omega'} \subset \Omega$, Ω' bounded, such that the support of φ_n (denoted by $\operatorname{supp} \varphi_n$) and the support of φ are included in $\overline{\Omega'}$, and for all $k \geq 0$, $\lim_{n\to\infty} \varphi_n = \varphi$ in $C^k(\overline{\Omega'})$.

Following L. Schwartz [1], we will for some time denote the space $C_0^\infty(\Omega)$ by $\mathscr{D}(\Omega)$. The *space of distributions* on Ω – the dual of $\mathscr{D}(\Omega)$ – will be denoted by $\mathscr{D}'(\Omega)$. For $f \in \mathscr{D}'(\Omega)$, we denote the value of f at the point $\varphi \in \mathscr{D}(\Omega)$ by $\langle \varphi, f \rangle$.

The derivative $D^i f$ of the distribution f is again a distribution, defined by the formula

$$\langle \varphi, D^i f \rangle = (-1)^{|i|} \langle D^i \varphi, f \rangle, \quad \varphi \in \mathscr{D}(\Omega). \tag{2.3}$$

If f_n is a sequence in $\mathscr{D}'(\Omega)$, we say that $\lim_{n\to\infty} f_n = f$ in $\mathscr{D}'(\Omega)$ if for all $\varphi \in \mathscr{D}(\Omega)$ we have:

$$\lim_{n\to\infty} \langle \varphi, f_n \rangle = \langle \varphi, f \rangle.$$

Let $L^1_{loc}(\Omega)$ be the space of locally integrable functions on Ω. We define the imbedding $L^1_{loc}(\Omega) \subset \mathscr{D}'(\Omega)$ by:

$$\langle \varphi, f \rangle = \int_\Omega \varphi f \, \mathrm{d}x, \quad \varphi \in \mathscr{D}(\Omega), \quad f \in L^1_{loc}.$$

Obviously we have:

Proposition 1.1. *If $f_1, f_2 \in L^1_{loc}(\Omega)$ and if for every $\varphi \in \mathscr{D}(\Omega)$*

$$\langle \varphi, f_1 \rangle = \langle \varphi, f_2 \rangle, \tag{2.4}$$

then $f_1 = f_2$ almost everywhere in Ω.

Indeed: It follows from (2.4) that for every interval $I \subset \Omega$, we have:

$$\int_I f_1 \, \mathrm{d}x = \int_I f_2 \, \mathrm{d}x.$$

□

Let B be a Banach space such that $\mathscr{D}(\Omega) \subset B$; B is called *normal* if $\mathscr{D}(\Omega)$ is dense in B.

If $\mathscr{D}(\Omega) \subset B$ algebraically and topologically, then $B' \subset \mathscr{D}'(\Omega)$, algebraically and topologically. B' is a subspace of distributions.

Exercise 1.1. If $f \in C^k(\Omega)$, then, for $|i| \leq k$, $D^i f$ in the classical sense and $D^i f$ in the sense of distributions coincide.

Hereafter if $f \in \mathscr{D}'(\Omega)$, $D^i f$ will denote the derivative in the distribution sense.

2.1.2 The Space $L^p(\Omega)$, Mean Continuity

Let $1 \leq p < \infty$. We denote by $L^p(\Omega)$ the space of $p-$integrable functions on Ω with the norm:

$$|f|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p}; \tag{2.5}$$

this space is a Banach space, it has a countable basis, it is reflexive for $p > 1$. The following *mean continuity* property holds:

Theorem 1.1. *Let Ω be an open set in $\mathbb{R}^N$, $f \in L^p(\Omega)$, $f(x) = 0$ for $x \notin \Omega$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|z| < \delta \Longrightarrow \left(\int_\Omega |f(x+z) - f(x)|^p \, \mathrm{d}x \right)^{1/p} < \varepsilon.$$

Proof. Let us assume Ω bounded and $\varepsilon > 0$. There exists $\nu <$ meas (Ω) such that

$$M \subset \Omega, \quad \text{meas } (M) < \nu \Longrightarrow \left(\int_M |f(x)|^p \, \mathrm{d}x \right)^{1/p} < \varepsilon/3.$$

According to Lusin's theorem, there exists a closed set $F \subset \Omega$, meas $(F) >$ meas $(\Omega) - (1/2)\nu$ such that f is continuous on F. Then there exists $\delta > 0$ such that

$$|z| < \delta \Longrightarrow x \in F, \quad x + z \in F, \quad |f(x+z) - f(x)| < \frac{\varepsilon}{3(\text{meas } (\Omega))^{1/p}}.$$

For z fixed, $|z| < \delta$, we denote

$$H_z = \{y \in \mathbb{R}^N, \ y = x + z, \ x \in F\}, \quad F_z = F \cap H_z = F - (F - H_z).$$

We can choose δ sufficiently small such that $H_z \subset \Omega$. Then we have:

$$\begin{aligned} \text{meas } (F_z) &> \text{meas } (\Omega) - \frac{\nu}{2} - \left[\text{meas } (\Omega) - \left(\text{meas } (\Omega) - \frac{\nu}{2} \right) \right] \\ &= \text{meas } (\Omega) - \nu, \end{aligned}$$

i.e.

$$\text{meas}\,(\Omega - F_z) < \nu.$$

We get:

$$\left(\int_{\Omega} |f(x+z) - f(x)|^p \,\mathrm{d}x\right)^{1/p} \le \left(\int_{F_z} |f(x+z) - f(x)|^p \,\mathrm{d}x\right)^{1/p}$$
$$+ \left(\int_{\Omega - F_z} |f(x+z) - f(x)|^p \,\mathrm{d}x\right)^{1/p} < \text{meas}\,(F_z)^{1/p} \frac{\varepsilon}{3(\text{meas}\,(\Omega))^{1/p}}$$
$$+ \left(\int_{\Omega - F_z} |f(x+z)|^p \,\mathrm{d}x\right)^{1/p} + \left(\int_{\Omega - F_z} |f(x)|^p \,\mathrm{d}x\right)^{1/p} < \varepsilon;$$

if Ω is unbounded, for any $\varepsilon > 0$, we can find a ball $K(r)$ with radius $r > 1$ such that

$$\left(\int_{\Omega\ -K(r-1)} |f(x)|^p \,\mathrm{d}x\right)^{1/p} < \varepsilon/3.$$

Then we can repeat the proof given previously for the bounded set $\Omega \cap K(r-1)$ with $\varepsilon/3$ and $\delta \le 1$. □

2.1.3 The Regularizing Operator

Let $h > 0$. We define the regularizing kernel by:

$$\omega(x,h) = \begin{cases} \exp(|x|^2/(|x|^2 - h^2)) & \text{for } |x| < h, \\ 0 & \text{for } |x| \ge h. \end{cases}$$

The kernel is a function in $C^\infty(\mathbb{R}^N)$. The *regularizing operator* is the operator mapping $L^p(\Omega)$ into itself and defined by

$$f_h(x) = \frac{1}{\kappa h^N} \int_{\Omega} \omega(x-y,h) f(y) \,\mathrm{d}y, \tag{2.6}$$

where

$$\kappa = \int_{|x|<1} \omega(x,1) \,\mathrm{d}x = \frac{1}{h^N} \int_{|x|<h} \omega(x,h) \,\mathrm{d}x.$$

Of course, we can use also other regularizing kernels; Sect. 2.5.5.

By an immediate computation we get:

$$f_h(x) = \frac{1}{\kappa} \int_{|z|<1} \omega(z,1) f(x+hz) \,\mathrm{d}z$$

where $f(x) = 0$ for $x \notin \Omega$.

In what follows, we will use the following notation: let B_1, B_2 be two Banach spaces, and T a bounded linear mapping defined on B_1 with values in B_2; we shall write $T \in [B_1 \to B_2]$.

Theorem 1.2. *The operator which defines $f_h(x)$ by (2.6), has the following properties: it belongs to $[L^p(\Omega) \to L^p(\Omega)]$, $f_h \in C^\infty(\overline{\Omega})$ (and also $\in C^\infty(\mathbb{R}^N)$), and $\lim_{h\to 0} f_h = f$ in $L^p(\Omega)$.*

Proof. We have to prove:

$$|f_h|_{L^p(\Omega)} \leq C|f|_{L^p(\Omega)}, \quad \lim_{h\to 0} f_h = f \text{ in } L^p(\Omega); \tag{2.7}$$

the other properties are clear. We always set $f(x) = 0$ for $x \notin \Omega$; then we have:

$$\begin{aligned}\int_\Omega &|f_h(x) - f(x)|^p \,\mathrm{d}x \\ &= \int_\Omega \left| \frac{1}{\kappa} \int_{|z|<1} \omega(z,1) f(x+hz)\,\mathrm{d}z - \frac{1}{\kappa} \int_{|z|<1} \omega(z,1) f(x) dz \right|^p \mathrm{d}x \\ &\leq \int_\Omega \left(\frac{1}{\kappa} \int_{|x|<1} \omega(z,1) |f(x+hz) - f(x)|\,\mathrm{d}z \right)^p \mathrm{d}x \equiv I(h).\end{aligned}$$

If $p = 1$ then the Fubini theorem implies:

$$I(h) \leq c \int_{|z|<1} dz \int_\Omega |f(x+hz) - f(x)|\,\mathrm{d}x. \tag{2.8}$$

If $p > 1$ then using the Hölder inequality and the Fubini theorem we get:

$$I(h) \leq c \int_\Omega \mathrm{d}x \int_{|z|<1} |f(x+hz) - f(x)|^p \,\mathrm{d}z = c \int_{|z|<1} dz \int_\Omega |f(x+hz) - f(x)|^p \,\mathrm{d}x; \tag{2.9}$$

the result follows according to (2.8), (2.9) and Theorem 1.1. The inequality in (2.7) can be obtained by a trivial modification of the proof. □

2.1.4 Compactness Condition

The following theorem is due to Kolmogorov.

Theorem 1.3. *Let Ω be a bounded domain, $M \subset L^p(\Omega)$. M is precompact if and only if:*

$$M \text{ is a bounded set,} \tag{2.10}$$

$$\text{the functions } f \in M \text{ are mean-equicontinuous.} \tag{2.11}$$

Proof. First, let us prove that the conditions are sufficient; as previously we set $f(x) = 0$ for $x \notin \Omega$. According to (2.8), (2.9) we have:

$$|f_h - f|_{L^p(\Omega)} \le c \sup_{|z|<h} \left(\int_\Omega |f(x+z) - f(x)|^p \,\mathrm{d}x \right)^{1/p}. \tag{2.12}$$

Let $\varepsilon > 0$; we can find $\delta > 0$ such that $h < \delta \Longrightarrow |f_h - f|_{L^p(\Omega)} < \varepsilon/2$. Let $M_h = \{f_h, f \in M, h \text{ fixed}\}$. The functions f_h are bounded by the same constant and they are equicontinuous, hence M_h is a relatively compact set in $C^0(\overline{\Omega})$, and there exists an $\varepsilon/2(\text{meas }(\Omega))^{1/p}$-net, say S_ε, in $C^0(\overline{\Omega})$. It follows from (2.12) that S_ε is an ε-net in $L^p(\Omega)$.

Now we prove the necessity: If M is relatively compact then (2.9) holds. Let $f_1, f_2, \dots, f_k$ be an $(\varepsilon/3)$-net in M. According to Theorem 1.1, each f_i, $i = 1, 2, \dots, k$ is mean continuous, and there exists $\delta > 0$ such that

$$|z| < \delta \Longrightarrow (\int_\Omega |f(x+z) - f(x)|^p \,\mathrm{d}x < (\varepsilon/3)^p, \quad i = 1, 2, \dots, k.$$

Hence, if $f \in M$, there exists an index i such that $|f - f_i|_{L^p(\Omega)} < \varepsilon/3$, and

$$|z| < \delta \Longrightarrow \left(\int_\Omega |f(x+z) - f(x)|^p \,\mathrm{d}x \right)^{1/p} < \varepsilon.$$

□

Exercise 1.2. Prove Theorem 1.1 using the fact that for Ω bounded we have $L^p(\Omega) = \overline{C(\overline{\Omega})}$.

Exercise 1.3. Prove Theorem 1.3 for Ω unbounded with the following additional condition: for each $\varepsilon > 0$ there exists $r > 0$ such that

$$f \in M \Longrightarrow \int_{\Omega - \overline{K(r)}} |f(x)|^p \,\mathrm{d}x < \varepsilon,$$

where $K(r)$ is a ball with center at the origin and with radius r.

2.2 The Spaces $W^{k,p}(\Omega)$

2.2.1 A Property of the Regularizing Operator

Let us recall that $L^p_{loc}(\Omega), p \ge 1$ is the space of complex-valued functions defined on Ω which are locally $p-$integrable on Ω (i.e. on every compact set in Ω). The following proposition is obvious:

Proposition 2.1. *Let $f_i \in \mathscr{D}'(\Omega), i = 1,2$. Then*

$$D^\alpha(f_1+f_2) = D^\alpha f_1 + D^\alpha f_2, \quad D^\alpha(\lambda f_i) = \lambda D^\alpha f_i, \quad D^\alpha(D^\beta f_i) = D^{\alpha+\beta} f_i.$$

Theorem 2.1. *Let $u \in \mathscr{D}'(\Omega)$, $\overline{\Omega}^* \subset \Omega$, Ω^* bounded. Suppose that $D^\alpha u \in L^p_{loc}(\Omega)$, $p \geq 1$. Then there exists $h_0 > 0$ such that for $h \leq h_0$, $x \in \Omega^*$ we have:*

$$D^\alpha u_h(x) = (D^\alpha u)_h(x), \tag{2.13}$$

$$\lim_{h\to 0} D^\alpha u_h = D^\alpha u \text{ in } L^p(\Omega^*). \tag{2.14}$$

Proof. Indeed: if $\varphi_h(y) = (1/\kappa h^N)\omega(x-y,h)$, then $u_h(x) = \langle \varphi_h, u\rangle$ for $h \leq h_0$, $x \in \Omega^*$, with $h_0 \leq \operatorname{dist}(\Omega^*, \partial\Omega)$. Using the definition from 2.1.1, it follows that

$$\begin{aligned} D^\alpha u_h(x) &= (-1)^{|\alpha|}\langle D^\alpha \varphi_h, u\rangle = \langle \varphi_h, D^\alpha u\rangle \\ &= \frac{1}{\kappa h^N}\int_\Omega \omega(x-y,h) D^\alpha u(y)\,\mathrm{d}y = (D^\alpha u)_h(x); \end{aligned}$$

(2.14) is a direct consequence of Theorem 1.2. □

2.2.2 *The Absolute Continuity*

Let Ω be a domain in $\mathbb{R}^N$, P a line verifying $P \cap \Omega \neq \emptyset$. A function defined almost everywhere in Ω is said *absolutely continuous on the line P* if it is continuous on each closed interval of $P \cap \Omega$.

Theorem 2.2. *Suppose $u \in L^1_{loc}(\Omega)$ and $\partial u/\partial x_i \in L^p(\Omega)$, $p \geq 1$. This function changed on a set of measure zero is absolutely continuous on almost all lines parallel to the axis x_i.*[2] *Let us denote by $[\partial u/\partial x_i]$ the usual derivative and by $\partial u/\partial x_i$ the distribution derivative. Then we have almost everywhere $[\partial u/\partial x_i] = \partial u/\partial x_i$.*

Conversely, if $u \in L^1_{loc}(\Omega)$ is absolutely continuous on almost all lines parallel to the axis x_i with $[\partial u/\partial x_i] \in L^p(\Omega)$, then we have $\partial u/\partial x_i = [\partial u/\partial x_i]$.

Proof. Let us prove the second part: If $\varphi \in \mathscr{D}(\Omega)$, by integration by parts we get $\langle \varphi, [\partial u/\partial x_i]\rangle = \langle \varphi, \partial u/\partial x_i\rangle$. For the first part, let $\Omega = \cup_{j=1}^\infty C_j$, where C_j are cubes; this cover is locally finite, which is always possible. Let C be one of these cubes, and $\psi \in \mathscr{D}(\Omega)$ such that $\psi(x) = 1$ for $x \in C$. Let us put $v = u\psi; v \in L^1(\Omega)$. Obviously $\partial v/\partial x_i = (\partial u/\partial x_i)\psi + u(\partial \psi/\partial x_i)$. Put $v = \partial v/\partial x_i = 0$ for $x \notin \Omega$. Let K be a cube big enough such that $\overline{\Omega} \subset K$.

---

[2]The set of all intersections of parallel hyperplanes where u is not absolutely continuous, with the hyperplane $x_i = 0$, is a set M such that $\operatorname{meas}_{(N-1)} M = 0$.

Let us define $v^*(x)$ *by*:

$$v^*(x) = \int_{-\infty}^{x_i} \frac{\partial v}{\partial x_i}(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_N)\,\mathrm{d}\xi \tag{2.15}$$

for the points $(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N)$, where

$$\int_{-\infty}^{\infty} \left| \frac{\partial v}{\partial x_i}(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_N) \right| \mathrm{d}\xi < \infty.$$

Let $\chi \in \mathscr{D}(\Omega)$ be a function with $\chi(x) = 1$ on $\operatorname{supp} v$.

For all $\varphi \in \mathscr{D}(\Omega)$, we have:

$$\begin{aligned}\int_\Omega \varphi v^*\,\mathrm{d}x &= \int_\Omega \Big(\int_{x_i}^{\infty} \varphi(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_N)\,\mathrm{d}\xi\Big)\frac{\partial v}{\partial x_i}(x)\,\mathrm{d}x \\ &= \int_\Omega \Big(\int_{x_i}^{\infty} \varphi\,\mathrm{d}\xi\Big)\chi\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \int_\Omega \varphi v\,\mathrm{d}x,\end{aligned}$$

then almost everywhere $v(x) = v^*(x)$; it is clear that v^* is absolutely continuous on almost all lines parallel to the axis x_i, and almost everywhere $[\partial v^*/\partial x_i] = \partial v/\partial x_i$. But since almost everywhere on C $v^*(x) = v(x)$, the result follows. □

Remark 2.1. According to Theorem 2.2, a function f not absolutely continuous on $[0,1]$ (or continuous) which has almost everywhere a derivative such that $[\mathrm{d}f/\mathrm{d}x] \in L^1(0,1)$, satisfies $[\mathrm{d}f/\mathrm{d}x] \neq \mathrm{d}f/\mathrm{d}x$. The well known example is a monotone function continuous on $(0,1)$, $f(0) = 0$, $f(1) = 1$, $\mathrm{d}f/\mathrm{d}x = 0$ almost everywhere.

2.2.3 *The Spaces* $W^{k,p}(\Omega)$

For an integer $k \geq 0$, and $p \geq 1$, we denote by $W^{k,p}(\Omega)$[3] the subspace of functions $f \in L^p(\Omega)$ such that for $|\alpha| \leq k, D^\alpha u \in L^p(\Omega)$. On $W^{k,p}(\Omega)$ we define a norm by:

$$|u|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p dx\Big)^{1/p}, \tag{2.16}$$

and, if $p = 2, W^{k,2}(\Omega)$ is an Hilbert space for the scalar product defined by (1.1.2). The membership of u in $W^{k,p}(\Omega)$ is a local property, indeed.

[3] If $u \in W^{k,2}(\Omega)$ by the definition from 1.1.1, then in general it is the same as in the definition given here; the converse is not true. Later we shall see that for $\Omega \in \mathfrak{N}^0$ the two definitions coincide. In N.G. Meyers, J. Serrin [1], it is proved that $C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Proposition 2.2. *Let $\Omega_i, i = 1,2,\dots,l$, be domains satisfying $\cup_{i=1}^{l}\Omega_i \supset \Omega$. If $u \in W^{k,p}(\Omega_i)$, $i = 1,2,\dots,l$, then $u \in W^{k,p}(\Omega)$, and*

$$|u|_{W^{k,p}(\Omega)} \leq c\sum_{i=1}^{l}|u|_{W^{k,p}(\Omega_i)}.$$

Proof. Let $|\alpha| \leq k$. We denote by g_i the derivatives $D^\alpha u$ in Ω_i; it follows immediately from the definition of $D^\alpha u$ that if $\Omega_i \cap \Omega_j \neq \emptyset$, then $g_i = g_j$ almost everywhere in $\Omega_i \cap \Omega_j$. We define $g(x) = g_i(x)$ in Ω_i, and let $\varphi \in C_0^\infty(\Omega)$. According to Proposition 1.2.3, there exist functions $\psi_i \in C_0^\infty(\Omega_i)$ such that

$$x \in \text{supp}\ \varphi \Longrightarrow \sum_{i=1}^{l}\psi_i(x) = 1.$$

We have:

$$\langle D^\alpha\varphi, u\rangle = \langle D^\alpha(\sum_{i=1}^{l}\psi_i\varphi), u\rangle = (-1)^{|\alpha|}\sum_{i=1}^{l}\langle\psi_i\varphi, g_i\rangle$$
$$= (-1)^{|\alpha|}\sum_{i=1}^{l}\langle\psi_i\varphi, g\rangle = (-1)^{|\alpha|}\langle\varphi, g\rangle.$$

□

Remark 2.2. According to Theorem 2.2, there is a definition equivalent to the previous: $u \in W^{1,p}(\Omega)$, if $u \in L^p(\Omega)$ and if, after a modification of u on a set of zero measure, u remains absolutely continuous on almost all lines parallel to the x_1 axis and if $[\partial u/\partial x_1] \in L^p(\Omega)$ (cf. Theorem 2.2). By another modification $[\partial u/\partial x_2] \in L^p(\Omega)$, etc.

There exists another definition for $p = 2$, $k = 1$ due to B. Levi related to Remark 2.2: $W^{1,p}(\Omega)$ is the subspace in $L^p(\Omega)$ of functions u which, and after a modification on a set of zero measure, remain absolutely continuous on almost all lines parallel to the axes $x_1, x_2, \dots, x_N$; the derivatives $[\partial u/\partial x_i] \in L^p(\Omega)$.

Here an adaptation of a theorem of J. Deny, J.L. Lions [1]:

Theorem 2.3. *For $W^{1,p}(\Omega)$, the definition from 2.2.3 and the definition of B. Levi are equivalent.*

Proof. If $u \in W^{1,p}(\Omega)$ by the B. Levi definition, then it is the same as by our definition according to Theorem 2.2. Let $u \in W^{1,p}(\Omega)$; we use the steps used in the proof of Theorem 2.2: $u\psi = v \in W^{1,p}(K)$, where K is a cube $(-l,l)^N$ sufficiently large such that $\text{supp}\, v \subset K$. According to Theorem 2.1, $\lim_{h\to 0} v_h = v$ in $L^p(\Omega)$, $\lim_{h\to 0} \partial v_h \partial x_i = \partial v/\partial x_i$ in $L^p(K)$, $i = 1,2,\dots,N$. Let us set for $i = 1,2,\dots,N$:

$$g_i(x) = \int_{-\infty}^{x_i}\frac{\partial v}{\partial x_i}(x_1,\dots,x_{i-1}\xi, x_{i+1},\dots,x_N)\,\mathrm{d}\xi. \tag{2.17}$$

Then $g_i(x)$ is an absolutely continuous function on almost all lines parallel to the axis x_i; we have:

$$\lim_{h\to 0}\underbrace{\int_{-l}^{l}\cdots\int_{-l}^{l}}_{(N-1)\text{-times}}\left(\int_{-l}^{l}\left|\frac{\partial v_h}{\partial x_1}-\frac{\partial v}{\partial x_1}\right|\mathrm{d}x_1\right)\mathrm{d}x'=0,$$

then we can find a sequence h_n, $\lim_{n\to\infty} h_n = 0$, such that for almost all lines parallel to the axis x_1

$$\lim_{n\to\infty}\int_{-l}^{l}\left|\frac{\partial v_{h_n}}{\partial x_1}-\frac{\partial v}{\partial x_1}\right|\mathrm{d}x_1=0.$$

We deduce that $\lim_{n\to\infty} v_{h_n}(x) = g_1(x)$ on almost all lines parallel to x_1. The same property holds for $i = 2$: we can extract a subsequence h_{n_m} of the sequence h_n such that $\lim_{m\to\infty} v_{h_{n_m}}(x) = g_2(x)$ on almost all lines parallel to the axis x_2, etc. Step by step we construct a sequence v_{h_s}, such that $\lim_{s\to\infty} v_{h_s}(x) = g_i(x)$ on almost all lines parallel to $x_1, x_2, \ldots, x_N$; it is clear that $\lim_{s\to\infty} v_{h_s} = v^*(x) = v(x)$ almost everywhere in Ω. We conclude as in the proof of Theorem 2.2. □

Exercise 2.1. If $u \in W^{1,p}(\Omega)$, prove that $|u| \in W^{1,p}(\Omega)$ and $||u||_{W^{1,p}(\Omega)} \le |u|_{W^{1,p}(\Omega)}$.

Hint: use Theorem 2.3.

2.2.4 The Spaces $W^{k,p}(\Omega)$ (Continuation)

Proposition 2.3. *The space $W^{k,p}(\Omega)$ is a Banach space with a countable basis, reflexive for $p > 1$.*

Proof. The space $W^{k,p}(\Omega)$ is complete, this is a consequence of the definition from 2.2.3; $W^{k,p}(\Omega)$ is a closed subspace of the space $[L^p(\Omega)]^s$ which has a countable basis and which is reflexive if $p > 1$. Here, s is the number of indices $|\alpha| \le k$. □

Proposition 2.4. *Let u_i be a sequence in $\mathscr{D}'(\Omega)$, with $|D^\alpha u_i|_{L^p(\Omega)} \le c_1$, $p > 1$, and $\lim_{i\to\infty} u_i = u$ in the sense of distributions. Then $D^\alpha u \in L^p(\Omega)$, and $|D^\alpha u|_{L^p(\Omega)} \le c_1$.*

Proof. For $\varphi \in C_0^\infty(\Omega)$, we have:

$$\lim_{i\to\infty}\langle\varphi, D^\alpha u_i\rangle = \lim_{i\to\infty}(-1)^{|\alpha|}\langle D^\alpha\varphi, u_i\rangle = (-1)^{|\alpha|}\langle D^\alpha\varphi, u\rangle,$$

$$\langle\varphi, D^\alpha u_i\rangle = \int_\Omega \varphi D^\alpha u_i\,\mathrm{d}x.$$

But $\overline{\mathscr{D}(\Omega)} = L^q(\Omega)$ for $q \ge 1$, hence $\lim_{i\to\infty} D^\alpha u_i = g$ weakly in $L^p(\Omega)$, and $\langle\varphi, g\rangle = \lim_{i\to\infty}\langle\varphi, D^\alpha u_i\rangle = \lim_{i\to\infty}(-1)^{|\alpha|}\langle D^\alpha\varphi, u_i\rangle = (-1)^{|\alpha|}\langle D^\alpha\varphi, u\rangle$. □

Remark 2.3. If $p = 1$, Proposition 2.4 is true if the sequence $D^\alpha u_i$ is weakly compact.

2.2.5 The Spaces $W_0^{k,p}(\Omega)$

We denote $W_0^{k,p}(\Omega) = \overline{\mathscr{D}(\Omega)}$, the closure of $\mathscr{D}(\Omega)$ with respect to the norm of $W^{k,p}(\Omega)$, and $W^{-k,q}(\Omega) = (W_0^{k,p}(\Omega))'$ the dual space of $W_0^{k,p}(\Omega)$.

Proposition 2.5. *Suppose $p > 1$; then every function $f \in W^{-k,q}(\Omega)$ can be written (not uniquely) in the following form:*

$$f = \sum_{|\alpha| \leq k} D^\alpha f_\alpha, \tag{2.18}$$

where

$$f_\alpha \in L^q(\Omega) \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Let s be as in the proof of Proposition 2.3. We have $W_0^{k,p}(\Omega) \subset [L^p(\Omega)]^s$. According to the Hahn-Banach theorem, f can be extended on $[L^p(\Omega)]^s$. But $([L^p(\Omega)]^s)' = [L^q(\Omega)]^s$, hence

$$v \in [L^p(\Omega)]^s \Longrightarrow fv = \sum_{i=1}^{s} \int_\Omega v_i g_i \,\mathrm{d}x$$

with $g_i \in L^q(\Omega)$. If $v \in W_0^{k,p}(\Omega)$, we get:

$$fv = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \int_\Omega D^\alpha v f_\alpha \,\mathrm{d}x$$

with $f_\alpha \in L^q(\Omega)$; then

$$v \in \mathscr{D}(\Omega) \Longrightarrow fv = \langle v, \sum_{|\alpha| \leq k} D^\alpha f_\alpha \rangle.$$

□

Exercise 2.2. If $p > 1$, prove that the closed unit ball in $W^{k,p}(\Omega)$ is weakly compact.

Remark 2.4. If Ω is a domain such that its complement $\complement\Omega$ has a positive measure, then for $k \geq 1$ we cannot have $W_0^{k,p}(\Omega) = W^{k,p}(\Omega)$; details can be found in J.L. Lions [5]. But if $\Omega = \mathbb{R}^N$ we have:

Proposition 2.6. $W_0^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$.

Proof. Let $\varphi \in \mathscr{D}(\mathbb{R}^N)$ satisfy $\varphi(x) = 1$ for $|x| \leq 1$, $\varphi(x) = 0$ for $|x| \geq 2$. If $u \in W^{k,p}(\mathbb{R}^N)$, let us put $u_r(x) = u(x)\varphi(x/r)$. Clearly $\lim_{r\to\infty} u_r = u$ in $W^{k,p}(\mathbb{R}^N)$. Using Theorem 2.1, we get $\lim_{h\to 0} u_{rh} = u_r$ in $W^{k,p}(\mathbb{R}^N)$ (we use (2.6)); but $u_{rh} \in \mathscr{D}(\mathbb{R}^N)$. □

2.3 Imbedding Theorems

2.3.1 The Lipschitz Transform

Lemma 3.1. *Let Ω, O be two bounded open sets, T a one-to-one continuous mapping, $T : O \to \Omega$, with a Lipschitz inverse, i.e.*

$$|T^{-1}(x) - T^{-1}(y)| \le c|x-y|. \tag{2.19}$$

Let $u \in L^p(\Omega), p \ge 1$. Then $v(y) = u(T(y)) \in L^p(O)$, and we have:

$$|v|_{L^p(O)} \le c|u|_{L^p(\Omega)}. \tag{2.20}$$

Proof. Using the regularizing operator (we put $u(x) = 0$ for $x \notin \Omega$), we get $\lim_{h\to 0} u_h = u$ in $L^p(\Omega)$. Let S_d be a rectangular lattice in $\mathbb{R}^N$ formed by cubes with sidelength d; let us consider cubes $C_1, C_2, \dots, C_{m_d}$ whose closures are contained in O; we have:

$$\int_O |u_h(T(y))|^p \,\mathrm{d}y = \lim_{d\to 0} \sum_{i=1}^{m_d} d^n \inf_{y\in C_i} |u_h(T(y))|^p. \tag{2.21}$$

According to (2.19) we have:

$$\text{meas}\,(C_i) \le c_i \,\text{meas}\,(T(C_i)). \tag{2.22}$$

Indeed: if y_i is the center of C_i, and if $\partial T(C_i)$ is the boundary of $T(C_i)$, we get $T^{-1}(\partial T(C_i)) \subset \partial C_i$. Using (2.19) if $x \in \partial T(C_i)$, and denoting $x_i = T(y_i)$, we have $|x - x_i| \ge c_2|T^{-1}(x) - y_i| \ge (1/2)dc_2$, hence $T(C_i)$ contains a ball with center x_i and radius $(1/2)dc_2$ and we have (2.22). Furthermore,

$$d^n \sum_{i=1}^{m_d} \inf_{y\in C_i} |u_h(T(y))|^p \le c_1 \sum_{i=1}^{m_d} \text{meas}\,(T(C_i)) \inf_{y\in C_i} |u_h(T(y))|^p \le c_1 \int_\Omega |u_h(x)|^p \,\mathrm{d}x, \tag{2.23}$$

and then

$$\int_O |u_h(T(y))|^p \,\mathrm{d}y \le c_1 \int_\Omega |u_h(x)|^p \,\mathrm{d}x. \tag{2.24}$$

Now $\lim_{h\to 0} u_h = u$ in $L^p(\Omega)$, and we can extract a subsequence, say u_{h_i}, such that $\lim_{i\to\infty} u_{h_i} = u$ almost everywhere in Ω. It follows from (2.19) that $\lim_{i\to\infty} u_{h_i}(T(y)) = u(T(y))$ almost everywhere in O. The Fatou lemma gives (2.24). □

Lemma 3.2. *Let Ω, O be two bounded open sets, and T and T^{-1} one-to-one Lipschitz mappings, $T : O \to \Omega$. Let $u \in W^{1,p}(\Omega)$, $p \ge 1$. We have $u(T(y)) \in W^{1,p}(O)$, and if we set $v(y) = u(T(y))$ we get:*

$$|v|_{W^{1,p}(O)} \leq c|u|_{W^{1,p}(\Omega)}. \tag{2.25}$$

Proof. Let us put $u \equiv 0$ outside of Ω, and let u_h be the regularized function. The function $v^{(h)}(y) = u_h(T(y))$ is a Lipschitz function in $\overline{O}$, hence *a posteriori* it is Lipschitz on all lines parallel to $y_1, y_2, \dots, y_N$. Then we have in the usual sense:

$$\frac{\partial v^{(h)}}{\partial y_i} = \sum_{j=1}^{N} \frac{\partial u_h}{\partial x_j}\frac{\partial x_j}{\partial y_i}; \tag{2.26}$$

according to Theorem 2.2 it holds in the sense of distributions. We have $\lim_{h\to 0} \partial u_h/\partial x_j = \partial u/\partial x_j$ in $L^p(\Omega^*)$ for $\overline{\Omega^*} \subset \Omega, \lim_{h\to 0} u_h = u$ in $L^p(\Omega)$. Let us denote $O^* = T^{-1}(\Omega^*)$. According to the previous lemma, $\partial v^{(h)}/\partial y_i$ is a Cauchy sequence in $L^p(O^*)$. We get:

$$\frac{\partial v}{\partial y_i} = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial y_i}, \tag{2.27}$$

and then, using again the previous lemma, we have $v \in W^{1,p}(O)$, and hence the inequality (2.25). □

2.3.2 Density of $C^\infty(\overline{\Omega})$ in $W^{k,p}(\Omega)$

In Chap. 1, we introduced another definition of $W^{k,2}(\Omega)$ (cf. the definition in 1.1.1); we can generalize it for $p \geq 1$.

Problem 3.1. Characterize the domains such that $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$.

We know that these two definitions are equivalent under certain conditions (cf. E. Gagliardo [2]), V.P. Iljin [1]); let us denote by $C^\infty_{\overline{\Omega}}(\mathbb{R}^N)$ the space of restrictions to $\overline{\Omega}$ of functions in $C^\infty(\mathbb{R}^N)$ (clearly we have $C^\infty_{\overline{\Omega}}(\mathbb{R}^N) \subset C^\infty(\overline{\Omega})$). We have:

Theorem 3.1. *Let $\Omega \in \mathfrak{N}^0$. There is $\overline{C^\infty_{\overline{\Omega}}(\mathbb{R}^N)} = W^{k,p}(\Omega)$.*

Proof. Using the notations introduced in 1.2.4, we set $u_r = u\varphi_r$. We get immediately $u_r \in W^{k,p}(\Omega)$; using the local charts (x'_r, x_{rN}), let us define $u_{r\lambda}$, $r \leq m$ by $u_{r\lambda}(x'_r, x_{rN}) = u_r(x'_r, x_{rN} + \lambda)$. If λ is sufficient small, we have $u_{r\lambda} \in W^{k,p}(\Omega)$, and according to Theorem 1.1 $\lim_{\lambda\to 0} u_{r\lambda} = u_r$ in $W^{k,p}(\Omega)$. We have $u_{r\lambda} \in W^{k,p}(\Omega_\lambda)$, with $\Omega_\lambda \supset \overline{\Omega}$. It follows from Theorem 2.1 that $\lim_{h\to 0} u_{r\lambda h} = u_{r\lambda}$ in $W^{k,p}(\Omega)$. If $r = m+1$, we have $\lim_{h\to 0} u_{m+1,h} = u_{m+1}$ in $W^{k,p}(\Omega)$, so $u_{\lambda h} = \sum_{r=1}^{m+1} u_{r\lambda h}$, and we get $\lim_{h\to 0, \lambda\to 0} u_{\lambda h} = u$ in $W^{k,p}(\Omega)$. □

Example 3.1. Let Ω be the disc in $\mathbb{R}^2$ with center at origin and radius 1, without the segment $0 \leq x_1 \leq 1, x_2 = 0$. Then $\overline{C^\infty(\overline{\Omega})} \neq W^{k,p}(\Omega)$, if $k \geq 1$, $p \geq 2$ (the result is true if $p \leq 2$, but at the moment, we are not able to prove it). Indeed, according to Theorem 1.1.2 we can define traces "from top" and "from bottom" on the segment mentioned. If we have $\overline{C^\infty(\overline{\Omega})} = W^{k,p}(\Omega)$, these traces will coincide, but this is not possible in polar coordinates (r,Θ) if we consider the function $r^k\Theta \in W^{k,p}(\Omega)$.

A bounded domain is called *starshaped with respect to the origin* if there exists a positive continuous function on the unit sphere, say $h(x/|x|)$, such that $\Omega = \{x \in \mathbb{R}^N, |x| < h(x/|x|)\}$. We have (cf. V.I. Smirnov [1]):

Theorem 3.2. *Let Ω be a starshaped domain with respect to the origin. Then $W^{k,p}(\Omega) = \overline{C^\infty_{\overline{\Omega}}(\mathbb{R}^N)}$.*

Proof. Indeed: let $u \in W^{k,p}(\Omega)$ and put $u_\lambda(x) = u(\lambda x)$, $0 < \lambda < 1$. According to Theorem 1.1, $\lim_{\lambda\to 1} u_\lambda = u$ in $W^{k,p}(\Omega)$. Denoting $\Omega_\lambda = \{x \in \mathbb{R}^N, x = y/\lambda, y \in \Omega\}$, we have $u_\lambda \in W^{k,p}(\Omega_\lambda)$, but $\overline{\Omega} \subset \Omega_\lambda$. Obviously, for every $\varepsilon > 0$, there exist λ, $h > 0$ such that $|u_{\lambda h} - u|_{W^{k,p}(\Omega)} < \varepsilon$. □

In the statements of Theorems 3.1, 3.2, the properties of Ω are used; but it is possible to generalize these theorems in the following form:

Theorem 3.3. *Let Ω be a bounded domain such that there exists a sequence of domains Ω_n, $n = 1,2,\ldots$ such that $\overline{\Omega} \subset \Omega_n$, $\Omega_n \supset \Omega_{n+1}$; $\cap_{n=1}^\infty \Omega_n = \Omega$, and let us assume that for every $u \in W^{k,p}(\Omega)$, there exists $u_n \in W^{k,p}(\Omega_n)$ such that $\lim_{n\to\infty} |u_n - u|_{W^{k,p}(\Omega)} = 0$. Then $\overline{C^\infty_{\overline{\Omega}}(\mathbb{R}^N)} = W^{k,p}(\Omega)$.*

Proof. Let $u_n = 0$ outside of Ω_n; according to Theorem 2.1 we can find a sequence h_n such that $\lim_{n\to\infty} |u_{nh_n} - u_n|_{W^{k,p}(\Omega)} = 0$. □

2.3.3 *The Gagliardo Lemma*

Let us prove a lemma due to E. Gagliardo [2]:

Lemma 3.3. *Let C be the cube $(-1,1)^N$, let C_i denote its faces $(-1,1)^{N-1}$ for $x_i = 0$. Let $f_i \in L^{N-1}(C_i)$, $i = 1,2,\ldots,N$, and define f_i in C by $f_i(x) = f_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N)$. Then*

$$\int_C \Big|\prod_{i=1}^N f_i\Big| dC \leq \prod_{i=1}^N \Big(\int_{C_i} |f_i|^{N-1} dC_i\Big)^{1/(N-1)}. \tag{2.28}$$

Proof. We use a recurrence process over N. For $N = 2$ inequality (2.28) is a simple consequence of Fubini's theorem:

$$\int_C |f_1||f_2| dC = \int_{-1}^1 |f_1(x_2)|\,\mathrm{d}x_2 \int_{-1}^1 |f_2(x_1)|\,\mathrm{d}x_1.$$

Let be $N > 2$, and let us assume that (2.28) holds for $N-1$; we have:

$$I = \int_C |f_1 f_2 \dots f_N| dC = \int_{C_1} |f_1| dC_1 \int_{-1}^{1} |f_2 f_3 \dots f_N| \,\mathrm{d}x_1.$$

Using the Hölder inequality for $p_i = N-1$, $i = 1,2,\dots,N-1$, we get:

$$I \le \int_{C_1} |f_i| dC_1 \prod_{i=2}^{N} \Big(\int_{-1}^{1} |f_i|^{N-1} \,\mathrm{d}x_1 \Big)^{1/(N-1)}. \tag{2.29}$$

Now using again the Hölder inequality for $p = N-1$, $q = (N-1)/(N-2)$ in (2.29), we get:

$$I \le \big(\int_{C_1} |f_1| dC_1 \big)^{1/(N-1)} \big(\int_{C_1} \prod_{i=2}^{N} \big(\int_{-1}^{1} |f_i|^{N-1} \,\mathrm{d}x_1 \big)^{1/(N-2)} dC_1 \big)^{(N-2)/(N-1)}. \tag{2.30}$$

Using now the recurrence hypothesis, denoting $C_{1i}, i = 2,\dots,N$, the projections of C_1 on the hyperplane $x_i = 0$, we finally get:

$$\int_{C_1} \prod_{i=2}^{N} \big(\int_{-1}^{1} |f_i|^{N-1} \,\mathrm{d}x_1 \big)^{1/(N-2)} dC_1$$
$$\le \prod_{i=2}^{N} \big(\int_{C_{1i}} dC_{1i} \int_{-1}^{1} |f_i|^{N-1} \,\mathrm{d}x_1 \big)^{1/(N-2)} = \prod_{i=2}^{N} \big(\int_{C_i} |f_i|^{N-1} dC_i \big)^{1/(N-2)}.$$

Then taking into account (2.30), we get (2.28). □

2.3.4 The Sobolev Imbedding Theorems

Now, let us start with the first of the *imbedding theorems*; basically these theorems are due to Sobolev (cf. S.L. Sobolev [1].) Here we use an adaptation of the method of Gagliardo (cf. E. Gagliardo [2]). Let us recall that if for two Banach spaces $B_1, B_2, B_1 \subset B_2$ is an imbedding algebraically and topologically, then this means that each element in B_1 is an element of B_2, and for every $x \in B_1$, $|x|_{B_2} \le c|x|_{B_1}$. We use the notation introduced in 1.2.4.

Theorem 3.4. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1 \le p < N$. If $1/q = 1/p - 1/N$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$ algebraically and topologically.*

Proof. It is sufficient to prove that $u_r = u\varphi_r \in L^q(V_r)$. We define a mapping T on the cube $C = \{|y_i| < 1, i = 1,2,\dots,N\}$, $T : C \to V_r$ using $x = (x'_r, x_{rN})$, $y = (y'_r, y_{rN})$; for simplicity we omit the index r, and we set:

$$x' = \alpha y', \quad x_N = (\beta/2) y_N + a(\alpha y') + \beta/2. \tag{2.31}$$

The mapping T and its inverse are lipschitzian between $\overline{C}$ and $\overline{V}$. According to Lemma 3.2 it is sufficient to prove that $v \in L^q(C)$ where $v(y) = u(T(y))$, and the corresponding inequality. Let $v \in W^{1,p}(C)$, and equal zero in a neighborhood of the sides of the cube, except the side $y_N = 1$, $|y_i| < 1$, $i = 1,2,\dots,N-1$, cf. Fig. 2.1

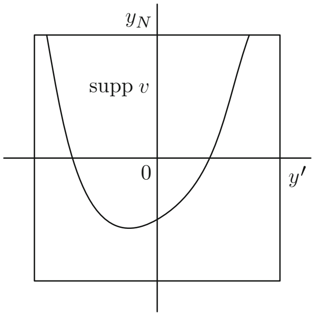

Fig. 2.1

Theorem 2.1 implies the existence of a sequence v_l, $\lim_{l\to\infty} v_l = v$ in $W^{1,p}(C), v_l \in C^\infty(\overline{C})$ with the support mentioned. We prove first the following inequality:

$$|v_l|_{L^q(C)} \le c|v_l|_{W^{1,p}(C)}, \tag{2.32}$$

then we can pass to the limit as $l \to \infty$ and we obtain:

$$|v|_{L^q(C)} \le c|v|_{W^{1,p}(C)}. \tag{2.33}$$

Let $v \in C^\infty(\overline{\Omega})$ with the support mentioned and let us consider the function:

$$y_i \to |v(y)|^{(Np-p)/(N-p)}$$

as a function of the variable y_i (all other local charts are fixed). On the interval $(-1,1)$ we have almost everywhere:

$$\frac{\partial}{\partial y_i}|v(y)|^{(Np-p)/(N-p)} \le \frac{Np-p}{N-p}|v|^{(Np-N)/(N-p)}\left|\frac{\partial v}{\partial y_i}\right|; \tag{2.34}$$

in fact, it is sufficient to use the inequality

$$\left|\frac{d|f|}{dt}\right| \le \left|\frac{df}{dt}\right| \tag{2.35}$$

which holds for an absolutely continuous function almost everywhere on $(-1,1)$. Using (2.34) we get:

$$\sup_{|y_i|\le 1} |v(y)|^{(Np-p)/(N-p)} \le \frac{Np-p}{N-p}\int_{-1}^{1} |v|^{(Np-N)/(N-p)} \left|\frac{\partial v}{\partial y_i}\right| \mathrm{d}y_i. \tag{2.36}$$

Let C_i be the projection of C on the hyperplane $y_i = 0$, $p > 1$. Using Hölder's inequality we get from (2.36),

$$\begin{aligned}
&\int_{C_i} \sup_{|y_i|\le 1} |v(y)|^{(Np-N)/(N-p)} dC_i \\
&\le \frac{Np-p}{N-p}\left(\int_C |v|^{Np/(N-p)} dC\right)^{(p-1)/p} \left(\int_C \left|\frac{\partial v^p}{\partial y_i}\right| dC\right)^{1/p} \\
&\le \frac{Np-p}{N-p}\left(\int_C |v|^{Np/(N-p)} dC\right)^{(p-1)/p} |v|_{W^{1,p}(C)}.
\end{aligned} \tag{2.37}$$

Taking into account Lemma 3.3 and (2.37), it follows that

$$\begin{aligned}
\int_C |v|^{Np/(N-p)} dC &\le \int_C \prod_{i=1}^{N} \sup_{|y_i|<1} |v|^{p/(N-p)} dC \\
&\le \prod_{i=1}^{N}\left(\int_{C_i} \sup_{|y_i|<1} |v|^{(Np-p)/(N-p)} dC_i\right)^{1/(N-1)} \\
&\le \left(\frac{Np-p}{N-p}\right)^{N/(N-1)} \left(\int_C |v|^{Np/(N-p)} dC\right)^{(Np-N)/(Np-p)} |v|^{N/(N-1)}_{W^{1,p}(C)},
\end{aligned} \tag{2.38}$$

and hence

$$\left(\int_C |v|^{Np/(N-p)} dC\right)^{(N-p)/Np} \le \frac{Np-p}{N-p}|v|_{W^{1,p}(C)}. \tag{2.39}$$

Going back to the index r, for $r = m+1$ we cover U_{m+1} by a finite number of cubes and using a partition of unity we again arrive at the inequality (2.39), and we have the result for $p > 1$. If $p = 1$ then (2.36) becomes:

$$\sup_{|y_i|<1} |v(y)| \le \int_{-1}^{1} \left|\frac{\partial v}{\partial y_i}\right| \mathrm{d}y,$$

and by Lemma 3.3, we obtain

$$\int_C |v|^{N/(N-1)} dC \le \int_C \prod_{i=1}^{N} \sup_{|y_i|<1} |v|^{1/(N-1)} dC$$

$$\le \prod_{i=1}^{N}\left(\int_{C_i} \sup_{|y_i|<1} |v| dC_i\right)^{1/(N-1)} \le \prod_{i=1}^{N}\left(\int_C \left|\frac{\partial v}{\partial y_i}\right| dC\right)^{1/(N-1)} \le |v|^{N/(N-1)}_{W^{1,1}(C)},$$

which completes the proof. □

Remark 3.1. Theorem 3.4 is obviously true for a finite union of domains of the type $\mathfrak{N}^{0,1}$.

A domain has the *interior cone property* if there exists a fixed cone such that each point of Ω is the vertex of this cone appropriately placed into Ω. We can easily prove that such a domain can be decomposed into a finite union of domains of the type $\mathfrak{N}^{0,1}$, cf. E. Gagliardo [2].

Remark 3.2. In J.L. Lions [5], Theorem 3.4 is proved for $\Omega = \mathbb{R}^N$. The proof follows the same ideas.

Example 3.2. The number $q = Np/(N-p)$ in Theorem 3.4 is the best possible: the function $u(x) = |x|^{-(N/2)+1}\ln^{-1}|x|$ defined on the ball $\Omega = \{x \in \mathbb{R}^N, |x| < 1/2\}$, $N \geq 3$, is in $W^{1,2}(\Omega)$; on the other hand,

$$\int_\Omega |u|^{(2N/(N-2))+\varepsilon}\,\mathrm{d}x = \infty \quad \text{for} \quad \varepsilon > 0.$$

Theorem 3.5. *Let $\Omega \in \mathfrak{N}^{0,1}, p = N$. Obviously $W^{1,p}(\Omega) \subset L^q(\Omega)$ algebraically and topologically for any $q, 1 \leq q < \infty$.*

Theorem 3.6. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$, $kp < N$. Put $1/q = 1/p - k/N$. Then $W^{k,p}(\Omega) \subset L^q(\Omega)$ algebraically and topologically.*

Proof. We proceed by recurrence with respect to k: the theorem is true if $k = 1$; we assume that it is true for $k-1$. Then $D^\alpha u \in W^{1,p}(\Omega)$, $|\alpha| \leq k-1$, hence $D^\alpha u \in L^{q^*}(\Omega)$ with $1/q^* = 1/p - 1/N \Longrightarrow u \in W^{k-1,q^*}(\Omega) \Longrightarrow u \in L^q(\Omega)$. □

Obviously, we have also

Theorem 3.7. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$, $kp = N$. Then $W^{k,p}(\Omega) \subset L^q(\Omega)$ algebraically and topologically for any q, $1 \leq q < \infty$.*

2.3.5 The Sobolev Imbedding Theorems (Continuation)

Theorem 1.1.11 is a particular case of an imbedding theorem if $kp > N$. Now we follow the ideas of C.B. Morrey [3]. Hereafter we shall say that for a *Banach space B, $B \subset C^0(\overline{\Omega})$ algebraically and topologically* if every function $f \in B$ (where B is a subspace of measurable functions on Ω) can be modified on a set of measure zero in such a way that this modified function is absolutely continuous on $\overline{\Omega}$; moreover we have:

$$\max_{x\in\overline{\Omega}} |f(x)| \leq c|f|_B.$$

Theorem 3.8. *Let $\Omega \in \mathfrak{N}^{0,1}, p \geq 1, kp > N$, and denote*

$$\mu \begin{cases} = k - (N/p) & \textit{if} \quad k - (N/p) < 1, \\ < 1 & \textit{if} \quad k - (N/p) = 1, \\ = 1 & \textit{if} \quad k - (N/p) > 1. \end{cases}$$

Then $W^{k,p}(\Omega) \subset C^{0,\mu}(\overline{\Omega})$ *algebraically and topologically.*

Proof. We proceed as in the proof of Theorem 3.4: we consider u_r, $r \leq m$; obviously $u_r \in W^{k,p}(V_r)$, and

$$|u_r|_{W^{k,p}(V_r)} \leq c_1 |u|_{W^{k,p}(\Omega)}. \tag{2.40}$$

Let us consider the case $k - (N/p) < 1$; we have $(k-1)p < N$, and thus, by Theorem 3.6,

$$|u_r|_{W^{1,q}(V_r)} \leq c_2 |u|_{W^{k,p}(V_r)} \tag{2.41}$$

where $1/q = 1/p - (k-1)/N$. For simplicity we omit the index r. If we use the mapping (2.31) we have to consider $u \in W^{1,q}(C)$, and due to Theorem 3.1, it is sufficient to assume that $u \in C^\infty(\overline{C})$, and to prove the inequality

$$|u|_{C^{0,\mu}(\overline{C})} \leq c_3 |u|_{W^{1,q}(C)}. \tag{2.42}$$

To do this, let $y_{[1]}, y_{[2]} \in \overline{C}$; it is always possible to find a cube C_ρ with faces parallel to the faces of C such that $y_{[1]}, y_{[2]} \in \overline{C_\rho} \subset \overline{C}$ and with sides of length equal to ρ, $\rho \leq |y_{[1]} - y_{[2]}| \leq \sqrt{N}\rho$, cf. Fig. 2.2 ($N = 2$).

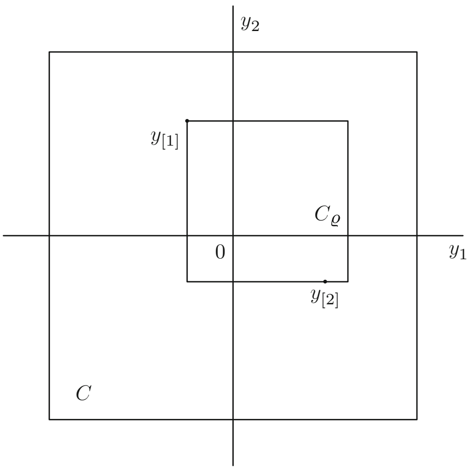

Fig. 2.2

Let $y \in C_\rho$. We have:

$$|u(y)-u(y_{[j]})| = \left| \int_0^1 \sum_{i=1}^N \frac{\partial u}{\partial y_i}(y_{[j]}+t(y-y_{[j]}))(y_i - y_{[j]i})\,\mathrm{d}t \right|$$
$$\leq c_4\rho \int_0^1 \sum_{i=1}^N \left| \frac{\partial u}{\partial y_i}(y_{[j]}+t(y-y_{[j]})) \right| \mathrm{d}t,$$

and then using the change of variables $z = y_{[j]} + t(y - y_{[j]})$, we get:

$$\begin{aligned} \left| \frac{1}{\rho^N} \int_{C_\rho} u(y)\,\mathrm{d}y - u(y_{[j]}) \right| &\leq \frac{1}{\rho^N} \int_{C_\rho} |u(y)-u(y_{[j]})|\,\mathrm{d}y \\ &\leq c_4 \rho^{-N+1} \int_0^1 t^{-N} dt \int_{C_{\rho t}} \sum_{i=1}^N \left| \frac{\partial u}{\partial y_i}(z) \right| dz, \end{aligned} \tag{2.43}$$

where

$$C_{\rho t} = \{ z \in C_\rho, z = y_{[j]} + t(y-y_{[j]}), y \in C_\rho \}.$$

Now from the Hölder inequality and from (2.43),

$$\left| \frac{1}{\rho^N} \int_{C_\rho} u(y)\,\mathrm{d}y - u(y_{[j]}) \right| \leq c_5 |u|_{W^{1,q}(C)} \rho^{k-(N/p)},$$

hence

$$|u(y_{[1]}) - u(y_{[2]})| \leq 2c_5 |u|_{W^{1,q}(C)} \rho^{k-(N/p)} \leq 2c_5 |u|_{W^{1,q}(C)} |y_{[1]} - y_{[2]}|^{k-(N/p)}. \tag{2.44}$$

Let $y_{[0]}, y \in C$; we have:

$$|u(y_{[0]})| \leq |u(y)| + \int_0^1 \sum_{i=1}^N \left| \frac{\partial u}{\partial y_i}(y_{[0]} + t(y-y_{[0]}))(y_i - y_{[0]i}) \right| \mathrm{d}t,$$

then by integration with respect to y over C we obtain as above:

$$|u(y_{[0]})| \leq c_6 |u|_{W^{1,q}(C)}, \tag{2.45}$$

and with (2.44) we have (2.42).

Concerning u_{m+1} it is sufficient to cover U_{m+1} by a finite number of cubes contained in Ω, and using a partition of unity, we obtain again (2.42). Hence, the case $k-(N/p) < 1$ is proved.

If $k-(N/p) = 1$, we use $p' < p$, and the result follows.

If $k-(N/p) > 1$, there exists a positive integer m such that $k-(N/p)-1 \leq m < k-(N/p) \Longrightarrow 0 < k-m-(N/p) \leq 1$, then $D^\alpha u \in W^{k-m,p}(\Omega)$, $|\alpha| \leq m; D^\alpha u = v$

is continuous, the case $0 < k - (N/p) < 1$ having been proved, thus $u \in C^{0,1}(\overline{\Omega})$; the theorem is proved completely. □

Remark 3.3. As an immediate consequence of the previous theorems, we obtain imbedding theorems for the derivatives. For instance, if $u \in W^{k,p}(\Omega)$, $\Omega \in \mathfrak{N}^{0,1}$, $(k-m)p < N \Longrightarrow D^\alpha u \in L^q(\Omega)$ with $1/q = 1/p - (k-m)/N$ and $|\alpha| \le m$.

Example 3.3. Theorem 3.4 does not hold if for instance Ω is a domain in $\mathbb{R}^2$ and its boundary contains a sharp cuspidal point: Assume

$$\Omega = \{x \in \mathbb{R}^2, 0 < x_1 < 1, |x_2| < x_1^{2p} \exp(-p/x_1)\}, \quad 1 \le p < 2,$$

and $u(x) = \exp(1/x_1)$. We have:

$$\int_\Omega |u|^p \,\mathrm{d}x = 2\int_0^1 \exp(p/x_1)\exp(-p/x_1)x_1^{2p}\,\mathrm{d}x_1 < \infty,$$

$$\int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^p \mathrm{d}x = 2\int_0^1 \frac{1}{x_1^{2p}}\exp(p/x_1)\exp(-p/x_1)x_1^{2p}\,\mathrm{d}x_1 < \infty,$$

on the other hand, if $q > p$,

$$\int_\Omega |u|^q \,\mathrm{d}x = 2\int_0^1 \exp(q/x_1)\exp(-p/x_1)x_1^{2p}\,\mathrm{d}x_1 = \infty,$$

Concerning the imbedding theorems if $\Omega = \mathbb{R}^N$ cf. J.L. Lions [5]; if Ω is unbounded, cf. J. Deny, J.L. Lions [1]; for Ω unbounded and the estimate of type (1.1.4); cf. also the previous paper.

2.3.6 Extension, the Nikolskii Method

Let $u \in W_0^{k,p}(\Omega)$. We define the *extension of u on* $\mathbb{R}^N$: let $u = \lim_{n\to\infty}\varphi_n$, $\varphi_n \in C_0^\infty(\Omega)$. φ_n is a Cauchy sequence in $W^{k,p}(\Omega)$, thus in $W^{k,p}(\mathbb{R}^N)$, the sequence converges to $v \in W^{k,p}(\mathbb{R}^N)$. Obviously, the restriction operator (denote it by R) which maps a function from $W^{k,p}(\mathbb{R}^N)$ onto its restriction on Ω satisfies $Rv = u$. We write $v = Pu$ where P is the extension operator $P \in [W_0^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)]$; P is linear and continuous.

Corollary 3.1. *Let Ω be a bounded domain. Then Theorems 3.4-3.7 hold if we replace $W^{k,p}(\Omega)$ by $W_0^{k,p}(\Omega)$.*

Proof. Indeed, take a cube $C \supset \overline{\Omega}$; for the extension P, we have $P \in [W_0^{k,p}(\Omega) \to W^{k,p}(C)]$. □

If $\Omega \subset \Omega_1$ and $u \in W^{k,p}(\Omega_1)$, we have $u \in W^{k,p}(\Omega)$ and we can formulate a question:

Problem 3.2. For what domains Ω does $R(W^{k,p}(\mathbb{R}^N)) = W^{k,p}(\Omega)$?

A.P. Calderon [2] proved that this equality holds for $\Omega \in \mathfrak{N}^{0,1}$, $p > 1$; we shall prove it in the case $\Omega \in \mathfrak{N}^{k-1,1}, p \geq 1$ and for $\Omega \in \mathfrak{N}^{0,1}$, $p = 2$.

Lemma 3.4. *Let Ω, O be two bounded open sets and T a one-to-one continuous mapping, $T : O \to \Omega$. We assume that T is $(k-1)$-times continuously differentiable in $\overline{O}$, the derivatives of order $k-1$ are lipschitzian in $\overline{O}$; we assume also T^{-1} lipschitzian in $\overline{\Omega}$. Let $u \in W^{k,p}(\Omega), v(y) = u(T(y))$. Then we have:*

$$|v|_{W^{k,p}(O)} \leq c|u|_{W^{k,p}(\Omega)}. \tag{2.46}$$

Proof. Let $u \in W^{k,p}(\Omega)$, we put $u = 0$ outside of Ω, and let u_h be the regularized function. Let $v_{[h]}(y) = u_h(T(y))$. We can compute the derivatives of order $\leq (k-1)$ of $v_{[h]}$ in the usual sense; for $|\alpha| = k-1, D^\alpha v_{[h]}$ is an absolutely continuous function on all lines parallel to the axes y_i. For $|\alpha| \leq k$, the derivatives $D^\alpha v_{[h]}$, taken in the distribution sense, are bounded on $\overline{\Omega}$. If $|\alpha| \leq k$, $D^\alpha v_{[h]}$ is a Cauchy sequence in $L^p(\Omega^*)$ due to Lemma 3.1. We deduce $v \in W^{k,p}(\Omega^*)$; according to formulae as (2.26) for $D^\alpha v, |\alpha| \leq k$, and according to Lemma 3.1 the result follows. □

The following theorem is based on an idea of S.M. Nikolskii, cf. V.M. Babich [1].

Theorem 3.9. *Let $\Omega \in \mathfrak{N}^{k-1,1}$, $1 \leq p < \infty$. The extension operator P exists and maps $W^{k,p}(\Omega)$ linearly and continuously into $W^{k,p}(\mathbb{R}^N)$.*

Proof. Let $u_r = u\varphi_r$, the function u_{m+1} belongs to $W_0^{k,p}(\Omega)$, hence we extend it by zero outside of Ω. Let us consider $u_r, r \leq m$, and omit the index r. Let T be the mapping of the prism

$$K = \{y \in \mathbb{R}^N, y = (y', y_N), |y_i| < \alpha, i = 1, 2, \ldots, N-1, 0 < y_N < \beta\}$$

to V (cf. 1.2.4, Chap. 1), defined by

$$x' = y', \quad x_N = y_N + a(y'). \tag{2.47}$$

All hypotheses in Lemma 3.4 are satisfied, $u \in W^{k,p}(K)$ and the inequality (2.46) holds. We denote:

$$K' = \{y \in \mathbb{R}^N, |y_i| < \alpha, i = 1, 2, \ldots, N-1, |y_N| < \beta\},$$

$$K_+ = \{y \in \mathbb{R}^N, |y_i| < \alpha, i = 1, 2, \ldots, N-1, 0 < y_N < k\beta\},$$

$$K_- = \{y \in \mathbb{R}^N, |y_i| < \alpha, i = 1, 2, \ldots, N-1, -\beta < y_N < 0\}.$$

Let us extend u by zero outside of K on K_+, and put for $y_N < 0$

$$u(y', y_N) = \sum_{i=1}^{k} \lambda_i u(y', -iy_N), \tag{2.48}$$

where the coefficients λ_i are defined by

$$1 = \sum_{i=1}^{k} (-i)^\alpha \lambda_i, \quad \alpha = 0, 1, \ldots, k-1. \tag{2.49}$$

The determinant of this linear system is not equal to zero, thus the λ_i's are uniquely determined. Clearly $u \in W^{k,p}(K_-)$, but we have also $u \in W^{k,p}(K')$. To see this let us consider $D^\alpha u$, $|\alpha| \leq k$. Assume that $u \in C^\infty(K)$; according to Theorem 3.1, if we pass to the limit we get $u \in W^{k,p}(K)$. Let $\varphi \in C_0^\infty(K')$, and let us consider:

$$\int_{K'} D^\alpha \varphi(y) u(y)\,\mathrm{d}y = \int_K D^\alpha \varphi(y) u(y)\,\mathrm{d}y + \sum_{i=1}^{k} \lambda_i \int_{K_-} D^\alpha \varphi(y) u(y', -iy_N)\,\mathrm{d}y. \tag{2.50}$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and let us set $\alpha = \alpha' + \alpha''$, $\alpha'' = (0, \ldots, 0, \alpha_N)$. Now using integration by parts in (2.50) we get:

$$\begin{aligned}
\int_{K'} D^\alpha \varphi(y) u(y)\,\mathrm{d}y &= (-1)^{|\alpha'|} \int_K \frac{\partial^{\alpha_N}}{\partial y_N^{\alpha_N}} D^{\alpha'} u(y)\,\mathrm{d}y \\
&\quad + (-1)^{|\alpha'|} \sum_{i=1}^{k} \lambda_i \int_{K_-} \frac{\partial^{\alpha_N} \varphi(y)}{\partial y_N^{\alpha_N}} D^{\alpha'} u(y', -iy_N)\,\mathrm{d}y \\
&= (-1)^{|\alpha|} \int_K \varphi(y) D^\alpha u(y)\,\mathrm{d}y + (-1)^{|\alpha|} \int_{K_-} \varphi(y) \sum_{i=1}^{k} \lambda_i (-i)^{\alpha'} D^\alpha u(y', -iy_N)\,\mathrm{d}y \\
&\quad + \sum_{j=1}^{\alpha_N} (-1)^j \Big(\sum_{i=1}^{k} \lambda_i (-i)^{j-1} - 1\Big) \int_\Delta \frac{\partial^{\alpha_N - j} \varphi(y', 0)}{\partial y_N^{\alpha_N - j}} \frac{\partial^{j-1}}{\partial y_N^{j-1}} D^{\alpha'} u(y', 0)\,\mathrm{d}y.
\end{aligned} \tag{2.51}$$

According to (2.49), the last term of (2.51) is equal to zero, then

$$|u|_{W^{k,p}(K')} \leq c |u|_{W^{k,p}(K)}. \tag{2.52}$$

The mapping (2.47) and its inverse satisfy the hypotheses of Lemma 3.4 for $\Omega = U$, $O = K'$.

We obtain the extension of u on U such that $u \in W^{k,p}(U)$ and u is equal to zero in a neighborhood of ∂U. If we put $u = 0$ outside of U we have constructed an *extension operator*. Now we go back to the index r, and denote this extension operator by P_r; we have the result if we put $Pu = \sum_{r=1}^{m+1} P_r u_r$. □

2.3.7 Extension, the Calderon Method

We describe the method of A.P. Calderon only for $p = 2$: we shall introduce some simplifications as in the work of J. Nečas [9]. We shall finish with remarks concerning the case $p \neq 2$.

Let $a(x')$ be a lipschitzian function in the cube $\Delta = \{|x_i| < \alpha, i = 1,2,\dots,N-1\}$. Let us denote by $C(y)$ the cone $x_N > y_N$, $|x' - y'| < \kappa(x_N - y_N)$. We can find κ sufficiently small such that the set $\Lambda = \{x \in \mathbb{R}^N, x' \in \Delta, x_N = a(x')\}$ has an empty intersection with $C((x', a(x')))$, $x' \in \Delta$. Let us denote $K_\infty = \cup_{x' \in \Delta} C((x', a(x')))$; we choose $\gamma > 0$ such that $x' \in \overline{\Delta} \Longrightarrow a(x') < \gamma$, and define $K = \{x \in K_\infty, x_N < \gamma\}$; cf. Fig. 2.3.

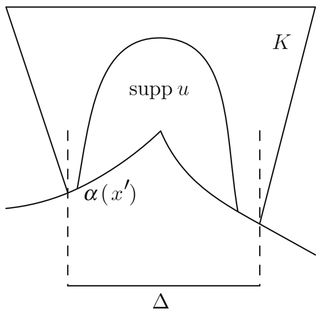

Fig. 2.3

Let $u \in C^\infty(\overline{K})$ be a function with support in $K \cup \Lambda$, cf. Fig. 2.3. We denote by S the unit sphere with center at the origin O and let $\sigma \in S \cap C(0)$. Now for $x \in K$, we have:

$$\int_0^\infty t^{k-1} \frac{d^k}{dt^k}(\exp(-t)u(x+t\sigma))\,\mathrm{d}t = (-1)^k (k-1)!u(x). \tag{2.53}$$

Let us denote $\alpha! = \alpha_1!\alpha_2!\dots\alpha_N!$, $\sigma^\alpha = \sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\dots\sigma_N^{\alpha_N}$; we get:

$$\frac{d^k}{dt^k}(\exp(-t)u(x+t\sigma)) = \exp(-t)(-1)^k \sum_{|\alpha| \le k} D^\alpha u(x+t\sigma)\sigma^\alpha \frac{|\alpha|!}{\alpha!}\binom{k}{|\alpha|}(-1)^{|\alpha|}. \tag{2.54}$$

Now let $\nu(\sigma)$ be a real infinitely differentiable function on S with support in $S \cap C(0)$, $\nu(\sigma) \ge 0$, $\int_S \nu(\sigma)\,\mathrm{d}S = 1$. From (2.53) and (2.54) we get:

$$\begin{aligned} u(x) = &\frac{1}{(k-1)!}\int_S \nu(\sigma) dS \int_0^\infty t^{k-1} \exp(-t)(-1)^k \\ &\times \sum_{|\alpha| \le k} (-1)^{|\alpha|} \frac{|\alpha|!}{\alpha!}\binom{k}{|\alpha|} \sigma^\alpha D^\alpha u(x+t\sigma) dt, \end{aligned}$$

and the substitution $y = x + t\sigma$ leads to

$$u(x) = \frac{1}{(k-1)!} \int_{R^N} \left[\sum_{|\alpha| \le k} \frac{|\alpha|!}{\alpha!} (\binom{k}{|\alpha|}) \frac{(x-y)^\alpha}{|x-y|^{|\alpha|+N-k}} D^\alpha u(y) \right] \times \exp(-|x-y|) \mu \left(\frac{x-y}{|x-y|} \right) \mathrm{d}y, \tag{2.55}$$

where $\mu(x) = \nu(-x)$; formula (2.55) is an adaptation of the Sobolev identity, cf. S.L. Sobolev [1].

If $f \in L^1(\mathbb{R}^N)$, we denote by $\hat{f}(\xi)$ the Fourier transform of f defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} \exp(-\mathrm{i}(x,\xi)) f(x)\,\mathrm{d}x;$$

if $f \in L^2(\mathbb{R}^N), \hat{f}$ is defined by the Plancherel method (cf. S. Bochner, K. Chandrasekharan [1]). We have

Lemma 3.5.

$$u \in W^{k,2}(\mathbb{R}^N) \Longleftrightarrow \int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 (1+|\xi|^2)^k\,\mathrm{d}\xi < \infty,$$

and

$$c_1 |u|_{W^{k,2}(\mathbb{R}^N)} \le \left(\int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 (1+|\xi|^2)^k\,\mathrm{d}\xi \right)^{1/2} \le c_2 |u|_{W^{k,2}(\mathbb{R}^N)}.$$

Indeed, it is sufficient to use the Parseval identity (cf. the book of S. Bochner, K. Chandrasekharan [1]); for $f, g \in L^2(\mathbb{R}^N)$, we have:

$$\int_{\mathbb{R}^N} f\overline{g}\,\mathrm{d}x = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{f}\overline{\hat{g}}\,\mathrm{d}\xi,$$

by the definition of $W^{k,2}(\mathbb{R}^N)$ in 2.2.3 and by the well known properties of the Fourier transform such as

$$\varphi \in C_0^\infty(\mathbb{R}^N) \Longrightarrow \widehat{\left(\frac{\partial \varphi}{\partial x_i}\right)}(\xi) = i\xi_i \hat{\varphi}(\xi), \text{ etc.}$$

Remark 3.4. If $\Omega = \mathbb{R}^N$, then from Lemma 3.5 follows easily a particular case of Theorem 3.8, i.e.: if $2k > N$, then $W^{k,2}(\mathbb{R}^N) \subset C^0(\mathbb{R}^N)$ algebraically and topologically. This follows from

$$\int_{\mathbb{R}^N} |\hat{f}(\xi)|\,\mathrm{d}\xi \le \left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 (1+|\xi|^2)^k\,\mathrm{d}\xi \right)^{1/2} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|\xi|^2)^k} \right)^{1/2},$$

and from the inverse transform:

$$f(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{f}(\xi) \exp(\mathrm{i}(x,\xi))\, \mathrm{d}\xi.$$

With the notations of this section, $K, \dots$, we have:

Lemma 3.6. *Let $M \subset K \cup \Lambda$ be a closed set, and $u \in W^{k,2}(K)$,* $\operatorname{supp} u \subset M$. *Let us denote by $W \subset W^{k,2}(\Omega)$ the set of functions with support in M. Then there exists a mapping $P \in [W \to W^{k,2}(\mathbb{R}^N)]$ such that $RPu = u$.*

Proof. Let $u \in W \cap C^\infty(\overline{K})$. Let us put for $|\alpha| \le k$:

$$f_\alpha(x) = \begin{cases} D^\alpha u(x) & \text{for} \quad x \in K, \\ 0 \quad \text{for} \quad x \notin K. \end{cases}$$

For $x \in \mathbb{R}^N$, we define:

$$\begin{aligned} v(x) = \frac{1}{(k-1)!} \int_{\mathbb{R}^N} &\left[\sum_{|\alpha| \le k} \frac{|\alpha|!}{\alpha!} \binom{k}{|\alpha|} \frac{(x-y)^\alpha}{|x-y|^{|\alpha|+N-k}} f_\alpha(y) \right] \times \\ &\times \exp(-|x-y|)\mu\left(\frac{x-y}{|x-y|}\right) \mathrm{d}y = \sum_{|\alpha| \le k} \int_{\mathbb{R}^N} I_\alpha(x-y) f_\alpha(y)\, \mathrm{d}y. \end{aligned} \tag{2.56}$$

It is clear that $\int_{\mathbb{R}^N} |I_\alpha(x)|\, \mathrm{d}x < \infty$, $|\alpha| \le k$, thus according the to theorem about the Fourier transform of a convolution, we get:

$$\hat{v}(\xi) = \sum_{|\alpha| \le k} \hat{I}_\alpha(\xi) \hat{f}_\alpha(\xi). \tag{2.57}$$

For $|\alpha| \le k$, we have

$$|\hat{I_\alpha}(\xi)| \le C_1 (1 + |\xi|^2)^{-k/2}. \tag{2.58}$$

Indeed, it is sufficient to consider:

$$\begin{aligned} \hat{J}_\alpha(\xi) &= \frac{1}{(k-1)!} \int_{\mathbb{R}^N} \frac{x^\alpha}{|x|^{|\alpha|+N-k}} \exp(-|x|)\mu\left(\frac{x}{|x|}\right) \exp(-\mathrm{i}(x,\xi)\, \mathrm{d}x \\ &= \int_S \frac{\sigma^\alpha \mu(\sigma)}{(1 + \mathrm{i}(\sigma,\xi))^k}\, \mathrm{d}S \end{aligned}$$

Clearly we have:

$$\max_{|\xi| \le 1} |\hat{J}_\alpha(\xi)| \le c_2; \tag{2.59}$$

let $|\xi| \ge 1$. Let us consider the vector $\eta = \xi/|\xi|$. Without loss of generality we can assume the support of μ sufficiently small such that if $\mu(\sigma) \ne 0$, then $(\sigma, \sigma_0) > 1/2$, where $\sigma_0 = (0,0,\dots,-1)$; then we can find $\varepsilon > 0$ such that if $|\eta_N| \le \varepsilon$, $\eta \notin \operatorname{supp}\mu$, and if $|\eta_N| > \varepsilon$, $\sigma \in \operatorname{supp}\mu, |(\sigma,\eta)| \ge c_3$. In the case $|\eta_N| \le \varepsilon$ let us introduce a new system of charts generated by vectors $\sigma^1,\sigma^2,\dots,\sigma^N$, such that $-(\sigma^N, \sigma_0) = (1-\eta_N^2)^{1/2}$, $\sigma^1 = \eta$, where the coordinates are $\tau_1,\tau_2,\dots,\tau_N$. We put $\tau = (\tau',\tau_N)$ and obtain:

$$\hat{f}_\alpha(\xi) = \int_{|\tau'|\le 1} \frac{\lambda(\tau')\,\mathrm{d}\tau'}{(1+\mathrm{i}\tau_1|\xi|)^k},$$

with λ indefinitely differentiable if $|\tau'| < 1$, and with $\operatorname{supp}\lambda$ in $|\tau'| < 1$.

Using integration by parts, we get:

$$\begin{aligned}\int_{|\tau'|\le 1} \frac{\lambda(\tau')\,\mathrm{d}\tau'}{(1+\mathrm{i}\tau_1|\xi|)^k} &= \frac{1}{(k-1)!}\frac{1}{(\mathrm{i}|\xi|)^{k-1}}\int_{|\tau'|<1}\frac{(\partial^{k-1}\lambda/\partial\tau_1^{k-1})\,\mathrm{d}\tau'}{1+\mathrm{i}\tau_1|\xi|}\\ &= \frac{1}{(k-1)!}\frac{1}{(\mathrm{i}|\xi|)^{k-1}}\int_{|\tau'|<1}\frac{(\partial^{k-1}\lambda/\partial\tau_1^{k-1})(1-\mathrm{i}\tau_1|\xi|)\,\mathrm{d}\tau'}{1+\tau_1^2|\xi|^2}.\end{aligned} \tag{2.60}$$

We have:

$$\left|\int_{|\tau'|<1}\frac{(\partial^{k-1}\lambda/\partial\tau^{k-1})\,\mathrm{d}\tau'}{1+\tau_1^2|\xi|^2}\right| \le c_4\int_{-1}^{1}\frac{\mathrm{d}\tau_1}{1+\tau_1^2|\xi|^2} \le \frac{c_5}{|\xi|}. \tag{2.61}$$

To simplify the notation, let us set $\delta = \partial^{k-1}\lambda/\partial\tau_1^{k-1}$; we get:

$$\begin{aligned}\int_{|\tau'|<1}\frac{\delta(\tau')|\xi|\tau_1\,\mathrm{d}\tau'}{1+\tau_1^2|\xi|^2} &= \int_{|\tau'|<1}\frac{\delta(0,\tau_2,\dots,\tau_{N-1})|\xi|\tau_1\,\mathrm{d}\tau'}{1+\tau_1^2|\xi|^2}\,\mathrm{d}\tau'\\ &\quad+\int_{|\tau'|<1}\frac{\delta(\tau')-\delta(0,\tau_2,\dots,\tau_{N-1})}{1+\tau_1^2|\xi|^2}|\xi|\tau_1\,\mathrm{d}\tau'.\end{aligned} \tag{2.62}$$

The first integral in the right hand side is equal zero, and using in the second integral the inequality $|\delta(\tau')-\delta(0,\tau_2,\dots,\tau_{N-1})| \le c_6|\tau_1|$ we get

$$\left|\int_{|\tau'|<1}\frac{\delta(\tau')-\delta(0,\tau_2,\dots,\tau_{N-1})}{1+\tau_1^2|\xi|^2}|\xi|\tau_1 d\tau'\right| \le \frac{c_7}{|\xi|}. \tag{2.63}$$

Inequality (2.58) follows from (2.60) to (2.63).

If $|\eta_N| > \varepsilon$, then $|(\sigma,\eta)| \ge c_3$ for $\sigma \in \operatorname{supp}\mu$, and in this case (2.58) again holds.

Finally, according to (2.57), (2.58) and Lemma 3.5, we get:

$$|v|_{W^{k,2}(\mathbb{R}^N)} \le c_9|u|_{W^{k,2}(K)}, \tag{2.64}$$

and (2.56) defines the extension operator P. □

Now we can prove

Theorem 3.10. *Let $\Omega \in \mathfrak{N}^{0,1}$. Then there exists $P \in [W^{k,2}(\Omega) \to W^{k,2}(\mathbb{R}^N)]$, such that $RPu = u$.*

Proof. We use always φ_r, V_r, etc. as in 1.2.4; for $u\varphi_r$, $r \leq m$, we use Lemma 3.6, with $K_r \subset V_r$, such that supp $\varphi_r \subset K_r \cup \Lambda_r$. Let P_r be as in Lemma 3.6, for $r = m+1$ let us define P_{m+1} by 2.3.6; and if we put: $Pu = \sum_{r=1}^{m+1} P_r u$, the result holds. □

Remark 3.5. Theorem 3.9 is true also in the general case $p > 1$, $p \neq 2$. For the proof we can use again (2.56), and in addition the theory of singular operators due to A.P. Calderon, A. Zygmund [1], cf. J.L. Lions [5], or (2.57) and in addition the multiplier theorems, cf. S.G. Mikhlin [4], B. Malgrange [2].

The Nikolskii and Calderon extensions are different at one point: the Nikolskii extension operator can be the same for $W^{k,p}(\Omega)$, $k = 1, 2, \ldots, \kappa$, the Calderon extension depends on k which can be in some situations inconvenient, for instance if we use interpolation, cf. J.L. Lions [5].

Let us formulate an unsolved problem:

Problem 3.3. Is Theorem 3.10 true for $p = 1$?

The questions concerning the existence of extension operators P are closely related with the whole theory of Sobolev spaces $W^{k,p}(\Omega)$; if P exists for one $W^{k,p}(\Omega)$ we can restrict our consideration solely to the case of $W^{k,p}(\mathbb{R}^N)$ as far as concerns density theorems (cf. Theorem 3.1), imbeddings, compactness of imbedding operators, and their consequences such as trace theorems etc. We can also formulate *converse* problems, such as, e.g.,

Problem 3.4. If Ω is bounded and if Theorem 3.4 holds, does P exist?

In some particular cases the answer is negative, for instance: if $\Omega \in \mathfrak{N}^0$, Theorem 3.1 is true, but in general, P does not exist, according to Example 3.3. If P exists, we can see that Ω must be *almost* in $\mathfrak{N}^{0,1}$, etc.

2.3.8 The Spaces $W^{k,p}(\Omega)$, k Non-integer

It is natural, in particular in problems of traces, cf. Chap. 1, Theorem 1.2 and the two following sections, to introduce the spaces $W^{k,p}(\Omega)$, $p \geq 1$, where k is a real number ≥ 0, in general not integer. The definition of these spaces is due to S. Aronszajn [3], L.N. Slobodetskii [1], see also N. Aronszajn, K.T. Smith [1]:

Let Ω be a domain in $\mathbb{R}^N$, $k \geq 0$, $p \geq 1$. If *k is not integer, the space $W^{k,p}(\Omega)$* is a subspace of $W^{[k],p}(\Omega)$, where $[k]$ is the integer part of k, of functions such that for $|\alpha| = [k]$

$$\int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+p(k-[k])}} \,\mathrm{d}x\,\mathrm{d}y < \infty; \tag{2.65}$$

the norm in $W^{k,p}(\Omega)$ is defined as

$$|u|_{W^{k,p}(\Omega)} = \Big(|u|^p_{W^{[k],p}(\Omega)} + \sum_{|\alpha|=[k]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+p(k-[k])}} \mathrm{d}x\mathrm{d}y\Big)^{1/p}. \quad (2.66)$$

For $k \geq 0$ we define also $W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$.

If $k < 0$, we put $W^{k,p}(\Omega) = (W_0^{-k,q}(\Omega))'$, the dual space of $W_0^{-k,q}(\Omega)$, $1/p + 1/q = 1$.

Clearly we have:

Proposition 3.1. *The space $W^{k,p}(\Omega), k \geq 0$, integer, is a Banach space, separable, and reflexive for $p > 1$.*

Proof. $W^{k,p}(\Omega)$ is obviously complete. We can consider this space as a closed subspace of the product $[L^p(\Omega)]^s \times [L^p(\Omega \times \Omega)]^t$ {where s is as in Proposition 2.2 and t is the number of indices $\alpha, |\alpha| = k$} of elements of the following form:

$$\left(u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^{[k]} u}{\partial x_N^{[k]}}, \frac{\frac{\partial^{[k]} u(x)}{\partial x_1^{[k]}} - \frac{\partial^{[k]} u(y)}{\partial x_1^{[k]}}}{|x-y|^{N/p+(k-[k])}}, \dots, \frac{\frac{\partial^{[k]} u(x)}{\partial x_N^{[k]}} - \frac{\partial^{[k]} u(y)}{\partial x_N^{[k]}}}{|x-y|^{N/p+(k-[k])}} \right).$$

This space is separable, and reflexive if $p > 1$, thus the same is true for $W^{k,p}(\Omega)$. □

Actually, the theory of the spaces $W^{k,p}(\Omega)$ can be extended to other spaces whose definition is closely related to $W^{k,p}(\Omega)$; we have the Nikolskii spaces $H^{k,p}$, the Besov spaces $B^{k,p}$ (cf. O.V. Besov [1, 2]), the Lizorkin spaces $L^{k,p}$, the weighted spaces $W_\alpha^{k,p}$, the Morrey spaces $W^{k,p,\lambda}$, etc. For details and a survey cf. S.M. Nikolskii [2], E. Magenes [4], the following section and Chap. 7; concerning the spaces $W^{k,p,\lambda}$ cf. S. Campanato [6, 7].

2.4 The Problem of Traces

2.4.1 Lemmas

We have solved this problem partly in Theorem 1.1.2. Here in this section,we consider the problem in more detail. In fact, most of these theorems are due to S.L. Sobolev [1]; here we shall follow the ideas of E. Gagliardo [1]. We obtain slightly stronger results than S.L. Sobolev. It is important to observe that we are interested in traces on $(N-1)-$dimensional manifolds; for smaller dimension cf. the monograph of S.L. Sobolev [1]; if $p = 2$, it is possible to consult K. Maurin [1].

Let $\Omega \in \mathfrak{N}^{0,1}$ and f a function defined almost everywhere on $\partial\Omega$ which means that $f(x'_r, a_r(x'_r))$ is defined almost everywhere in Δ_r, $r = 1, 2, \dots, m$, cf. 1.2.4. If $f(x'_r, a_r(x'_r)) \in L^p(\Delta_r)$, $r = 1, 2, \dots, m$, $p \geq 1$, we say that $f \in L^p(\partial\Omega)$ with the following norm:

$$|f|_{L^p(\partial\Omega)} = \left[\sum_{r=1}^{m} \int_{\Delta_r} |f(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r \right]^{1/p}. \tag{2.67}$$[4]

Theorem 4.1. *The space $L^p(\partial\Omega)$ is a separable Banach space.*

Proof. We must prove that $L^p(\partial\Omega)$ is complete. Indeed, let f_i be a Cauchy sequence. We have $\lim_{i\to\infty} f_i(x'_r, a_r(x'_r)) = f_{[r]}(x'_r, a_r(x'_r))$; let $U_r \cap U_s \cap \partial\Omega = M \neq \emptyset$. Let P_r (resp. P_s) be the projection of M onto the plane $x_{rN} = 0$ (resp. $x_{sN} = 0$), and T the mapping $T : \overline{P}_s \to \overline{P}_r$ defined for $x'_s \in P_s$ by $T(x'_s) = x'_r$ (e.g. x'_r and x'_s are projections of the same point on m to Δ_r and Δ_s, resp). If it is necessary we can rotate the coordinate system (x'_s, x_{sN}) (resp. (x'_r, x_{rN})) to get:

$$x_{ri} = x_{si} + \lambda_i, \quad i = 1, 2, \ldots, N-2,$$
$$x_{rN-1} = x_{sN-1}\cos\varphi - x_{sN}\sin\varphi + \lambda_{N-1}, x_{rN} = x_{sN-1}\sin\varphi + x_{sN}\cos\varphi + \lambda_N, \tag{2.68}$$

where φ is the angle between the axes x_{sN}, x_{rN}. The mapping $T : \overline{P_s} \to \overline{P_r}$ is one-to-one and lipschitzian as well as T^{-1}; indeed:

$$\begin{aligned} |x'_r - y'_r|^2 &= \sum_{i=1}^{N-2} (x_{ri} - y_{ri})^2 + (x_{rN-1} - y_{rN-1})^2 \\ &= \sum_{i=1}^{N-2} (x_{si} - y_{si})^2 + [(x_{sN-1} - y_{sN-1})\cos\varphi - (a_s(x'_s) - a_s(y'_s))\sin\varphi]^2 \\ &\le 2|x'_s - y'_s|^2 + |(a_s(x'_s) - a_s(y'_s)|^2 \le (2 + c_1)|x'_s - y'_s|^2. \end{aligned}$$

We prove by the same approach that T^{-1} is lipschitzian. Without loss of generality we can assume that $f_i(x'_s, a_s(x'_s))$ (resp. $f_i(x'_r, a_r(x'_r))$) converges almost everywhere in Δ_s (resp. in Δ_r), as $i \to \infty$; we get:

$$f_{[s]}(x'_s, a_s(x'_s)) = f_{[r]}(x'_r, a_r(x'_r))$$

almost everywhere on P_s (resp. P_r), where $x'_r = T(x'_s)$. □

We will use also a fundamental lemma:

Lemma 4.1. *Let f be a function defined almost everywhere on $\partial\Omega$, different from zero at most only for $x'_r \in \Delta_r$, r fixed. Let*

$$\int_{\Delta_r} |f(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r < \infty.$$

Then $f \in L^p(\partial\Omega)$, and

[4] In Chap. 3 we shall consider surface integrals in more detail. We shall define $(\int_{\partial\Omega} |f|^p dS)^{1/p}$ precisely by a norm equivalent to (2.67): we shall deduce that $L^p(\partial\Omega)$ does not depend on the systems of local coordinates (x'_r, x_{rN}).

$$|f|_{L^p(\partial\Omega)} \le c\left(\int_{\Delta_r} |f(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r\right)^{1/p}.$$

Proof. We proceed as in the proof of Theorem 1.1; we use $T : P_s \to P_r$, and Lemma 3.1. □

2.4.2 Imbedding Theorems

Theorem 4.2. *Let* $\Omega \in \mathfrak{N}^{0,1}$, $1 \le p < N$, $1/q = 1/p - [1/(N-1)](p-1)/p$. *Then there exists exactly one mapping* $Z \in [W^{1,p}(\Omega \to L^q(\partial\Omega))]$ *such that* $u \in C^\infty(\overline{\Omega}) \Longrightarrow Zu = u$.

Proof. It suffices to consider $u \in C^\infty(\overline{\Omega})$, use a partition of unity as in 1.2.4, and investigate the functions $u_r, r = 1, 2, \ldots, m$. To simplify the notation we omit the index r. Let us put

$$v(x', a(x')) = |u(x', a(x'))|^{(Np-p)/(N-p)}.$$

We have:

$$v(x', a(x')) = -\int_{a(x')}^{a(x')+\beta} \frac{\partial v}{\partial x_N}(x', \eta) \, \mathrm{d}\eta,$$

then

$$|u(x', a(x'))|^{(Np-p)/(N-p)} \le \frac{Np-p}{N-p} \int_{a(x')}^{a(x')+\beta} |u(x', \eta)|^{(Np-p)/(N-p)} \left| \frac{\partial u}{\partial x_N}(x', \eta) \right| \mathrm{d}\eta,$$

and for $p > 1$

$$\int_\Delta |u(x', a(x'))|^{(Np-p)/(N-p)} \, \mathrm{d}x' \le \frac{Np-p}{N-p}\left(\int_V |u|^{Np/(N-p)} \, \mathrm{d}x\right)^{(p-1)/p} \left(\left|\int_V \frac{\partial u}{\partial x_N}\right|\right)^{1/p}; \qquad (2.69)$$

this inequality also holds for $p = 1$. According to Theorem 3.4, we obtain for $u \in C^\infty(\overline{\Omega})$

$$|u|_{L^q(\partial\Omega)} \le c|u|_{W^{1,p}(\Omega)} \qquad (2.70)$$

and the result follows from Theorem 3.1. □

Using the terminology introduced in Chap. 1, we call Zu *the trace of* u*;* to simplify we shall write u instead of Zu.

Exercise 4.1. Use Theorem 3.8 and prove, with the hypotheses of Theorem 4.2, that $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega) \Longrightarrow Zu = u$.

Theorem 4.3. *Let $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{1,p}(\Omega)$, $1 \le p < N$. Then, after a modification on a set of measure zero, u is absolutely continuous for almost all $x'_r \in \Delta_r$ on the interval $a_r(x'_r) \le x_{rN} \le a_r(x'_r) + \beta$, and $u(x'_r, a_r(x'_r)) = (Zu)(x'_r, a_r(x'_r))$ almost everywhere in Δ_r.*

Proof. Using Theorem 3.9, we extend u from V_r to U_r; then we use Theorem 2.2. As in the previous proof, after a modification of u on a set of measure zero, we obtain the inequality (2.69). □

Remark 4.1. If Ω is a smooth domain (cf. 1.2.3) or if $\partial\Omega$ is sufficiently smooth, in Theorem 4.3 we can use G_r instead of U_r (cf. 1.2.4). Then we obtain that for almost all interior normals, the function $u(\sigma_r, t)$ is absolutely continuous for $0 \le t \le \delta$, and modulo a set of measure zero, its limit value coincides with the trace of u.

Theorem 4.4. *Let us consider $u \in W^{1,p}(V)$, $1 \le p < N, a(x')$ a lipschitzian function in $\overline{\Delta}$, cf. 1.2.4. Then if $1/q = 1/p - [1/(N-1)](p-1)/p$, we have in the sense of traces*

$$\lim_{\varepsilon \to 0} \int_\Delta |u(x', a(x') + \varepsilon) - u(x', a(x'))|^q \, \mathrm{d}x' = 0, \tag{2.71}$$

$$\int_\Delta |u(x', a(x') + \varepsilon) - u(x', a(x'))|^p \, \mathrm{d}x' \le c\varepsilon^{p-1} |u|^p_{W^{1,p}(V_\varepsilon)}, \tag{2.72}$$

where

$$V_\varepsilon = \{x \in \mathbb{R}^N, x = (x', x_N), x' \in \Delta,\ a(x') < x_N < a(x') + \varepsilon\},$$

Proof. Let $V' = \{x \in \mathbb{R}^N, x = (x', x_N), x' \in \Delta, a(x') < x_N < a(x') + \beta/2\}$, and $u(x', x_N + \varepsilon) - u(x', x_N)$ in V', $\varepsilon < \beta/2$. Using inequality (2.69), we get (2.71) according to Theorem 1.1.

To obtain (2.72) we restrict ourselves only to the case $u \in C^\infty(\overline{V})$; we have:

$$u(x', a(x') + \varepsilon) - u(x', a(x')) = \int_{a(x')}^{a(x')+\varepsilon} \frac{\partial u}{\partial x_N}(x', \xi) \, \mathrm{d}\xi,$$

then

$$|u(x', a(x') + \varepsilon) - u(x', a(x'))|^p \le \varepsilon^{p-1} \int_{a(x')}^{a(x')+\varepsilon} \left| \frac{\partial u}{\partial x_N}(x', \xi) \right|^p \mathrm{d}\xi,$$

and (2.72) follows by integration with respect to x'. □

Let $\Omega_n, \Omega \in \mathfrak{N}^k$. We shall say that $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^k$, if $\Omega_n \subset \Omega$ and if $\partial\Omega_n$ and $\partial\Omega$ are described by the same system of charts; let a_{rn} be the corresponding functions defined on the charts. Let us assume that

$$\lim_{n\to\infty} |a_{rn} - a_r|_{C^{k,1}(\overline{\Delta_r})} = 0.$$

Let Ω_n, $\Omega \in \mathfrak{N}^{k,1}$. We shall say that $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^{k,1}$ if $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^k$, if

$$\lim_{n\to\infty} |a_{rn} - a_r|_{W^{k+1,2}(\Delta_r)} = 0$$

and if $|a_{rn}|_{C^{k,1}(\Delta r)} \le$ const.

If $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^{0,1}$, and if $g_n \in L^q(\partial\Omega_n)$, $g \in L^q(\partial\Omega)$, we shall say that $\lim_{n\to\infty} g_n = g$ in $L^q(\partial\Omega)$, if

$$\lim_{n\to\infty} \sum_{r=1}^{m} \int_{\Delta_r} |g_n(x'_r, a_{rn}(x'_r)) - g(x'_r, a_r(x'_r))|^q \, \mathrm{d}x'_r = 0.$$

We have:

Theorem 4.5. *Let Ω_n, $\Omega \in \mathfrak{N}^{0,1}$, $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^{0,1}$.*[5] *Let $u \in W^{1,p}(\Omega)$, $1 \le p < N$. Then if $Zu = g$ on $\partial\Omega$ (resp. $Zu = g_n$ on $\partial\Omega_n$), we have $\lim_{n\to\infty} g_n = g$ in L^{q_1}, $1/q_1 > 1/q = 1/p - [1/(N-1)](p-1)/p$.*

Proof. We consider again V_r as above, and prove as in (2.72):

$$\lim_{n\to\infty} \int_{\Delta_r} |u(x'_r, a_r(x'_r)) - u(x', a_{rn}(x'_r))|^p \, \mathrm{d}x'_r = 0.$$

If $p = 1$, everything is proved. Let $p > 1$; in this case according to (2.69), we get:

$$\int_{\Delta_r} |u(x', a_{rn}(x'_r))|^q \, \mathrm{d}x'_r \le c_1;$$

then

$$\begin{aligned}
&\int_{\Delta_r} |u(x'_r, a(x'_r)) - u(x', a_{rn}(x'_r))|^{q_1} \, \mathrm{d}x'_r \\
&\le \int_{\Delta_r} |u - u_n|^{[q/(q_1-q)]/(q-1)} |u - u_n|^{(q-q_1)/(q-1)} \, \mathrm{d}x'_r \qquad (2.72\,\mathrm{bis}) \\
&\le \left(\int_{\Delta_r} |u - u_n|^q \, \mathrm{d}x'_r \right)^{(q_1-1)/(q-1)} \left(\int_{\Delta_r} |u - u_n| \, \mathrm{d}x'_r \right)^{(q-q_1)/(q-1)},
\end{aligned}$$

where $u_n = u(x'_r, a_{rn}(x'_r))$. □

Obviously we have:

Theorem 4.6. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p = N$. Then Theorems 4.2, 4.4 hold for arbitrary $q \ge 1$.*

Theorem 4.7. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \ge 1$, $kp < N$. Let $1/q = 1/p - [1/(N-1)] \times (kp - 1)/p$. Then the mapping Z from Theorem 4.2 satifies $Z \in [W^{k,p}(\Omega) \to L^q(\partial\Omega)]$.*

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[5] In this theorem it is sufficient that $\lim_{n\to\infty} \Omega_n = \Omega$ in the following sense:

$$\lim_{n\to\infty} |a_{rn} - a_r|_{C^0(\overline{\Delta_r})} = 0, \ |a_{rn}|_{C^{0,1}(\overline{\Delta_r})} \le \text{const.}$$

Indeed, if $u \in W^{k,p}(\Omega)$, then $(\partial u/\partial x_i) \in W^{k-1,p}(\Omega)$, thus we can apply Theorem 3.6: $u \in W^{1,q^*}(\Omega)$ with $1/q^* = 1/p - (k-1)/N$. Then we use Theorem 4.2. □

It is immediately clear that we have:

Theorem 4.8. *Let $\Omega \in \mathfrak{N}^{0,1}$, $kp = N$. Then the mapping Z defined in Theorem 4.2 satisfies: $Z \in [W^{k,p}(\Omega) \to L^q(\partial\Omega)]$ for every $q \geq 1$.*

2.4.3 Two Trace Theorems

Let $p < N$. In Sect. 2.5 we shall see that for $\Omega \in \mathfrak{N}^{0,1}$, the space $L^q(\partial\Omega)$ with

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{N-1}\frac{p-1}{p}$$

is not a trace space but a larger space: the mapping $Z : W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is not surjective.

Nevertheless we have:

Theorem 4.9. *Let $\Omega \in \mathfrak{N}^{0,1}, p \geq 1$. Then $\overline{Z(W^{1,p}(\Omega))} = L^p(\partial\Omega)$.*

Proof. Let $f \in L^p(\partial\Omega)$ and set, for $x \in \partial\Omega, f_r(x) = f(x)\varphi_r(x)$. We have that

$$f_r \in L^p(\partial\Omega), \quad \sum_{r=1}^{m} f_r = f.$$

It is sufficient to prove our theorem for f_r, $r = 1, 2, \ldots, m$. For simplicity we omit the index r. Fix $\varepsilon > 0$. The function $f(x', a(x'))$ belongs to $L^p(\Delta)$, thus there exists $\varphi \in C_0^\infty(\Delta)$ such that $|f - \varphi|_{L^p(\Delta)} < \varepsilon$; let us set $v(x', x_N) = \varphi(x')$, and let $\psi \in C_0^\infty(U)$ be such that $x \in \partial\Omega \Longrightarrow v(x)\psi(x) = v(x)$. Then $Z(v\psi) = \varphi$. □

In Chap. 1 we introduced the subspace:

$$V = \{v \in W^{k,2}(\Omega), v = \frac{\partial v}{\partial n} = \frac{\partial^2 v}{\partial n^2} \ldots = \frac{\partial^{k-1} v}{\partial n^{k-1}} = 0 \text{ on } \partial\Omega\}.$$

Let us recall that $W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$. We have:

Theorem 4.10. *Let $\Omega \in \mathfrak{N}^{0,1}, p \geq 1$. Then $W_0^{1,p}(\Omega) = W \equiv \{v \in W^{1,p}(\Omega),$ $v = 0$ on $\partial\Omega\}$.*

Proof. As $u = 0$ on $\partial\Omega$, we have $u_r = u\varphi_r = 0$ on $\partial\Omega, r = 1, 2, \ldots, m$. For u_{m+1}, it is clear that $u_{m+1} \in W_0^{1,p}(\Omega) : \lim_{h\to 0} u_{m+1,h} = u_{m+1}$ in $W^{1,p}(\Omega)$ and $u_{m+1,h} \in C_0^\infty(\Omega)$. Let now $r \leq m$; for simplicity we omit the index r. We have $u \in W^{1,p}(V)$, and set $u = 0$ in U outside of V. Then $u \in W^{1;p}(U)$: Indeed, using the transformation (2.47), denoting

$$K = \{y \in \mathbb{R}^N, |y|_i < \alpha, i = 1,2,\dots,N-1, |y_N| < \beta\},$$
$$K_+ = \{y \in \mathbb{R}^N, |y|_i < \alpha, i = 1,2,\dots,N-1, 0 < y_N < \beta\},$$
$$K_- = \{y \in \mathbb{R}^N, |y|_i < \alpha, i = 1,2,\dots,N-1, -\beta < y_N < 0\},$$

and setting $v(y) = u(T(y))$ we have that $v \in W^{1,p}(K_+)$ and according to Lemma 3.2, we get:

$$v(y',0) = 0. \tag{2.73}$$

If $\psi \in C_0^\infty(K)$, we get:

$$\int_{K_+} \frac{\partial \psi}{\partial y_i} v \, \mathrm{d}y = -\int_{K_+} \psi \frac{\partial v}{\partial y_i} \, \mathrm{d}y, \quad i = 1,2,\dots,N-1, \tag{2.74}$$

$$\int_{K_+} \frac{\partial \psi}{\partial y_N} v \, \mathrm{d}y = -\int_{\Delta} \psi(y',0) v(y',0) \, \mathrm{d}y' - \int_{K_+} \psi \frac{\partial v}{\partial y_N} \, \mathrm{d}y. \tag{2.75}$$

It follows from (2.73)–(2.75) that $v \in W^{1,p}(K) \Longrightarrow u \in W^{1,p}(U)$ by the transformation (2.46) extended to K. We denote $u_\lambda(x',x_N) = u(x',x_N - \lambda)$. For $\lambda > 0$ sufficiently small, $u_\lambda \in W^{1,p}(V)$, with $\operatorname{supp} u_\lambda \subset V$. Then we have $u_\lambda \in W_0^{1,p}(V)$, and since $\lim_{\lambda \to 0} u_\lambda = u$, we get: $W \subset W_0^{1,p}(\Omega)$. Obviously $W_0^{1,p}(\Omega) \subset W$, and the result follows. □

Let $\Omega \in \mathfrak{N}^{k-1,1}$. Denote by $W^{k,p}(\partial\Omega)$, $p \geq 1$, $k \geq 1$ integer, the subspace of $L^p(\partial\Omega)$ of functions for which $f(x'_r, a_r(x'_r)) = f_r \in W^{k,p}(\Delta_r)$, $r = 1,2,\dots,m$. On $W^{k,p}(\partial\Omega)$ we introduce the norm:

$$|f|_{W^{k,p}(\partial\Omega)} = \left(\sum_{r=1}^m |f_r|^p_{W^{k,p}(\Delta_r)} \right)^{1/p},$$

where $f_r(x'_r) = f(x'_r, a_r(x'_r))$. $W^{k,p}(\partial\Omega)$ is a separable Banach space.

2.4.4 Some Other Properties of Traces

Lemma 4.2. *Let $\Omega \in \mathfrak{N}^{0,1}$. The exterior (or interior) normal exists almost everywhere on $\partial\Omega$.*

Proof. It is sufficient to prove that the function a_r, $r = 1,2,\dots,$ has almost everywhere in Δ_r a total differential. We again omit the index r. Let M be a countable set on the unit sphere $|x'| = 1$ dense on this sphere; we assume that M contains the points of intersection of all axes of coordinates with the sphere. Let $m \in M$. The function $a(x')$ has almost everywhere in Δ a derivative $\partial a/\partial m$; as M is countable, there exists $B \subset \Delta$, $\operatorname{meas} B = \operatorname{meas} \Delta$, such that the function $a(x')$ has for $x' \in B, m \in M$ a derivative $(\partial a/\partial m)$; we denote this derivative $a_m(x')$. According to Theorem 2.2, $a_m(x')$ is the distributional derivative and hence, if $m \in M, x \in B$,

$$a_m(x') = \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x') m_i. \tag{2.76}$$

This equality holds for all $x \in B$ and all directions. Indeed: Let n be a normed vector and $m_{[j]} \in M$ with $\lim_{j\to\infty} m_{[j]} = n$ in $\mathbb{R}^{N-1}$. Fix $\varepsilon > 0$. We have:

$$\left| \frac{a(x'+tn) - a(x')}{t} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x') n_i \right| < \varepsilon$$

for t sufficiently small; indeed, according to $a \in C^{0,1}(\overline{\Delta})$, there exists a sequence $m_{[j]}$, and a number $\delta > 0$ such that for $|t| < \delta$

$$\left| \frac{a(x'+tm_{[j]}) - a(x'+tn)}{t} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x') m_{[j],i} + \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x') n_i \right| < \varepsilon/2.$$

Let $m_{[j]}$ be fixed; using (2.76) for $m_{[j]} \in M$, the result follows. If moreover $x \in B$, then

$$\lim_{t\to 0} \left(\sup_{|n|=1} \left| \frac{a(x'+tn) - a(x')}{t} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x') n_i \right| \right) = 0. \tag{2.77}$$

If (2.77) does not hold, there would exist an $\varepsilon > 0$, a sequence $t_k, \lim_{k\to\infty} t_k = 0$, and normed vectors $n_{[k]}$ with $\lim_{k\to\infty} n_{[k]} = n$ such that

$$\left| \frac{a(x'+t_k n_{[k]}) - a(x')}{t_k} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x') n_{[k]_i} \right| \geq \varepsilon;$$

this would imply that for k sufficient large

$$\left| \frac{a(x'+t_k n) - a(x')}{t_k} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x') n_i \right| \geq \varepsilon/2,$$

and this inequality is a contradiction to (2.76). But (2.77) implies the existence of the total differential at x. □

Let $\Omega \in \mathfrak{N}^{0,1}$. If $u \in W^{k,p}(\Omega)$ then $D^\alpha u \in L^p(\partial\Omega)$, $|\alpha| \leq k-1$, and we define the exterior normal derivative of order $l \leq k-1$ by:

$$\frac{\partial^l u}{\partial n^l} = \sum_{|\alpha|=l} \frac{l!}{\alpha!} D^\alpha u n^\alpha, \tag{2.78}$$

where n is the exterior normal, $n = (n_1, n_2, \ldots, n_N)$ and $n^\alpha = n_1^{\alpha_1} n_2^{\alpha_2} \ldots n_N^{\alpha_N}$.

Theorem 4.11. *Let $\Omega \in \mathfrak{N}^{0,1}$; if $1 \le p < N$, put $1/q = 1/p - [1/(N-1)](p-1)/p$; if $p = N$, put $q \ge 1$. There exists a unique mapping $Z \in [W^{2,p}(\Omega) \to W^{1,q}(\partial\Omega)]$ such that $u \in C^\infty(\overline{\Omega}) \Longrightarrow Zu = u$.*

Proof. Fix $u \in C^\infty(\overline{\Omega})$. Then we have $u(x', a_r(x'_r)) \in W^{1,q}(\Delta_r)$, and according to Theorem 2.2, if we denote $v(x'_r) = u(x', a_r(x'_r))$, we have:

$$\frac{\partial v}{\partial x_{ri}} = \frac{\partial u}{\partial x_{ri}} + \frac{\partial u}{\partial x_{rN}} \frac{\partial a_r}{\partial x_{ri}}, \quad i = 1,2,\ldots,N-1. \tag{2.79}$$

We have $\partial u/\partial x_{ri} \in W^{1,p}(V_r), i = 1,2,\ldots,N$.

Using Theorems 4.2, 4.5 and (2.79) we get:

$$|u|_{W^{1,q}(\partial\Omega)} \le c|u|_{W^{2,p}(\Omega)}.$$

□

Remark 4.2. It is easy to see that for $u \in W^{2,p}(\Omega)$, (2.79) holds for the derivatives $\frac{\partial u}{\partial x_{ri}}, i = 1,2,\ldots,N$, considered in the sense of traces.

Remark 4.3. If $\Omega \in \mathfrak{N}^{k-2,1}$, $k \ge 2$, we have with the same notations as in Theorem 4.11 that $Z \in [W^{k,p}(\Omega) \to W^{k-1,q}(\partial\Omega)]$.

Theorem 4.12. *If $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{2,p}(\Omega)$, $p \ge 1$, $u = \partial u/\partial n = 0$ on $\partial\Omega$, then $u \in W_0^{2,p}(\Omega)$.*

Proof. For $r = 1,2,\ldots,m$, we have according to (2.79)

$$0 = \sum_{i=1}^{N-1} \frac{\partial u}{\partial x_{ri}} \frac{\partial a_r}{\partial x_{ri}} - \frac{\partial u}{\partial x_{rN}}, \tag{2.80}$$ [6]

$$0 = \frac{\partial u}{\partial x_{rN}} + \frac{\partial u}{\partial x_{rN}} \frac{\partial a_r}{\partial x_{ri}}, \quad 1,2,\ldots,N-1. \tag{2.81}$$

Starting from (2.80), (2.81) we can compute the derivatives $\partial u/\partial x_{ri}$, $i = 1,2,\ldots,N$, and we get a homogeneous linear system with nonzero determinant, thus $\partial u/\partial x_{ri} = 0$ on $\partial\Omega$. As in Theorem 4.10, the function u, extended by zero outside of Ω, is in $W^{2,p}(\mathbb{R}^N)$; the remaining part of the proof is the same as in Theorem 4.10. □

Theorem 4.13. *If $\Omega \in \mathfrak{N}^{k,1}$, $u \in W^{k,p}(\Omega)$ with $u = \partial u/\partial n = \ldots = \partial^{k-1}u/\partial n^{k-1} = 0$ on $\partial\Omega$, then $u \in W_0^{k,p}(\Omega)$.*

Proof. For $k \le 2$ this theorem was proved for $\Omega \in \mathfrak{N}^{0,1}$; hence we can assume $k \ge 3$. It is sufficient to prove that u extended by zero outside of Ω is in $W^{k,p}(\mathbb{R}^N)$.

[6] Clearly $n = \left(1 + \sum_{i=1}^{N-1} (\partial a_r/\partial x_{ri})^2\right)^{-1/2} \left(\partial a_r/\partial x_{r1}, \partial a_r/\partial x_{r2} \ldots, \partial a_r/\partial x_{r(N-1)}, -1\right)$.

To do this we use the domains $G_1, G_2, \dots, G_M$, ψ_i, $i = 1,2,\dots,M$, given in 2.2.4, and the mapping (1.35) from Chap. 1; this mapping is smooth, one-to-one from $\overline{\Delta} \times (-\delta, \delta)$ to $\overline{G_r}$, the inverse has the same properties, they are generated by functions in $C^{k-1,1}(\overline{\Delta})$. Using Lemma 3.4 and $u_r = u\psi_r$ in G_r, after the mapping we get $u_r \in W^{k,p}(C_+)$ with $C_+ = \Delta \times (0, \delta)$. For $t = 0$ we have:

$$u_r = \frac{\partial u_r}{\partial t} = \dots = \frac{\partial^{k-1} u_r}{\partial t^{k-1}} = 0,$$

then $D^\alpha u_r = 0$ for $t = 0, |\alpha| \le k-1$. The function u extended by zero if $\sigma \in \Delta$, $t < 0$, is in $W^{k,p}(C)$ with $C = \Delta \times (-\delta, \delta)$. By the mapping inverse to (1.35), we get $u_r \in W^{k,p}(G_r)$, $r = 1,2,\dots,M$. Obviously $u_{M+1} \in W_0^{k,p}(\Omega)$, so

$$u = \sum_{r=1}^{M+1} u_r \in W^{k,p}(\mathbb{R}^N).$$

□

The notion of smoothness almost everywhere for $\partial\Omega$ was introduced in 1.2.3.

Theorem 4.14. *Let $u \in W^{k,p}(\Omega)$, $\partial\Omega$ almost everywhere smooth, $u = \partial u/\partial n = \dots = \partial^{k-1}/\partial n^{k-1} = 0$ on $\partial\Omega$ in the sense of traces. Then $u \in W_0^{k,p}(\Omega)$.*

Proof. It is sufficient to prove that $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| \le k-1$. Let $y \in \partial\Omega$ be a regular point. We use the set U from 1.2.4 which corresponds to that point, and proceed as the proof of the previous theorem. □

Problem 4.1. Is Theorem 4.14 true if $\Omega \in \mathfrak{N}^{0,1}$?

Remark 4.4. Let $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{2,p}(\Omega)$. We can prove that the subspace of $W^{1,p}(\partial\Omega) \times L^p(\partial\Omega)$, generated by $(u, \partial u/\partial n), u \in W^{2,p}(\Omega)$, is dense in this space. Cf. Chap. 5.

Exercise 4.2. Let $\Omega \in \mathfrak{N}^{1,1}$, $u \in W^{3,p}(\Omega)$, $u = \partial u/\partial n = \partial^2 u/\partial n^2 = 0$ on $\partial\Omega$ in the sense of traces. Prove that $u \in W_0^{3,p}(\Omega)$.

2.5 The Problem of Traces (Continuation)

2.5.1 Application of the Fourier Transform

In the previous section we pointed out that the spaces $L^p(\partial\Omega)$ in Theorem 4.2 are larger than the trace spaces. For instance, if $\Omega \in \mathfrak{N}^{0,1}$, the natural topology of the space of traces of $W^{1,p}(\Omega)$ is the topology of the quotient space $W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$. On the other hand, this approach is rather formal and does not give a characterization of the trace space.

If $\Omega = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N,\ x_N > 0\}$, $p = 2$, $k \geq 1$, we can easily characterize the trace space using the Fourier transform, cf. for instance J.L. Lions [4]. To do this we introduce for $\Omega = \mathbb{R}^N$ the notion of $W^{k,2}(\mathbb{R}^N)$ with k an arbitrary real number; later we shall see that the new definition coincides for $k \geq 0$ with the definition given in 2.3.8.

If $k \geq 0$, $W^{k,2}(\mathbb{R}^N)$ *is defined as the subset of* $L^2(\mathbb{R}^N)$ of functions $f(x)$ such that their Fourier transforms satisfy:

$$\left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 (1+|\xi|^2)^k \,\mathrm{d}\xi\right)^{1/2} < \infty. \tag{2.82}$$

The left hand side of (2.82) is a *norm*, moreover $W^{k,2}(\mathbb{R}^N)$ is an *Hilbert* space with the scalar product:

$$[f,g] = \int_{\mathbb{R}^N} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1+|\xi|^2)^k \,\mathrm{d}\xi. \tag{2.83}$$

If $k < 0$, we define $W^{k,2}(\mathbb{R}^N) = (W^{-k,2}(\mathbb{R}^N))'$. It is possible to define $W^{k,2}(\mathbb{R}^N)$ directly for all real k in the setting of tempered distributions and their Fourier transforms, cf. L. Schwartz [2].

Let us denote as usual $x = (x', x_N), \xi = (\xi', \xi_N)$.

Theorem 5.1. *Let k be an integer,* $k = 1, 2 \ldots$ *Then*

$$u \in W^{k,2}(\mathbb{R}^N_+) \Longrightarrow u(x',0) \in W^{k-1/2,2}(\mathbb{R}^{N-1}),$$

and if $g(x') = u(x',0)$*, we have:*

$$|g|_{W^{k-1/2,2}(\mathbb{R}^{N-1})} \leq c|u|_{W^{k,2}(\mathbb{R}^N_+)} \tag{2.84}$$

Proof. By immediate application of (1.1.10) we obtain $g \in L^2(\mathbb{R}^{N-1})$; we extend u to $\mathbb{R}^N$ using (2.48) and get obviously $|u|_{W^{k,2}(\mathbb{R}^N)} \leq c_1 |u|_{W^{k,2}(\mathbb{R}^N_+)}$. Then according to Lemma 3.5 it follows:

$$\left(\int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 (1+|\xi|^2)^k \,\mathrm{d}\xi\right)^{1/2} \leq c_2 |u|_{W^{k,2}(\mathbb{R}^N)}. \tag{2.85}$$

Due to Proposition 2.5, it is sufficient to prove (2.84) for $u \in C_0^\infty(\mathbb{R}^N)$; we have:

$$\hat{g}(\xi') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi)\,\mathrm{d}\xi_N,$$

and then

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} |\hat{g}(\xi)|^2 (1+|\xi'|^2)^{k-1/2}\,\mathrm{d}\xi' \\
&\quad = \frac{1}{4\pi^2}\int_{\mathbb{R}^{N-1}} \left|\int_{-\infty}^{\infty} \hat{u}(\xi)\,\mathrm{d}\xi_N\right|^2 (1+|\xi'|^2)^{k-1/2}\,\mathrm{d}\xi' \\
&\quad \le \frac{1}{4\pi^2}\int_{\mathbb{R}^{N-1}} \left[(1+|\xi'|^2)^{k-1/2}\int_{-\infty}^{\infty} |\hat{u}(\xi)|^2(1+|\xi|^2)^k\,\mathrm{d}\xi_N \times \right. \\
&\qquad\qquad \left. \times \int_{-\infty}^{\infty} (1+|\xi'|^2)^{-k}\,\mathrm{d}\xi_N\right]\mathrm{d}\xi'.
\end{aligned} \tag{2.86}$$

We have

$$\int_{-\infty}^{\infty} (1+|\xi'|^2)^{-k}\,\mathrm{d}\xi_N = \pi(1+|\xi'|^2)^{-k+1/2},$$

and the result follows from (2.86). □

Remark 5.1. Clearly we have: if $u \in W^{k,2}(\mathbb{R}^N_+)$ then we have for $l = 1,2,\dots,k-1$ that $(\partial^l u/\partial x_N^l)(x',0) \in W^{k-l-1/2,2} x'_N(\mathbb{R}^{N-1})$, with

$$\sum_{l=0}^{k-1} \left|\frac{\partial^i u}{\partial x_N^i}\right|_{W^{k-l-1/2,2}(\mathbb{R}^{N-1})} \le c|u|_{W^{k,2}(\mathbb{R}^N_+)}.$$

In what follows, $B_1, B_2, \dots, B_k$ will be Banach spaces; we denote $B_1 \times B_2 \times \cdots \times B_k$ the Cartesian product of B_i, i.e. the set of elements $u = (u_1, u_2, \dots, u_k)$, where $u_i \in B_i$; and we equip $\prod_{i=1}^k B_i$ with the norm $\sum_{i=1}^k |u_i|_{B_i}$ or some equivalent norm.

We have a "converse" of Remark 5.1:

Theorem 5.2. *There exists a mapping:*

$$Z \in \left[\prod_{l=0}^{k-1} W^{k-l-1/2,2}(\mathbb{R}^{N-1}) \to W^{k,2}(\mathbb{R}^N_+)\right]$$

such that if

$$g = (g_1, g_2, \dots, g_{k-1}) \in \prod_{l=0}^{k-1} W^{k-l-1/2,2}(\mathbb{R}^{N-1}),$$

then for $u = Zg$

$$\frac{\partial^l u}{\partial x_N^l}(x',0) = g_l(x').$$

Proof. We define:

$$Z_l \in [W^{k-l-1/2,2}(\mathbb{R}^{N-1}) \to W^{k,2}(\mathbb{R}^N_+)], \quad l = 1,2,\dots,k-1,$$

taking

$$Z_l g_l = u_l,$$

where

$$\hat{u}_l(\xi', x_N) = x_N^l \exp(-(1+|\xi'|)x_N)\hat{g}_l(\xi'). \tag{2.87}$$

Z_l is of the type mentioned; indeed, let $|\alpha| \le k$, let us consider $w_\alpha = D^\alpha u_l$ in $\mathbb{R}^N_+$ and set $w_\alpha = 0$ for $x_N < 0$. Let us denote $\alpha' = (\alpha_1, \dots, \alpha_{N-1}, 0)$, $\alpha'' = (0, \dots, 0, \alpha_N)$. Hence $\hat{w_\alpha}(\xi)$ is a finite sum of expressions like:

$$aI(\xi) = a\int_0^\infty \exp(-\mathrm{i}x_N\xi_N)(\xi')^{\alpha'}(1+|\xi'|)x_N^{l-j} \times \\ \times \exp(-(1+|\xi'|)x_N)\hat{g}_l(\xi')\,\mathrm{d}x_N,$$

where a is a constant, $j = 0, 1, \dots, \min(\alpha_N, l)$. We have:

$$I(\xi) = \frac{(l-j)!(\xi')^{\alpha'}(1+|\xi'|)^{\alpha_N-1}\hat{g}_l(\xi')}{(1+|\xi'|+\mathrm{i}\xi_N)^{l-j+1}},$$

and so

$$\begin{aligned}
&\int_{\mathbb{R}^N} |I(\xi)|^2\,\mathrm{d}\xi \\
&\le c_1 \int_{\mathbb{R}^{N-1}} \Bigg[|\xi|^{2|\alpha'|}(1+|\xi'|^{2\alpha_N-2j})|\hat{g}_l(\xi')|^2 \times \\
&\qquad \times \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi_N}{((1+|\xi'|)^2+\xi_N^2)^{l-j+1}} \Bigg] \mathrm{d}\xi' \\
&= c_2 \int_{\mathbb{R}^{N-1}} |\xi|^{2|\alpha'|}(1+|\xi'|)^{2\alpha_N-2l-1}|\hat{g}_l(\xi')|^2\,\mathrm{d}\xi' \\
&\le c_3 \int_{\mathbb{R}^{N-1}} (1+|\xi'|)^{2\alpha-2l-1}|\hat{g}_l(\xi')|^2\,\mathrm{d}\xi' \\
&\le c_4 \int_{\mathbb{R}^{N-1}} (1+|\xi'|)^{k-l-1/2}|\hat{g}_l(\xi')|^2\,\mathrm{d}\xi'.
\end{aligned}$$

Now we have $\partial^j u_l/\partial x_N^j(x',0) = 0$ for $j < l$. We construct Z as a linear combination of Z_l by setting

$$\begin{aligned}
Zg &= Z_0 g_0 + Z_1\left(g_1 - \frac{\partial}{\partial x_N} Z_0 g_0\right) \\
&\quad + \frac{1}{2!} Z_2 \left(g_2 - \frac{\partial^2}{\partial x_N^2} Z_0 g_0 - \frac{\partial^2}{\partial x_N^2} Z_1\left(g_1 - \frac{\partial}{\partial x_N} Z_0 g_0\right)\right) + \dots
\end{aligned} \tag{2.88}$$

□

2.5.2 Lemmas Based on the Hardy Inequality

The method used in 2.5.1 does not work if $p \neq 2$; here we generalize the Gagliardo approach, E. Gagliardo [1], assuming $p > 1$. Moreover we assume Ω bounded.

Let $k \geq 0$, $\Omega \in \mathfrak{N}^{[k]'-1,1}$, where $[k]'$ is the smallest integer such that $k \leq [k]'$. We define $W^{k,p}(\partial\Omega)$ as the subset of functions from $W^{[k],p}(\partial\Omega)$ such that $f_r(x'_r) = f(x'_r, a_r(x'_r)) \in W^{k,p}(\Delta_r), r = 1,2,\dots,m$; the norm is defined by:

$$|f|_{W^{k,p}(\partial\Omega)} = \left(\sum_{r=1}^{m} |f_r|^p_{W^{k,p}(\Delta_r)}\right)^{1/p}. \tag{2.89}$$

Remark 5.2. The space $W^{k,p}(\partial\Omega)$ is a separable Banach space and reflexive if $p > 1$; this is an immediate consequence of Proposition 3.1.

In this section we consider only the case $p > 1$. If $p = 1$ we have to use another approach, cf. E. Gagliardo [1].

Now we shall prove a *Hardy inequality*, cf. G.H. Hardy, J.L. Littlewood, G. Pólya [1].

Lemma 5.1. *Let be $f \in L^p(a,b)$, $-\infty < a < b < \infty$, $p > 1$. The following inequalities hold:*

$$\int_a^b \left[\frac{1}{x-a}\int_a^x |f(\xi)|\,\mathrm{d}\xi\right]^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_a^b |f(x)|^p\,\mathrm{d}x, \tag{2.90}$$

$$\int_a^b \left[\frac{1}{b-x}\int_x^b |f(\xi)|\,\mathrm{d}\xi\right]^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_a^b |f(x)|^p\,\mathrm{d}x. \tag{2.91}$$

Proof. Let us define:

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{for} \quad x \geq a+\varepsilon, \\ 0 & \text{for} \quad a < x < a+\varepsilon. \end{cases}$$

We have

$$\begin{aligned} &\int_a^b \frac{1}{(x-a)^p}\left[\int_a^x |f_\varepsilon(\xi)\,\mathrm{d}\xi|\right]^p \mathrm{d}x \\ &\qquad = \frac{1}{(b-a)^{p-1}}\frac{1}{p-1}\left[\int_a^b |f_\varepsilon(\xi)|\,\mathrm{d}\xi\right]^p \\ &\qquad\quad + \frac{p}{p-1}\int_a^b \frac{1}{(x-a)^{p-1}}\left(\int_a^x |f_\varepsilon(\xi)|\,\mathrm{d}\xi\right)^{p-1} |f_\varepsilon(x)|\,\mathrm{d}x \end{aligned}$$

$$\leq \frac{p}{p-1}\int_a^b \frac{1}{(x-a)^{p-1}}\left(\int_a^x |f_\varepsilon(\xi)|\,\mathrm{d}\xi\right)^{p-1} |f_\varepsilon(x)|\,\mathrm{d}x$$

$$\leq \frac{p}{p-1}\left(\int_a^b \frac{1}{(x-a)^{p}}\left(\int_a^x |f_\varepsilon(\xi)|\,\mathrm{d}\xi\right)^{p}\mathrm{d}x\right)^{(p-1)/p}\left(\int_a^b |f_\varepsilon(x)|^p\,\mathrm{d}x\right)^{1/p}.$$

This implies inequality (2.90) for the function f_ε, and consequently

$$\int_a^b\left[\frac{1}{x-a}\int_a^x |f_\varepsilon(\xi)|\,\mathrm{d}\xi\right]^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p\int_a^b |f_\varepsilon(x)|^p\,\mathrm{d}x$$
$$\leq \left(\frac{p}{p-1}\right)^p\int_a^b |f(x)|\,\mathrm{d}x.$$

Now, Fatou's lemma gives (2.90) for f. Inequality (2.91) can be proved in the same way. □

Lemma 5.2. *Let* $\Delta = \{x\in\mathbb{R}^2, 0<x_1<1, 0<x_2<x_1\}$, $u\in W^{1,p}(\Delta)$, $p>1$. *Then we have the inequality*

$$\int_0^1\int_0^1\left|\frac{u(t,t)-u(\tau,\tau)}{t-\tau}\right|^p dt\,\mathrm{d}\tau < c|u|^p_{W^{1,p}(\Delta)}. \tag{2.92}$$

Proof. According to Theorem 3.1, it is sufficient to consider $u\in C^\infty(\overline{\Delta})$. Let $0\leq \tau<t\leq 1$. If we denote $f(t)=u(t,t)$, then

$$\left|\frac{f(t)-f(\tau)}{t-\tau}\right| \leq \frac{1}{t-\tau}\int_\tau^t\left|\frac{\partial u}{\partial x_1}(x_1,\tau)\right|\mathrm{d}x_1 + \frac{1}{t-\tau}\int_\tau^t\left|\frac{\partial u}{\partial x_2}(t,x_2)\right|\mathrm{d}x_2,$$

and

$$\left|\frac{f(t)-f(\tau)}{t-\tau}\right|^p \leq 2^{p-1}\Bigg[\frac{1}{(t-\tau)^p}\left(\int_\tau^t\left|\frac{\partial u(x_1,\tau)}{\partial x_1}(x_1,\tau)\right|\mathrm{d}x_1\right)^p + \frac{1}{(t-\tau)^p}\left(\int_\tau^t\left|\frac{\partial u(t,x_2)}{\partial x_2}(t,\tau)\right|\mathrm{d}x_2\right)^p\Bigg]. \tag{2.93}$$

By integration with respect to t, $\tau<t<1$, and then to τ, $0<\tau<1$, we get according to Lemma 5.1

$$\int_0^1\mathrm{d}\tau\int_\tau^1\left|\frac{f(t)-f(\tau)}{t-\tau}\right|^p\mathrm{d}t$$
$$\leq 2^{p-1}\left(\frac{p}{p-1}\right)^p\Bigg[\int_0^1\mathrm{d}\tau\int_\tau^1\left|\frac{\partial u(x_1,\tau)}{\partial x_1}(x_1,\tau)\right|^p\mathrm{d}x_1 \tag{2.94}$$
$$+\int_0^1\mathrm{d}t\int_0^t\left|\frac{\partial u(t,x_2)}{\partial x_2}(t,\tau)\right|^p\mathrm{d}x_2\Bigg] \leq 2^{p-1}\left(\frac{p}{p-1}\right)^p|u|^p_{W^{1,p}(\Delta)}.$$

□

Lemma 5.3. *Let C be the cube $(-1,1)^{N-1}, p>1, u \in L^p(C)$; moreover let us assume:*

$$c_i^p = \underbrace{\int_{-1}^{1} \dots \int_{-1}^{1}}_{(N-2)times} \mathrm{d}x_1 \dots \mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1} \dots \mathrm{d}x_{N-1} \times \\ \times \left(\int_{-1}^{1} \int_{-1}^{1} \frac{|A_i(t)-A_i(\tau)|^p}{|t-\tau|^p}\,\mathrm{d}t\,\mathrm{d}\tau \right) < \infty, \tag{2.95}$$

where $A_i(t) = u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1})$.

In this case we have the inequality:

$$|u|_{W^{1-1/p,p}(C)} \le c[|u|_{L^p(C)} + \sum_{i=1}^{N-1} c_i^p]^{1/p}. \tag{2.96}$$

Proof. For $x, y \in C$. Denote $x_{[i]} = (y_1, \dots, y_i, x_{i+1}, \dots, x_{N-1})$, $i = 0, 1, \dots, N-1$. We have:

$$\int_C \int_C \frac{|u(x)-u(y)|^p}{|x-y|^{N-2+p}}\,\mathrm{d}x\,\mathrm{d}y \le c \sum_{i=1}^{N-1} \int_C \int_C \frac{|u(x_{[i]})-u(x_{[i-1]})|^p}{|x-y|^{N-2+p}}\,\mathrm{d}x\,\mathrm{d}y.$$

For instance, let us consider:

$$\begin{aligned} &\int_C \int_C \frac{|u(x_{[1]})-u(y)|^p}{|x-y|^{N-2+p}}\,\mathrm{d}x_1 \dots \mathrm{d}x_{N-1}\,\mathrm{d}y_1 \dots \mathrm{d}y_{N-1} \\ &\quad = \int_C \int_{-1}^{1} |u(x_{[1]})-u(x)|^p\,\mathrm{d}x\,\mathrm{d}y_1 \underbrace{\int_{-1}^{1} \dots \int_{-1}^{1}}_{(N-2)\text{times}} \frac{\mathrm{d}y_2 \dots \mathrm{d}y_{N-1}}{|x-y|^{N-2+p}} \\ &\quad \le c \int_C \int_{-1}^{1} \frac{|u(x_{[1]})-u(x)|^p}{|x_{[1]}-x|^p}\,\mathrm{d}x\,\mathrm{d}y_1 = c c_1^p. \end{aligned}$$

□

Exercise 5.1. For the cube C in the previous lemma, prove the converse of (2.96).

Let

$$[|u|_{L^p(C)} + \sum_{i=1}^{N-1} c_i^p]^{1/p}. \tag{2.96 bis}$$

Notice that (2.96 bis) defines an equivalent norm on $W^{k,p}(C)$ (this is a consequence of the previous exercise).

Exercise 5.2. Replace C by $\mathbb{R}^N$ and prove the equivalence of (2.96 bis) and (2.90).

2.5.3 Imbedding Theorems, Application of the Spaces $W^{1-1/p,p}(\partial\Omega)$

Theorem 5.3. *Let C be the cube $(-1,1)^N$, $p>1$. Let C_i be the faces $x_i=1$, $|x_j|<1$, $j\neq i$, and $u\in W^{1,p}(C)$. Then we have the inequality:*

$$|u|_{W^{(1-1/p),p}(C_i)} \le c|u|_{W^{1,p}(C)}, \quad i=1,2,\dots,N.$$

Proof. According to Theorem 3.1, it is sufficient to take $u\in C^\infty(\overline{C})$. Let i be fixed, $1\le i\le N$, $j\neq i$ (for instance $j<i$.) By Lemma 5.2, we have:

$$\underbrace{\int_{-1}^{1}\cdots\int_{-1}^{1}}_{(N-2)\text{times}} \mathrm{d}x_1\dots\mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1}\dots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\dots\mathrm{d}x_N\times$$
$$\times\left(\int_{-1}^{1}\int_{-1}^{1}\frac{\left|B_{ji}(t)-B_{ji}(\tau)\right|^p}{|t-\tau|^p}\,\mathrm{d}t\,\mathrm{d}\tau\right)$$
$$\le c\underbrace{\int_{-1}^{1}\cdots\int_{-1}^{1}}_{(N-2)\text{times}} \mathrm{d}x_1\dots\mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1}\dots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\dots\mathrm{d}x_N\times$$
$$\times\int_{-1}^{1}\int_{-1}^{1}\left(\left|\frac{\partial u}{\partial x_j}\right|^p+\left|\frac{\partial u}{\partial x_i}\right|^p\right)\mathrm{d}x_j\,\mathrm{d}x_i,$$

where $B_{ji}(t)=u(x_1,\dots x_{j-1},t,x_{j+1},\dots,x_{i-1},1,x_{i+1}\dots,x_N)$ and the result follows from Lemma 5.3. □

Theorem 5.4. *Let $u\in W^{k,p}(C)$; under the same hypotheses as that of Theorem 5.3, we have for $l\le k-1$*

$$\left|\frac{\partial^l u}{\partial x_i^l}\right|_{W^{k-l-1/p,p}(C_i)} \le c|u|_{W^{k,p}(C)} \quad i=1,2,\dots,N.$$

Indeed, $D^\alpha u\in W^{1,p}(C)$ for $|\alpha|\le k-1$, and the result follows from the previous theorem. □

2.5.4 Imbedding Theorems, Application of the Spaces $W^{1-1/p,p}(\partial\Omega)$ (Continuation)

Lemma 5.4. *Let $\Omega\in\mathfrak{N}^{0,1}, p\ge 1, k\ge 0$, $0\le\lambda<k$. Then $W^{k,p}(\Omega)\subset W^{\lambda,p}(\Omega)$ algebraically and topologically.*

Proof. The only nontrivial case is the case of λ non integer, k integer; the other cases are either trivial or consequences of the case considered. It is sufficient to investigate the case $0 < \lambda < 1, k = 1$. We set $u_r = u\varphi_r$, use the transformation (2.31), and then everything reduces to $\Omega = C = (-1,1)^N$. According to Theorem 3.1, we can assume $v \in C^\infty(\overline{C})$, and prove the inequality:

$$\int_C \int_C \frac{|v(x) - v(y)|^p}{|x-y|^{N+p\lambda}} \,\mathrm{d}x\,\mathrm{d}y \leq c_1 |v|^p_{W^{1,p}(C)}. \tag{2.97}$$

We use the Hölder inequality and write:

$$\begin{aligned}
\int_C \int_C \frac{|u(x) - u(y)|^p}{|x-y|^{N+p\lambda}} \,\mathrm{d}x\,\mathrm{d}y &= \int_C \int_C \frac{\left|\int_0^1 (d/dt) v(y + t(x-y))\,\mathrm{d}t\right|^p}{|x-y|^{N+p\lambda}} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq c_2 \int_C \int_C \frac{\left|\int_0^1 \sum_{i=1}^N (\partial v/\partial x_i)(y + t(x-y))\,\mathrm{d}t\right|^p}{|x-y|^{N+p\lambda-p}} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq c_3 \sum_{i=1}^N \int_C \int_C \int_0^1 \frac{|(\partial v/\partial x_i)(y + t(x-y))|^p}{|x-y|^{N+p\lambda-p}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}$$

We transform the set of points $(x,y,t) \in C \times C \times (0,1)$ to G by:

$$\tau = t, \quad \eta = y, \quad \xi = y + t(x-y).$$

We have for $i = 1, 2, \ldots, N$:

$$\begin{aligned}
\int_C \int_C \int_0^1 \frac{|(\partial v/\partial x_i)(y + t(x-y))|^p}{|x-y|^{N+p\lambda-p}} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &= \int_G \frac{|(\partial v)/\partial x_i(\xi)|^p \tau^{p(\lambda-1)}}{|\xi - \eta|^{N+p\lambda-p}} \,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}t \\
= \int_0^1 \tau^{p(\lambda-1)} \,\mathrm{d}\tau \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \mathrm{d}\xi \int_{G_{\tau,\xi}} \frac{\mathrm{d}\eta}{|\xi - \eta|^{N+p\lambda-p}}.
\end{aligned} \tag{2.98}$$

Since $|\xi(1-\tau)^{-1} - \eta| \leq c_4 \tau (1-\tau)^{-1}$, we have $|\xi - \tau| \leq c_5 \tau (1-\tau)^{-1}$, and

$$\begin{aligned}
&\int_0^1 \tau^{p(\lambda-1)} \,\mathrm{d}\tau \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \mathrm{d}\xi \int_{G_{\tau,\xi}} \frac{\mathrm{d}\eta}{|\xi - \eta|^{N+p\lambda-p}} \\
&\quad \leq c_6 \int_0^{1/2} \tau^{p(\lambda-1)} \,\mathrm{d}\tau \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \mathrm{d}\xi \int_{G_{\tau,\xi}} \frac{\mathrm{d}\eta}{|\xi - \eta|^{N+p\lambda-p}} \\
&\quad + c_6 \int_{1/2}^1 \tau^{p(\lambda-1)} \,\mathrm{d}\tau \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \int_{G_{\tau,\xi}} \frac{\mathrm{d}\eta}{|\xi - \eta|^{N+p\lambda-p}} \,\mathrm{d}\xi \leq c_7 \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \mathrm{d}\xi,
\end{aligned}$$

and finally (2.97) follows from (2.98). □

Lemma 5.5. *Let* $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{k,p}(\Omega)$, $p \geq 1$, $k \geq 0$; $h \in C^{[k]'-1,1}(\overline{\Omega})$. *Then* $hu \in W^{k,p}(\Omega)$, *and*

$$|uh|_{W^{k,p}(\Omega)} \leq c|u|_{W^{k,p}(\Omega)}|h|_{C^{[k]'-1,1}(\overline{\Omega})}.$$

Proof. If k is an integer the result is trivial. Let k be a non-integer; we have for $|\alpha| < [k]$:

$$D^\alpha(uh) = \sum_{|\beta| \leq |\alpha|} h_\beta D^\beta u,\ h_\beta \in C^{0,1}(\overline{\Omega}).$$

If $|\beta| \leq [k]$, then

$$\begin{aligned}\int_\Omega \int_\Omega & \frac{|h_\beta(x)D^\beta u(x) - h_\beta(y)D^\beta u(y)|^p}{|x-y|^{N+p(k-[k])}}\,\mathrm{d}x\,\mathrm{d}y \\ &\leq 2^{p-1}\int_\Omega \int_\Omega \frac{|h_\beta(x)(D^\beta u(x) - D^\beta u(y))|^p}{|x-y|^{N+p(k-[k])}}\,\mathrm{d}x\,\mathrm{d}y \\ &+ 2^{p-1}\int_\Omega \int_\Omega \frac{|D^\beta u(y)|^p|h_\beta(x) - h_\beta(y))|^p}{|x-y|^{N+p(k-[k])}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

The first integral on the right hand side is less than or equal to:

$$|u|^p_{W^{k,p}(\Omega)}|h|^p_{C^{[k]'-1,1}(\overline{\Omega})},$$

and the second integral is less than or equal to:

$$c_1\int_\Omega |D^\beta u(y)|^p\,\mathrm{d}y \int_\Omega \frac{\mathrm{d}x}{|x-y|^{N-1+p([k-[k]])}} \leq c_2 \int_\Omega |D^\beta u(y)|^p\,\mathrm{d}y.$$

□

Theorem 5.5. *Let* $\Omega \in \mathfrak{N}^{k-1,1}$, $u \in W^{k,p}(\Omega)$, $p > 1$, *k an integer. If* $l \leq k-1$, *then the following inequality holds:*

$$\left|\frac{\partial^l u}{\partial n^l}\right|_{W^{k-l-1/p,p}(\partial\Omega)} \leq c|u|_{W^{k,p}(\Omega)}.$$

Proof. It is sufficient to prove that $D^\alpha u \in W^{k-l-1/p,p}(\partial\Omega)$ and the corresponding inequality for $|\alpha| = l$; we have $D^\alpha u \in W^{k-l,p}(\Omega)$. Let us put $v = D^\alpha u, v_r(x'_r) = v(x'_r, a_r(x'_r))$. Differentiating v_r with respect to coordinates x'_r $|\beta|$-times, $|\beta| \leq k-l-1$, we obtain that $v \in W^{k-l-1,p}(\Delta_r)$; then, by the previous lemma, everything goes back to knowing whether $D^\beta v \in W^{1-1/p,p}(\Delta_r)$. But $D^\beta v$ is a linear combination of terms $aD^\gamma u, |\gamma| \leq |\beta|$, $a \in C^{0,1}(\overline{\Delta}_r)$, hence using once more the previous lemma it is sufficient to see that $D^\delta u \in W^{1-1/p,p}(\partial\Omega), |\delta| \leq k-1$, with the corresponding estimate.

Setting $w = D^\delta u$, we have $w \in W^{1,p}(\Omega)$. Using the transformation (2.31) and the inverse transformation which are at least lipschitzian, the result follows from Theorem 5.4. □

2.5.5 A Lemma

We have to prove the "converse" of Theorem 5.5. We do this with more restrictive conditions concerning the domains.

A function $R(z), z \in \mathbb{R}^N$ will be called a *regularizing kernel* if $R \in C^\infty(\mathbb{R}^N)$, with support contained in the closed unit ball with center at the origin. We shall write $R \in \mathfrak{R}_N$. If $\int_{\mathbb{R}^N} R(z)dz = 1$, we define the *regularizing operator* by:

$$u_h(x) = \int_{|z|<1} R(z)u(hz+x)\,\mathrm{d}z = \frac{1}{h^N}\int_{|y-x|<h} R\left(\frac{y-x}{h}\right)u(y)\,\mathrm{d}y.$$

Lemma 5.6. *Let P be the pyramid defined by: $P = \{x \in \mathbb{R}^N,\ 0 < x_N < 1,\ |x_i| < 1 - x_N,\ i = 1,2,\ldots,N-1\}$, $\varphi_0 \in W^{k-1/p,p}(C)$, $p > 1$, where $C = \{x' \in \mathbb{R}^{N-1}, |x_i| < 1\}$. There exists a mapping $Z \in [W^{k-1/p,p}(C) \to W^{k,p}(P)]$ such that if $Z\varphi_0 = u$, then $u = \varphi_0$ on the basis C of P.*

Proof. Let $R \in \mathfrak{R}_{N-1}$, $\int_{\mathbb{R}^{N-1}} R(z')\,\mathrm{d}z' = 1$, let us set $x = (x', x_N)$, and if $x \in P$:

$$u(x', x_N) = \frac{1}{x_N^{N-1}}\int_{|y'-x'|<x_N} R\left(\frac{y'-x'}{x_N}\right)\varphi_0(y')\,\mathrm{d}y'.$$

Clearly we have $u \in C^\infty(P)$. Let$|\alpha| \le k-1$ and consider $D^\alpha u$; we have:

$$D^\alpha u(x) = \sum_{\substack{|\beta|=|\alpha| \\ |\lambda|=(\alpha_N)}} c_{\beta\lambda}\int_{|z'|<1} R(z'){z'}^\lambda D^\beta \varphi_0(x_N z' + x)\,\mathrm{d}z'.$$

Now, if we proceed as in the proof of Theorem 1.2, we get the inequality:

$$\int_{|x'|<1-x_N} |D^\alpha u(x', x_N)|^p\,\mathrm{d}x' \le c\sum_{|\beta|=|\alpha|}\int_{|x'|<1} |D^\beta\varphi_0(x')|^p\,\mathrm{d}x',$$

which implies

$$|u|_{W^{k-1,p}(P)} \le c|\varphi_0|_{W^{k-1/p,p}(C)}. \tag{2.98 bis}$$

It remains to prove: if $R \in \mathfrak{R}_{N-1}$ and $f \in W^{k-1/p,p}(C)$, then for

$$v(x) = \int_{|z'|<1} R(z')f(x_N z' + x')\,\mathrm{d}z'$$

we have:

$$|v|_{W^{1,p}(P)} \le c|f|_{W^{1-1/p,p}(C)}.$$

First we consider $\partial v/\partial x_i, 1 \le i \le N-1$; without loss of generality let us consider $\partial v/\partial x_1$; we have:

$$\frac{\partial v}{\partial x_1} = -\frac{1}{x_N^N}\int_{|y'-x'|<x_N} \frac{\partial R}{\partial z_1}\left(\frac{y'-x'}{x_N}\right) f(y')\,\mathrm{d}y'.$$

We have:

$$\frac{1}{x_N^N}\int_{|y'-x'|<x_N} \frac{\partial R}{\partial z_1}\left(\frac{y'-x'}{x_N}\right)\mathrm{d}y' = 0,$$

so

$$\begin{aligned}\frac{\partial v}{\partial x_1}(x) &= \frac{1}{x_N^N}\int_{|y'-x'|<x_N} \frac{\partial R}{\partial z_1}\left(\frac{y'-x'}{x_N}\right)(f(x')-f(y'))\,\mathrm{d}y' \\ &= \int_{|z'|<1} \frac{\partial R}{\partial z_1}(z')\frac{f(x')-f(x_N z'+x')}{x_N}\,\mathrm{d}z'.\end{aligned}$$

From this, we get:

$$\begin{aligned}&\int_P \left|\frac{\partial v}{\partial x_1}(x)\right|^p \mathrm{d}x \le c_1\int_P \mathrm{d}x\int_{|z'|<1}\left|\frac{f(x')-f(x_N z'+x')}{x_N}\right|^p dz' \\ &= c_1\int_P x_N^{(-p-N+1)}\,\mathrm{d}x\int_{|y'-x'|<x_N}\frac{|f(x')-f(y')|^p}{|x'-y'|^{N-2+p}}|x'-y'|^{N-2+p}\,\mathrm{d}y' \\ &= c_1\int_{(\max\limits_{1\le i\le N-1}|x_i|)<1}\mathrm{d}x'\int_{|y'-x'|<1-(\max\limits_{1\le i\le N-1}|x_i|)}\frac{|f(x')-f(y')|^p}{|x'-y'|^{N-2+p}}\,\mathrm{d}y'\times \\ &\quad\times\int_{|x'-y'|<1}^{|y'-x'|<1-(\max\limits_{1\le i\le N-1}|x_i|)}|x'-y'|^{N-2+p}x_N^{-p-N+1}\,\mathrm{d}x_N \\ &\le \frac{2c_1}{N+p-2}\int_{(\max\limits_{1\le i\le N-1}|x_i|)<1}\mathrm{d}x'\int_{|y'-x'|<1-(\max\limits_{1\le i\le N-1}|x_i|)}\frac{|f(x')-f(y')|^p}{|x'-y'|^{N-2+p}}\,\mathrm{d}y',\end{aligned}$$

and then

$$\left|\frac{\partial v}{\partial x_1}\right|_{L^p(P)} \le c|f|_{W^{1-1/p,p}(C)}.$$

Now, let us consider

$$\begin{aligned}\frac{\partial v}{\partial x_N} = &-\frac{N-1}{x_N^N}\int_{|y'-x'|<x_N} R\left(\frac{y'-x'}{x_N}\right) f(y')\,\mathrm{d}y' \\ &-\frac{1}{x_N^N}\int_{|y'-x'|<x_N}\sum_{i=1}^{N-1}\frac{\partial R}{\partial z_i}\left(\frac{y'-x'}{x_N}\right)\frac{y_i-x_i}{x_N}f(y')\,\mathrm{d}y'.\end{aligned}$$

We have:

$$\frac{N-1}{x_N^N}\int_{|y'-x'|<x_N} R\left(\frac{y'-x'}{x_N}\right)\mathrm{d}y' + \frac{1}{x_N^N}\int_{|y'-x'|<x_N}\sum_{i=1}^{N-1}\frac{\partial R}{\partial z_i}\left(\frac{y'-x'}{x_N}\right)\frac{y_i-x_i}{x_N}\,\mathrm{d}y' = 0;$$

finally as above we get the inequality:

$$\left|\frac{\partial v}{\partial x_N}\right|_{L^p(P)} \le c|f|_{W^{1-1/p,p}(C)}.$$

Using Theorem 1.2, we see that $u=\varphi_0$ on the basis of P. □

2.5.6 *The Converse Theorem*

Lemma 5.7. *Let* $R\in\mathfrak{R}_{N-1}$*,* $\int_{\mathbb{R}^{N-1}} R(z')\,\mathrm{d}z' = 1$*,* $l\ge 0$ *an integer. We have:*

$$\frac{\partial^i}{\partial x_N^l}\left[x_N^{l-N+1}R\Big(\frac{y'-x'}{x_N}\Big)\right] = \frac{1}{x_N^{N-1}}H\Big(\frac{y'-x'}{x_N}\Big),$$

where $H\in\mathfrak{R}_{N-1}, \int_{\mathbb{R}^{N-1}} H(z')\,\mathrm{d}z' = l!$

Proof. We have to prove that $\int_{\mathbb{R}^{N-1}} H(z')\,\mathrm{d}z' = l!$. We have:

$$\begin{aligned}\int_{|x'-y'|<x_N}&\frac{\partial^i}{\partial x_N^l}\left[x^{l-N+1}R\Big(\frac{y'-x'}{x_N}\Big)\right]\mathrm{d}y\\ &= \frac{\partial^l}{\partial x_N^l}\int_{|x'-y'|<x_N} x_N^{l-N+1}R\Big(\frac{y'-x'}{x_N}\Big)\,\mathrm{d}y' = \frac{\partial^l}{\partial x_N^l}(x_N^l) = l!.\end{aligned}$$

□

Lemma 5.8. *Let P be the pyramid as in Lemma 5.6, and C its basis,* $\varphi_i\in W^{k-l-1/p,p}(C)$*,* l,k *integers such that* $0\le l\le k-1$*,* $p>1$*. Then there exists a mapping*

$$Z_l\in[W^{k-l-1/p,p}(C)\to W^{k,p}(P)]$$

such that $Z_l\varphi_l = u_l$ *satisfies:*

$$u_l = \frac{\partial u_l}{\partial x_N} = \ldots\frac{\partial^{l-1}u_l}{\partial x_N^{l-1}} = 0 \ \ on\ C, \tag{2.99a}$$

$$\frac{\partial^l u_l}{\partial x_N^l} = \varphi_l \ \ on\ C. \tag{2.99b}$$

Proof. Let $R \in \mathscr{R}_{N-1}, \int_{\mathbb{R}^{N-1}} R(z')\,\mathrm{d}z' = 1/l!$, and let us set:

$$u_l(x', x_N) = x_N^{l-N+1} \int_{|x'-y'|<x_N} R(\frac{y'-x'}{x_N})\varphi_l(y')\,\mathrm{d}y'.$$

Obviously, $u_l \in C^\infty(P)$. Let $|\alpha| \le k-l-1$; we get:

$$u_l(x', x_N) = x_N^l \int_{|z'|<1} R(z')\varphi_l(x_N z' + x')\,\mathrm{d}z',$$

thus

$$\left|D^\alpha u_l\right|_{L^p(P)} \le c|\varphi_l|_{W^{k-l-1/p,p}(C)}, \tag{2.100}$$

for the same reasons as that used previously in the proof of Lemma 5.6, starting from the proof of (2.98bis). Let $k-l \le |\alpha| \le k-1$; let us set $\alpha = \alpha' + \alpha''$, $|\alpha''| = k-l-1$. We have for $M \in \mathfrak{R}_{N-1}$:

$$D^{\alpha'} u_l(x', x_n) = x_N^{|\alpha|'-1} \int_{|z'|<1} M(z')\varphi_l(x_N z' + x')\,\mathrm{d}z'. \tag{2.101}$$

If we apply the operator $D^{\alpha''}$ to (2.101), we get (2.100) with $|\alpha| \le k-1$ for the same reasons as in the proof of (2.98bis). If $|\alpha| = k$, we use the ideas of the proof of Lemma 5.6 and we get:

$$Z_l \in [W^{k-l-1/p,p}(C) \to W^{k,p}(P)].$$

It is clear that (2.98) holds; concerning (2.99a, b), we use the previous lemma. □

Theorem 5.6. *Let P be the pyramid as in Lemma 5.6 and C its basis, k an integer, $k \ge 1$, $p > 1$, $\varphi_l \in W^{k-l-1/p,p}(C)$, $l = 0, 1, \ldots, k-1$. There exists a mapping:*

$$Z \in \left[\prod_{l=0}^{k-1} W^{k-l-1/p,p}(C) \to W^{k,p}(P)\right]$$

such that if $Z(\varphi_0, \varphi_1, \ldots, \varphi_{k-1}) = u$, then $\frac{\partial^l u}{\partial x_N^l} = \varphi_l$ on C.

Indeed: We use Lemmas 5.6 and 5.8 and proceed as in the construction of (2.88). □

2.5.7 The Converse Theorem (Continuation)

Let $k = 1$. We have:

Theorem 5.7. *Let $\Omega \in \mathfrak{N}^{0,1}, p > 1$. There exists a mapping $Z \in [W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)]$ such that for $h \in W^{1-1/p,p}(\partial\Omega), Zh = u$, we have $u = h$ on $\partial\Omega$.*

Proof. We use the partition of unity from 1.2.4 and put $h_r = h\varphi_r$, $1 \le r \le m$; to simplify we omit the index r. By projection onto the hyperplane $x_N = 0$, the function $h \in W^{(1-1/p),p}(\Delta)$. Without loss of generality we assume $\alpha = 1$ (in the definition of Δ in 1.2.4). We use Lemma 5.6 to construct $Z \in [W^{1-1/p,p}(\Delta) \to W^{1,p}(P)]$. Since $\operatorname{supp} h \subset \Delta$, with compact support, there exists $\psi \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp}\psi \subset P \cup C$ such that $h\psi = h$ on C. We write $Zh = h$; we have $h\psi \in W^{1,p}(K_+)$, $K_+ = \{x \in \mathbb{R}^N, |x_i| < 1, i = 1,2,\dots,N-1, 0 < x_N < 1\}$. We use the transformation (2.31), and going back to the index r starting from $h\psi$ we construct a function $v_r \in W^{1,p}(V_r)$, $\operatorname{supp} v_r \subset V_r \cup \Lambda_r$. Obviously, the function $v = \sum_{r=1}^m v_r$ gives the extension of h on Ω. □

Problem 5.1. Prove a theorem analogous to Theorem 5.7, but for $k \ge 2$, $\Omega \in \mathfrak{N}^{0,1}$ or for a nonsmooth boundary $\partial\Omega$. The problem can be posed in the following way: Let t be the number of indexes α, $|\alpha| = k-1$. We consider the closed subset $W \subset \left[W^{1-1/p,p}(\partial\Omega)\right]^t$ as the closure of the set M of elements $(D^{\alpha_{[1]}}u, D^{\alpha_{[2]}}u, \dots, D^{\alpha_{[t]}}u), u \in C^\infty(\overline{\Omega})$. Is the mapping $Z \in [W^{k,p}(\Omega) \to W]$, defined by $Zu = (D^{\alpha_{[1]}}u, D^{\alpha_{[2]}}u, \dots, D^{\alpha_{[t]}}u)$, surjective?

For $N = 2, \partial\Omega$ piecewise smooth, the solution with some modifications is given by G.N. Jakovlev [1]. Cf. also P. Grisvard [1].

If $\partial\Omega$ is sufficiently smooth, the problem is solved (cf. papers by L.N. Slobodetskii [1], S.V. Uspenskii [1]):

Theorem 5.8. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p > 1$, $h_l \in W^{k-l-1/p,p}(\partial\Omega)$, $l = 0, 1,2,\dots,k-1$. There exists a mapping,*

$$Z \in \left[\prod_{l=0}^{k-1} W^{k-l-1/p,p}(\partial\Omega) \to W^{k,p}(\Omega)\right],$$

such that if $Z((h_0,h_1,\dots,h_{k-1})) = u$, then on $\partial\Omega$ we have $\partial^l u/\partial n^l = h_l$, $l = 0,1,2,\dots,k-1$.

Proof. Due to the previous theorem it is sufficient to consider the case $k \ge 2$. We use $G_r, \psi_r, \dots$ as in 1.2.4, and the transformation (1.35) from Chap. 1; this transformation is one-to-one, with lipschitzian derivatives of order $\le k-1$ on $\overline{G}_r$, and the inverse transformation has the same properties on $\overline{K}_r$. Put $h_{lr} = h_l\psi_r$. By projection on the hyperplane $x_{rN} = 0$, we consider h_{lr} as a function of the variable σ; we have $h_{lr} \in W^{k-l-1/p,p}(\Delta_r)$.

Without loss of generality, we can assume in the definition of G_r that $\alpha = \delta = 1$. According to Theorem 5.6 we construct $u_r \in W^{k,p}(P)$ such that on C $\partial^l u_r/\partial x_N^l = (-1)^l h_{lr}$. Since $\operatorname{supp} h_{lr} \subset C$, there exists $\psi \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\psi \subset P \cup C$ such that $\partial^l(u_r\psi)/\partial x_N^l = (-1)^l h_{lr}$ on C. We have $u_r\psi \in W^{k,p}(K_+)$, $K_+ = \{y \in \mathbb{R}^N, y = (\sigma,t), |\sigma_i| < 1, 0 < t < 1\}$. Using the transformation 1.2.7, we get $v_r = u_r\psi \in W^{k,p}(G_r \cap \Omega)$; but according to the form of the support of ψ we have $v_r \in W^{k,p}(\Omega)$; on the other hand $\partial^l v_r/\partial n^l = h_{lr}$ on $\partial\Omega$, thus $v = \sum_{r=1}^M v_r$ gives the transformation such that $Z((h_0,h_1,\dots,h_{k-1})) = v$. □

2.5.8 Remarks

It is possible to investigate the spaces $W^{k,p}(\Omega)$, where k is noninteger, in more detail. Concerning the questions about extension and density, cf. J.L. Lions, E. Magenes [4], for imbedding theorems cf. S.V. Uspenskii [1, 2, 4]. There is a strong link between interpolation in the sense of J.L. Lions and traces, cf. J.L. Lions [7–10]. From our considerations on these questions it is possible to obtain some consequences, for instance:

Corollary 5.1. *Let C be an $(N-1)-$dimensional cube, $p > 1$. If $p < N$, $1/q = 1/p - [1/(N-1)][(p-1)/p]$, then $W^{1-1/p,p}(C) \subset L^q(C)$ algebraically and topologically; if $p = N$, q is an arbitrary real number ≥ 1; if $p > N$, then $W^{1-1/p,p}(C) \subset C^{0,\mu}(\overline{C})$ with $\mu = 1 - N/p$, algebraically and topologically.*

Remark 5.3. Let us set $k = 1 - 1/p$. Then in Corollary 5.1 for $p < N$ we have $1/q = 1/p - k/(N-1)$, and we have the formula from Theorem 3.4. It holds in other cases, cf. S.V. Uspenskii [1].

Let C be an $(N-1)-$dimensional cube, $u \in W^{k-1/p,p}(C), k \geq 1$ an integer, $p > 1$; according to Lemma 5.5 it is possible to extend u to the corresponding pyramid P so that $u \in W^{k,p}(P)$. $P \in \mathfrak{N}^{0,1} \Longrightarrow W^{k,p}(P) = \overline{C^\infty(\overline{P})} \Longrightarrow W^{k-1/p,p}(C) = \overline{C^\infty(\overline{C})}$, hence

Corollary 5.2. *Let C be an $(N-1)-$dimensional cube, $k \geq 1$ an integer, $p > 1$. Then $W^{k-1/p,p}(C) = \overline{C^\infty(\overline{C})}$.*

Let C_{N-1} be an $(N-1)$-dimensional cube, $u \in W^{k-1/p,p}(C_{N-1}), k \geq 1$ an integer, $p > 1$. According to Lemma 5.6 we can extend u to a cube C_N with C_{N-1} as its face. By the approach used in Theorem 3.8 it is possible to extend u to the whole $\mathbb{R}^N$ so that $|u|_{W^{k,p}(\mathbb{R}^N)} \leq c|u|_{W^{k,p}(C_N)}$. We get an extension of u to $\mathbb{R}^{N-1}, u \in W^{k-1/p,p}(\mathbb{R}^{N-1})$. We have also

Corollary 5.3. *Let C be an $(N-1)-$dimensional cube, $p > 1$, $k \geq 1$ an integer. Then there exists $P \in [W^{k,p}(C) \to W^{k-1/p,p}(\mathbb{R}^{N-1})]$, such that $RP = I$, where R is the restriction operator, $R \in [W^{k-1/p,p}(\mathbb{R}^N) \to W^{k-1/p,p}(C)]$, and I the identity operator on $W^{k-1/p,p}(C)$.*

Remark 5.4. A well known simple counterexample due to J. Hadamard states the existence of a continuous function on the unit circle which is not the trace of a function from $W^{1,2}(K)$ where K is the unit disc, cf. for instance S.G. Mikhlin [2]. L. De Vito [1] has constructed an absolutely continuous function with the same property. It is easy to see that a piecewise smooth function cannot be the trace of a function $u \in W^{1,2}(\Omega)$. For sufficient conditions implying that f belongs to $W^{k-1/p,p}(\partial\Omega)$, cf. S.M. Nikolskii [2, 5–8], G. Prodi [1], J. Nečas [11], etc.

Remark 5.5. If $p = 1$, $\Omega \in \mathfrak{N}^{0,1}$, we have obtained by Theorem 4.2 that the traces belong to $L^1(\partial\Omega)$. We have also the converse, i.e. $W^{1,1}(\Omega)/W_0^{1,1}(\Omega) = L^1(\partial\Omega)$; cf. E. Gagliardo [1].

Exercise 5.3. Prove that on the unit circle we have:

$$W^{1/2,2}(\partial\Omega) = \{u \in L^2(\partial\Omega), \quad \sum_{-\infty}^{\infty} n|a_n|^2 < \infty,\ a_n = \frac{1}{2\pi}\int_0^{2\pi} u(\vartheta)\exp(-in\vartheta)\,\mathrm{d}\vartheta\}.$$

Exercise 5.4. Prove directly that for $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{1,p}(\partial\Omega)$ we can extend u to Ω so that $u \in W^{1,p}(\Omega), p \geq 1$.

2.6 Compactness

2.6.1 The Kondrashov Theorem

In Chap. 1, we have proved a particular case of the compactness of the imbedding operator, cf. Chap. 1, Theorem 1.1.4. We shall give some generalizations; concerning the literature on the subject see W. Kondrashov [1], S.L. Sobolev [1], V.I. Smirnov [2], etc.

Theorem 6.1. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1 \leq p < N$, $1 \geq 1/q > 1/p - 1/N$. The identity mapping $I : W^{1,p}(\Omega) \to L^q(\Omega)$ is compact.*

Proof. Let $u_n \in W^{1,p}(\Omega)$ be a bounded sequence, $|u_n|_{W^{1,p}(\Omega)} \leq 1$. It is possible to find elements $v_n \in C^\infty(\overline{\Omega})$ such that $|u_n - v_n|_{W^{1,p}(\Omega)} < 1/n$, $|v_n|_{W^{1,p}(\Omega)} \leq 1$; it is sufficient to prove the existence of a subsequence of v_n, for simplicity we denote this subsequence again v_n, which converges in $L^q(\Omega)$. Let us put $1/q^* = 1/p - 1/N$, let $\varepsilon > 0$, $\Omega^* \subset \overline{\Omega^*} \subset \Omega, \Omega^*$ a subdomain such that

$$\text{meas}\,(\Omega - \Omega^*) < \left(\frac{\varepsilon}{3c_1}\right)^{q^*/(q^*-1)},$$

where $|v_n|_{L^{q^*}(\Omega)} \leq c_1$ (cf. Theorem 3.4). Let $\delta > 0$ be sufficient small such that $x \in \Omega^*$, $|z| < \delta \Longrightarrow x + z \in \Omega$; we have:

$$|v_n(x+z) - v_n(x)| = \left|\int_0^{|z|} \sum_{i=1}^{N} \frac{\partial v_n}{\partial x_i}\left(x + \frac{z}{|z|}t\right)\frac{z_i}{|z|}\,\mathrm{d}t\right| \leq \int_0^{|z|} \sum_{i=1}^{N} \left|\frac{\partial v_n}{\partial x_i}\left(x + \frac{z}{|z|}t\right)\right|\mathrm{d}t,$$

then it follows that

$$\int_{\Omega^*} |v_n(x+z) - v_n(x)|\,\mathrm{d}x = \int_0^{|z|} dt \int_\Omega \sum_{i=1}^{N} \left|\frac{\partial v_n}{\partial x_i}(y)\right| \mathrm{d}y \leq |z| \cdot |v_n|_{W^{1,1}(\Omega)}.$$

Let $|v_n|_{W^{1,1}(\Omega)} \leq c_2$, and let us choose $\delta < \varepsilon/3c_2$. Let us extend v_n by zero outside of Ω; then we have:

$$\int_\Omega |v_n(x+z)-v_n(x)|\,\mathrm{d}x$$
$$\leq \int_{\Omega-\Omega^*} |v_n(x+z)|\,\mathrm{d}x + \int_{\Omega-\Omega^*} |v_n(x)|\,\mathrm{d}x + \int_{\Omega^*} |v_n(x+z)-v_n(x)|\,\mathrm{d}x \leq \varepsilon.$$

If $q=1$ the sequence v_n has the same properties as that in Theorem 1.3. and the result is true in this case. If $1<q<q^*$, we have:

$$\int_\Omega |v_m-v_n|^q\,\mathrm{d}x = \int_\Omega |v_m-v_n|^{[q^*(q-1)]/(q^*-1)}|v_m-v_n|^{(q^*-q)/(q^*-1)}\,\mathrm{d}x$$
$$\leq \left(\int_\Omega |v_m-v_n|^{q^*}\,\mathrm{d}x\right)^{(q-1)/(q^*-1)} \left(\int_\Omega |v_m-v_n|\,\mathrm{d}x\right)^{(q^*-q)/(q^*-1)};$$

this finishes the general case. □

Corollary 6.1. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p\geq 1$, $kp<N$, k an integer, $1\geq 1/q > 1/p-k/N$. The identity mapping $I: W^{k,p}(\Omega)\to L^q(\Omega)$ is compact.*

Corollary 6.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p\geq 1$, $kp=N$, k an integer, $q\geq 1$ arbitrary. The identity mapping $I: W^{k,p}(\Omega)\to L^q(\Omega)$ is compact.*

Corollary 6.3. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p\geq 1$, $kp>N$, k an integer. The identity mapping $I: W^{k,p}(\Omega)\to C(\overline{\Omega})$ is compact.*

2.6.2 Traces

As far as concern traces, we have

Theorem 6.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1<p<N$, $1\geq 1/q > 1/p-[1/(N-1)](p-1)/p$. The mapping $Z\in [W^{1,p}(\Omega)\to L^q(\partial\Omega)]$, which defines the traces, is compact.*

Proof. As in proof of Theorem 6.1 it is sufficient to consider a sequence $v_n \in C^\infty(\overline{\Omega})$, $n=1,2,\ldots$, bounded in $W^{1,p}(\Omega)$. With a partition of unity as in 1.2.4 we take v_r, $\operatorname{supp} v_r \subset V_r\cup\Lambda_r$. For $1\leq p<\infty$ the function $p\to 1/p-[1/(N-1)](p-1)/p$ is decreasing, thus there exists exactly one value p^* such that $1/q = 1/p^*-[1/(N-1)](p^*-1)/p^*$. It is clear that it is sufficient to consider the case $q>1$. According to (2.69) with $1/q^* = 1/p^*-1/N$ (we omit the index r)

$$|v_n-v_m|^q_{L^q(\Lambda)} \leq c[|v_n-v_m|^q_{L^q(V)} + |v_n-v_m|^{(Np^*-N)/(N-p^*)}_{L^{q^*}(V)}|v_n-v_m|^q_{W^{1,p^*}(V)}]. \tag{2.102}$$

According to Theorem 6.1 we can extract a subsequence denoted also by v_n which converges in $L^{q^*}(V)$; but $q\leq q^*$, hence (2.102) implies the convergence of v_n in $L^q(V)$.

□

Exercise 6.1. Give a formulation of the previous theorem with the hypotheses $kp < N$ or $kp = N$ with $k \geq 1$ an integer.

Remark 6.1. Let $\Omega \in \mathfrak{N}^{0,1}$, $1 < p < N$. Then the mapping Z of $W^{1,p}(\Omega)$ into $L^q(\partial\Omega)$ with $1/q = 1/p - [1/(N-1)](p-1)/p$ is not surjective. Indeed: it is always possible to find a sequence v_n defined on $\partial\Omega$ and bounded by the same constant, with $|v_n|_{L^1(\partial\Omega)} = 1$, which converges weakly to zero in $L^1(\partial\Omega)$. If Z was surjective, we could extend v_n to Ω in such a way that v_n is bounded in $W^{1,p}(\Omega)$, and due to Theorem 6.2 there will exist a subsequence v_{n_k} which will converge in $L^1(\partial\Omega)$ to a limit equal to zero, and this is a contradiction to $|v_n|_{L^1(\partial\Omega)} = 1$.

2.6.3 The Lions Lemma, Another Theorem of Compactness

We can prove Lemma 5.1 from Chap. 1 under more general conditions (cf. J.L. Lions [4]):

Lemma 6.1. *Let B_i, $i = 1,2,3$, be three Banach spaces, $B_1 \subset B_2 \subset B_3$ algebraically and topologically. Assume that the identity mapping $I : B_1 \to B_2$ is compact. Then for every $\varepsilon > 0$ there exists $\lambda(\varepsilon)$ such that $u \in B_1 \Longrightarrow$*

$$|u|_{B_2} \leq \varepsilon |u|_{B_1} + \lambda(\varepsilon)|u|_{B_3}.$$

The proof is the same as that of Lemma 1.5.1.

Example 6.1. Let $B_1 = W^{1,p}(\Omega)$, $B_2 = L^q(\Omega)$, $B_3 = L^1(\Omega)$, $\Omega \in \mathfrak{N}^{0,1}$, $1/q > 1/p - 1/N$ if $p < N$, $q \geq 1$ if $p \geq N$. This satisfies the conditions of Lemma 6.1.

Theorem 6.3. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1 \leq p$, $1 \leq q \leq p$. The identity mapping $I : W^{1,p}(\Omega) \to L^q(\Omega)$ is compact.*

Proof. It is sufficient to consider a sequence $v_n \in C^\infty(\overline{\Omega})$ bounded in $W^{1,p}(\Omega)$, and to extract a subsequence which converges in $L^p(\Omega)$. Fix $\varepsilon > 0$; we can find $\Omega' \subset \overline{\Omega}' \subset \Omega$ such that

$$\left(\int_{\Omega - \Omega'} |v_n(x)|^p \, \mathrm{d}x \right)^{1/p} < \frac{\varepsilon}{3}. \tag{2.102 bis}$$

To do this we consider the open sets V_r, $r = 1, 2, \ldots, m$. If $a_r(x'_r) < x_N < a_r(x'_r) + \beta/2$, we have:

$$v_n(x'_r, x_{rN}) = v_n(x'_r, \tau) - \int_{x_{rN}}^{\tau} \frac{\partial v_n}{\partial x_{rN}}(x'_r, \xi_N) \, \mathrm{d}\xi_N, \quad a_r(x'_r) + \beta/2 < \tau < a_r(x'_r) + \beta. \tag{2.103}$$

From this we get (we omit the index r):

$$\begin{aligned}|v_n(x',x_N)|^p &\leq 2^{p-1}\left[|v_n(x',\tau)|^p + \left|\int_{x_N}^{\tau} \frac{\partial v_n}{\partial x_n}(x',\xi_N)\,\mathrm{d}\xi_N\right|^p\right] \\ &\leq 2^{p-1}\left[|v_n(x',\tau)|^p + \beta^{p-1}\int_{a(x')}^{a(x')+\beta}\left|\frac{\partial v_n}{\partial x_n}(x',\xi_N)\right|^p \mathrm{d}\xi_N\right].\end{aligned} \tag{2.104}$$

Now we integrate (2.104) with respect to $\tau \in (a(x')+\beta/2, a(x')+\beta)$ and obtain

$$\frac{\beta}{2}|v_n(x',x_N)|^p \leq 2^{p-1}\left[\int_{a(x')+\beta/2}^{a(x')+\beta}|v_n(x',\tau)|^p\,\mathrm{d}\tau + \frac{\beta^p}{2}\int_{a(x')}^{a(x')+\beta}\left|\frac{\partial v_n}{\partial x_n}(x',\xi_N)\right|^p \mathrm{d}\xi_N\right]; \tag{2.105}$$

now by integration of (2.105) with respect to $x' \in \Delta$ and to $x_N \in (a(x'), a(x')+\gamma)$, γ sufficiently small, we get:

$$\frac{\beta}{2}\int_{\Delta}\mathrm{d}x'\int_{a(x')}^{a(x')+\gamma}|v_n(x',x_N)|^p\,\mathrm{d}x_N \leq 2^{p-1}\left(1+\frac{\beta^p}{2}\right)\gamma|v_n|^p_{W^{1,p}(\Omega)};$$

this gives (2.102 bis) if γ is sufficiently small. Now let Ω'' be a subdomain satisfying $\Omega' \subset \Omega'' \subset \overline{\Omega''} \subset \Omega$ and $\delta > 0$ sufficiently small such that $x \in \Omega'', |z| < \delta \Longrightarrow x+z \in \Omega$. We have:

$$\begin{aligned}|v_n(x+z)-v_n(x)| &\leq \left|\int_0^{|z|}\sum_{i=1}^{N}\frac{\partial v_n}{\partial x_i}\left(x+\frac{z}{|z|}t\right)\frac{z_i}{|z|}\,\mathrm{d}t\right| \\ &\leq |z|^{1-1/p}\sum_{i=1}^{N}\left(\int_0^{|z|}\left|\frac{\partial v_n}{\partial x_i}\left(x+\frac{z}{|z|}t\right)\right|^p dt\right)^{1/p},\end{aligned}$$

hence

$$|v_n(x+z)-v_n(x)|^p \leq c|z|^{p-1}\sum_{i=1}^{N}\int_0^{|z|}\left|\frac{\partial v_n}{\partial x_i}\left(x+\frac{z}{|z|}t\right)\right|^p dt,$$

and

$$\int_{\Omega''}|v_n(x+z)-v_n(x)|^p\,\mathrm{d}x \leq c|z|^p\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial v_n}{\partial x_i}(y)\right|^p \mathrm{d}y \leq c|z|^p|v_n|^p_{W^{1,p}(\Omega)},$$

and thus

$$\left(\int_{\Omega''}|v_n(x+z)-v_n(x)|^p\,\mathrm{d}x\right)^{1/p} \leq c|z||v_n|_{W^{1,p}(\Omega)}.$$

Finally, let δ be sufficiently small such that $x \in \Omega - \Omega''$, $|z| < \delta \Longrightarrow x+z \in \Omega - \Omega'$, and

$$\left(\int_{\Omega''}|v_n(x+z)-v_n(x)|^p\,\mathrm{d}x\right)^{1/p} < \frac{\varepsilon}{3}.$$

Let us set $v_n(x) = 0$ for $x \notin \Omega$. We get:

$$\left(\int_\Omega |v_n(x+z) - v_n(x)|^p \,\mathrm{d}x\right)^{1/p} \le \left(\int_{\Omega''} |v_n(x+z) - v_n(x)|^p \,\mathrm{d}x\right)^{1/p} + \left(\int_{\Omega-\Omega''} |v_n(x+z)|^p \,\mathrm{d}x\right)^{1/p} + \left(\int_{\Omega-\Omega''} |v_n(x)|^p \,\mathrm{d}x\right)^{1/p} \le \varepsilon.$$

Now Theorem 1.3 gives the assertion. □

Exercise 6.2. Prove Theorem 6.3 for a domain Ω starshapped with respect to the origin.

Remark 6.2. For an arbitrary domain Ω, bounded or unbounded, the restriction operator $R : W^{1,p}(\Omega) \to L^q(\Omega')$ with Ω' bounded, $\overline{\Omega}' \subset \Omega$, and with q as in Theorem 6.1, is compact.

Exercise 6.3. Let Ω be a bounded domain such that for any $\varepsilon > 0$, we can find $\Omega' \subset \overline{\Omega'} \subset \Omega$ such that $|u|_{W^{1,p}(\Omega)} \le 1 \Longrightarrow |u|_{L^p(\Omega-\Omega')} < \varepsilon$. Then the identity mapping $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact.

Remark 6.3. If there exists an extension operator $P \in [W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)]$, then Theorem 6.3 is true for Ω bounded.

Problem 6.1. Characterize bounded domains for which the imbedding theorem $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact. We find only equivalent statements as for instance the existence of a spectrum having the form given in Theorem 1.5.1.

2.7 Quotient Spaces, Equivalent Norms

2.7.1 Equivalent Norms

The methods used in this paragraph are strongly related to these introduced in Chap. 1, Sect. 1.1 concerning the same questions, and also with the work of J. Deny, J.L. Lions [1], J.L. Lions [2]. As in 1.1.7 we denote by $P_{(k-1)}$ *the space of polynomials of degree* $\le k-1$. We shall consider only domains such that

$$v \in P_{(k-1)} \Longrightarrow |v|_{L^p(\Omega)} < \infty. \tag{2.106}$$

Lemma 7.1. *Let Ω be a domain satisfying (2.106), $p \ge 1$, $k \ge 1$ an integer. Then there exist functionals f_i, $i = 1,2,\ldots,l$, on $W^{k,p}(\Omega)$ such that $v \in P_{(k-1)}$ implies the equivalence:*

$$\sum_{i=1}^{l} |f_i v|^p = 0 \Longleftrightarrow v \equiv 0. \qquad (2.106\,bis)$$

Proof. There are many ways to construct f_i: for instance we take Ω^*, a bounded nonempty subdomain of Ω, and define:

$$f_\alpha v = \int_{\Omega^*} x^\alpha v(x)\,\mathrm{d}x, \quad |\alpha| \le k-1, \tag{2.107a}$$

or

$$f_\alpha v = \int_{\Omega^*} D^\alpha v(x)\,\mathrm{d}x, \quad |\alpha| \le k-1. \tag{2.107b}$$

□

Let us formulate the first theorem on equivalent norms in $W^{k,p}(\Omega)$:

Theorem 7.1. *Let $\Omega \in \mathfrak{N}^0$, f_i functionals satisfying (2.106 bis), $p \ge 1$, $k \ge 1$ an integer. We have the inequality:*

$$c_1|u|_{W^{k,p}(\Omega)} \le \left[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} + \sum_{i=1}^{l} |f_i u|^p\right]^{1/p} \le c_2|u|_{W^{k,p}(\Omega)}.$$

Proof. We have to prove the inequality:

$$c_1|u|_{W^{k,p}(\Omega)} \le \left[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} + \sum_{i=1}^{l} |f_i u|^p\right]^{1/p}. \tag{2.108}$$

We proceed by contradiction: suppose that (2.108) does not hold for any constant c_1, for instance for $c_1 = 1/n$. Then there exists a function $u_n \in W^{k,p}(\Omega)$, $|u_n|_{W^{k,p}(\Omega)} = 1$ such that

$$\frac{1}{n} > \left[\sum_{|\alpha|=k} |D^\alpha u_n|^p + \sum_{i=1}^{l} |f_i u_n|^p\right]^{1/p}. \tag{2.109}$$

From this we get, for $|\alpha| = k$:

$$\lim_{n\to\infty} D^\alpha u_n = 0 \text{ in } L^p(\Omega). \tag{2.110}$$

According to Theorem 6.3, the identity mapping $I : W^{k,p}(\Omega) \to W^{k-1,p}(\Omega)$ is compact, hence there exists a subsequence u_{n_m} of u_n which converges in $W^{k-1,p}(\Omega)$ and by (2.110) in $W^{k,p}(\Omega)$. Let $u = \lim_{m\to\infty} u_{n_m}$. We have $D^\alpha u = 0 \Longrightarrow u \in P_{(k-1)}$, but $P_{(k-1)}$ is of finite dimension and hence closed in $W^{k,p}(\Omega)$. Then, due to (2.108), we have:

$$\sum_{i=1}^{l} |f_i u|^p = 0 \Longrightarrow u = 0,$$

which is a contradiction to the fact that

$$\lim_{m\to\infty} |u_{n_m}|_{W^{k,p}(\Omega)} = |u|_{W^{k,p}(\Omega)} = 1.$$

□

Remark 7.1. Clearly Theorem 7.1 holds if the identity mapping $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact.

2.7.2 Quotient Spaces

Let $P \subset P_{(k-1)}$, and denote by $W^{k,p}(\Omega)/P$ the quotient space (cf. 1.1.7) with the topology associated with the usual norm:

$$\text{For } \tilde{u} \in W^{k,p}(\Omega)/P, \quad |\tilde{u}|_{W^{k,p}(\Omega)/P} = \inf_{u \in \tilde{u}} |u|_{W^{k,p}(\Omega)}. \tag{2.111}$$

Theorem 7.2. *Let $\Omega \in \mathfrak{N}^0$, and let us assume that the identity mapping $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. Then we have:*

$$c_1 |\tilde{u}|_{W^{k,p}(\Omega)/P_{(k-1)}} \leq \Big[\sum_{|\alpha|=k} \big[D^\alpha u|^p_{L^p(\Omega)} \big]^{1/p} \leq c_2 |\tilde{u}|_{W^{k,p}(\Omega)/P_{(k-1)}}. \tag{2.111 bis}$$

If $p = 2$, $W^{k,2}(\Omega)/P_{(k-1)}$ is a Hilbert space with the scalar product

$$(\tilde{v}, \tilde{u}) = \sum_{|\alpha|=k} \int_\Omega D^\alpha v D^\alpha \overline{u} \, \mathrm{d}x. \tag{2.112}$$

Proof. First, $W^{k,p}(\Omega)/P_{(k-1)}$ is complete with respect to the norm

$$\Big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} \Big]^{1/p}. \tag{2.113}$$

Indeed: let $\tilde{u}_n$ be a Cauchy sequence for (2.113). We can choose $u_n \in \tilde{u}_n$ such that $f_i u_n = 0$, $i = 1, 2, \ldots, l, f_i$ satisfying (2.106 bis); this is always possible by Lemma 7.1. We apply Theorem 7.1, hence u_n is a Cauchy sequence in $W^{k,p}(\Omega)$; let $\lim_{n \to \infty} u_n = u$, which implies $\lim_{n \to \infty} \tilde{u}_n = \tilde{u}$. Denote by B_1 the quotient space $W^{k,p}(\Omega)/P_{(k-1)}$ with the norm (2.111), and by B_2 the same space but with the norm (2.113). The identity mapping $I : B_1 \to B_2$ is continuous, due to the Banach theorem, cf. Chap. 1, Sect. 1.1, and the same property is true for inverse transformation. The result follows. □

Hereafter, when we shall use the mentioned Banach theorem we will not specify the spaces B_1, B_2, etc.

Theorem 7.3. *Let $\Omega \in \mathfrak{N}^0$, and assume that the identity mapping $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, $P \subset P_{(k-1)}$, f_i, $i = 1, 2, \ldots, l$, functionals on $W^{k,p}(\Omega), k \geq 1$ an integer, $p \geq 1$ such that*

$$\textit{for } v \in P_{(k-1)}, \quad \sum_{i=1}^{l} |f_i v|^p = 0 \Longleftrightarrow v \in P.$$

Then

$$c_1 |\tilde{u}|_{W^{k,p}(\Omega)/P} \le \left([\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)}]^p + \sum_{i=1}^{l} |f_i u|^p \right)^{1/p} \le c_2 |\tilde{u}|_{W^{k,p}(\Omega)/P}.$$

If $p = 2$, $W^{k,p}(\Omega)/P$ *is a Hilbert space with the scalar product:*

$$(\tilde{v}, \tilde{u}) = \sum_{|\alpha|=k} \int_\Omega D^\alpha v D^\alpha \overline{u} \, \mathrm{d}x + \sum_{i=1}^{l} f_i v \overline{f_i u}.$$

Proof. It is sufficient to prove that $W^{k,p}(\Omega)/P$ is complete with the norm

$$\big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} + \sum_{i=1}^{l} |f_i u|^p \big]^{1/p}. \tag{2.114}$$

Let $\tilde{u}_n$ be a Cauchy sequence. According to Theorem 7.2, we can find polynomials $p_n \in P_{(k-1)}$ such that $\lim_{n\to\infty}(u_n + p_n) = u$ in $W^{k,p}(\Omega)$. Since $\bar{u}_n$ is a Cauchy sequence with respect to the norm (2.114), the same holds for the sequence $\tilde{p}_n$. Clearly $(\sum_{i=1}^{l} |f_i u|^p)^{1/p}$ is a norm in $P_{(k-1)}/P$, hence $\lim_{n\to\infty} \tilde{p}_n = \tilde{p}$ in $W^{k,p}(\Omega)/P$, and $\lim_{n\to\infty} \tilde{u}_n = \tilde{u} - \tilde{p}$. □

Remark 7.2. Let $\Omega^* \subset \Omega$, Ω^* non empty, L the orthogonal complement of P in $P_{(k-1)}$ for the space $L^2(\Omega^*)$; $p_1, p_2, \dots, p_l$ a basis of L. Then the functionals $f_i u = \int_{\Omega^*} u p_i \, \mathrm{d}x$ satisfy the hypotheses of Theorem 7.3.

2.7.3 The Spaces $V^{k,p}(\Omega)$

Let us formulate a theorem on equivalent norms:

Theorem 7.4. *Let* $\Omega \in \mathfrak{N}^0$, Ω^* *a nonempty and open subset of* Ω. *Then we have the inequality*

$$|u|_{W^{k,p}(\Omega)} \le c \left[\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p \, \mathrm{d}x + \int_{\Omega^*} |u|^p \, \mathrm{d}x \right]^{1/p}. \tag{2.115}$$

Proof. It is sufficient to prove that $W^{k,p}(\Omega)$ with the norm (2.115) is complete. Let u_n be a Cauchy sequence. According to Theorem 7.2 there exist $p_n \in P_{(k-1)}$

such that $\lim_{n\to\infty}(u_n+p_n)=u$ in $W^{k,p}(\Omega)$. Since u_n+p_n is a Cauchy sequence for the norm (2.115), $\lim_{n\to\infty} p_n = p$ in $L^p(\Omega^*)$ and in $W^{k,p}(\Omega)$, which implies that $\lim_{n\to\infty} u_n = u-p$ in $W^{k,p}(\Omega)$. □

Fix $\Omega\subset\mathbb{R}^N$. Denote by $V^{k,p}(\Omega)$ the space of functions in $L^p_{loc}(\Omega)$ whose distributional derivatives of order k belong to $L^p(\Omega)$. Let $\Omega^*\subset\overline{\Omega}^*\subset\Omega$; Ω^* bounded; *on* $V^{k,p}(\Omega)$ *we define the norm by*

$$\left[\sum_{|\alpha|=k}|D^\alpha u|^p_{L^p(\Omega)}+|u|^p_{L^p(\Omega^*)}\right]^{1/p}. \tag{2.116}$$

Theorem 7.5. *The space $V^{k,p}(\Omega)$ is a Banach space. If we change Ω^* in such a way that $\overline{\Omega^*}\subset\Omega$, we obtain equivalent norms. Let $\Omega'\subset\overline{\Omega}'\subset\Omega$, Ω' bounded. Then the restriction operator R satisfies $R\in[V^{k,p}(\Omega)\to W^{k,p}(\Omega')]$.*

Proof. Let $u\in V^{k,p}(\Omega)$, and extend u by zero outside of Ω. We have $u_h\in C^\infty(\overline{\Omega})$ and $u_h\in W^{k,p}(\Omega')$. On the other hand $\lim_{h\to 0}u_h=u$ in $L^p(\Omega')$, and $\lim_{h\to 0}D^\alpha u_h = D^\alpha u$ in $L^p(\Omega')$ for $|\alpha|=k$. Without loss of generality we can assume $\Omega'\in\mathfrak{N}^0$. According to Theorem 7.4, $u\in W^{k,p}(\Omega')$, and we have:

$$|u|_{W^{k,p}(\Omega')}\le c_1|u|_{V^{k,p}(\Omega)}. \tag{2.117}$$

Using (2.117) we see that the topology of $V^{k,p}(\Omega)$ does not depend on Ω^*. Let u_n be a Cauchy sequence in $V^{k,p}(\Omega)$. Due to (2.117) there exists $u\in L^p_{loc}(\Omega)$, $u\in W^{k,p}(\Omega')$ for all bounded Ω', $\Omega'\subset\overline{\Omega}'\subset\Omega$, and $\lim_{n\to\infty}u_n=u$ in $L^p(\Omega')$, $\lim_{n\to\infty}D^\alpha u_n = D^\alpha u$ in $L^p(\Omega)$ for $|\alpha|=k$. □

Remark 7.3. It is a priori clear that we could assume in the definition of $V^{k,p}(\Omega)$ that $u\in L^1_{loc}(\Omega)$ and $D^\alpha u\in L^p(\Omega), |\alpha|=k$. It is sufficient to consider $u\in\mathscr{D}'(\Omega)$ with $D^\alpha u\in L^p(\Omega)$ and we obtain the same space, cf. J. Deny, J.L. Lions [1].

Theorem 7.6. *Let $\Omega\in\mathfrak{N}^0$. Then $W^{k,p}(\Omega)=V^{k,p}(\Omega)$ algebraically and topologically.*

Proof. It is sufficient to prove that $W^{k,p}(\Omega)=V^{k,p}(\Omega)$ algebraically. Let $|\alpha|=k-1$, and let us consider $v=D^\alpha u$. We have $v\in V^{1,p}(\Omega)$ due to Theorem 7.5. Let us consider v in V_r, $r=1,2,\dots,m$. For simplicity we omit the index r. Now using Theorem 2.2 change v on a set of measure zero in such a way that the function obtained is absolutely continuous in V on almost all lines parallel to the axis x_N. We have:

$$v(x',x_N)=\int_{y_N}^{x_N}\frac{\partial v}{\partial x_N}(x',\eta)\,\mathrm{d}\eta+v(x',y_N),\quad x_N,y_N\in(a(x'),a(x')+\beta).$$

From this relation we get:

$$|v(x',x_N)|^p \leq 2^{p-1}\left[\beta^{p-1}\int_{a(x')}^{a(x')+\beta}\left|\frac{\partial v}{\partial x_N}(x',\eta)\right|^p \mathrm{d}\eta + |v(x',y_N)|^p\right].$$

Integrating this inequality with respect to y_N on the interval $(a(x')+\beta/2, a(x')+\beta)$, we get:

$$\frac{\beta}{2}|v(x',x_N)|^p \leq 2^{p-2}\beta^p\int_{a(x')}^{a(x')+\beta}\left|\frac{\partial v}{\partial x_N}(x',\eta)\right|^p \mathrm{d}\eta + 2^{p-1}\int_{a(x')+\beta/2}^{a(x')+\beta}|v(x',y_N)|^p\,\mathrm{d}y_N. \tag{2.118}$$

Finally we integrate (2.118) with respect to x_N on $(a(x'), a(x')+\beta)$, and then with respect to x' on Δ, and get:

$$\begin{aligned}\frac{\beta}{2}\int_{\Delta}\mathrm{d}x'\int_{a(x')}^{a(x')+\beta}&|v(x',x_N)|^p\,\mathrm{d}x_N \\ &\leq 2^{p-2}\beta^{p+1}\int_{\Delta}\mathrm{d}x'\int_{a(x')}^{a(x')+\beta}\left|\frac{\partial v}{\partial x_N}(x',\eta)\right|^p \mathrm{d}\eta \\ &+ 2^{p-1}\beta\int_{\Delta}\mathrm{d}x'\int_{a(x')+\beta/2}^{a(x')+\beta}|v(x',y_N)|^p\,\mathrm{d}y_N.\end{aligned} \tag{2.119}$$

The estimates (2.119), (2.117) give $v \in W^{1,p}(\Omega)$; the result follows by recurrence. □

Remark 7.4. In R. Courant, D. Hilbert [1] we can find an example of a bounded domain such that $W^{k,p}(\Omega) \subset V^{k,p}(\Omega)$ holds strictly.

Remark 7.5. For Ω bounded, Theorem 7.1 holds with f_α as in (2.107a) or (2.107b) for $V^{k,p}(\Omega)$.

For other examples of equivalent norms on $W^{k,p}(\Omega)$ cf. Chap. 1, Sect. 1.1.

2.7.4 Nikodym Domains

Theorem 7.7. *Let $k \geq 1$ be an integer, $p \geq 1, \Omega$ a domain such that $v \in P_{(k-1)} \Longrightarrow \int_\Omega |v|^p\,\mathrm{d}x < \infty$. We have the inequality:*

$$c_1|\tilde{u}|_{V^{k,p}(\Omega)/P_{(k-1)}} \leq \Big[\sum_{|\alpha|=k}|D^\alpha u|^p_{L^p(\Omega)}\Big]^{1/p} \leq c_2|\tilde{u}|_{V^{k,p}(\Omega)/P_{(k-1)}}. \tag{2.120}$$

Proof. Let Ω_i be an increasing sequence of domains in $\mathfrak{N}^0$, $i = 1,2,\ldots$, such that $\lim_{i\to\infty}\Omega_i = \Omega$, and let $\tilde{u}_n$ be a Cauchy sequence for the norm (2.113).

According to Theorems 7.6 and 7.2, (2.120) holds for Ω_i, $i = 1,2,\dots$. There exists $\tilde{u}_{[i]} \in W^{k,p}(\Omega_i)/P_{(k-1)}$ such that $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}_{[i]}$ in $W^{k,p}(\Omega_i)/P_{(k-1)}$. It is clear that the restriction of $u_{[i+1]}$ to Ω_i is $u_{[i]}$, and hence there exists $u \in L^p_{loc}(\Omega)$ such that $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}$ in $W^{k,p}(\Omega_i)/P_{(k-1)}$, $i = 1,2,\dots$, and $\lim_{n\to\infty} D^\alpha u_n = D^\alpha u$ in $L^p(\Omega)$ for $|\alpha| = k$. Thus $V^{k,p}(\Omega)/P_{(k-1)}$ with the norm (2.113) is complete and the result follows from the Banach theorem. □

Remark 7.6. Taking into account Remark 7.5, we can proceed in the proof of Theorem 7.7 as in the proof of Theorem 7.2.

Exercise 7.1. Using Theorem 7.7 and the regularizing operator, prove that $u \in \mathscr{D}'(\Omega)$, $D^\alpha u \in L^p(\Omega)$, $|\alpha| = k$, imply $u \in V^{k,p}(\Omega)$.

A bounded open set Ω is called a *Nikodym domain* if for all $p \geq 1$, $V^{1,p}(\Omega) = W^{1,p}(\Omega)$.[7]

If we have to be more precise, we say that the bounded domain Ω is (k,p)-*Nikodym* if $V^{k,p}(\Omega) = W^{k,p}(\Omega)$.

According to Theorem 7.5 we get obviously

Proposition 7.1. *If Ω is a Nikodym domain, it is $(k,p)-$Nikodym for $k \geq 1$, $p \geq 1$.*

We have another characterization of Nikodym domains (for $p = 2$, cf. J. Deny, J.L. Lions [1]):

Theorem 7.8. *The domain Ω is a Nikodym domain if and only if for $\tilde{u} \in W^{1,p}(\Omega)/P_{(0)}$ the following inequality holds:*

$$c_1 |\tilde{u}|_{W^{1,p}(\Omega)/P_{(0)}} \leq \left[\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^p_{L^p(\Omega)} \right]^{1/p}. \tag{2.121}$$

Proof. If Ω is a Nikodym domain, the identity mapping

$$I : W^{1,p}(\Omega)/P_{(0)} \to V^{1,p}(\Omega)/P_{(0)}$$

is surjective. The norms $|\tilde{u}|_{W^{1,p}(\Omega)/P_{(0)}}$ and $|\tilde{u}|_{V^{1,p}(\Omega)/P_{(0)}}$ are equivalent, and we have (2.120) and then (2.121).

The relation (2.121) being satisfied, due to (2.120) it is sufficient to prove the density of $W^{1,p}(\Omega)/P_{(0)}$ in $V^{1,p}(\Omega)/P_{(0)}$. To do this, let $\tilde{u} \in V^{1,p}(\Omega)/P_{(0)}, u \in \tilde{u}$; without loss of generality we can assume u real. We define:

$$u_n = \begin{cases} u & \text{for} \quad |u| < n, \\ n & \text{for} \quad u \geq n, \\ -n & \text{for} \quad u \leq -n. \end{cases}$$

[7]The original definition was less restrictive: Ω is a Nikodym domain if $V^{1,2}(\Omega) = W^{1,2}(\Omega)$.

It follows from Theorem 2.3 that $u_n \in W^{1,p}(\Omega)$. For $F_n = \{x \in \Omega, |u(x)| \geq n\}$, we have obviously $\lim_{n\to\infty}$ meas $F_n = 0$, hence $\lim_{n\to\infty} \tilde{u_n} = \tilde{u}$ in $V^{1,p}(\Omega)/P_{(0)}$. □

Problem 7.1. Is in general $W^{k,p}(\Omega)/P_{(k-1)}$ dense in $V^{k,p}(\Omega)/P_{(k-1)}$ for instance for Ω bounded?

Exercise 7.2. For Ω a Nikodym domain prove the *Poincaré inequality:*

$$u \in W^{1,2}(\Omega) \Longrightarrow |u|^2_{L^2(\Omega)} - \frac{1}{\text{meas}\,\Omega}\left|\int_\Omega u(x)\,\mathrm{d}x\right|^2 \leq c\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2_{L^2(\Omega)}.$$

Remark 7.7. If Ω is $(k,p)-$Nikodym, then we have the inequality:

$$c_1|\tilde{u}|_{W^{k,p}(\Omega)/P_{(k-1)}} \leq \big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)}\big]^{1/p}.$$

We proceed as in the first part of the proof of Theorem 7.8.

Now we formulate a more general theorem concerning equivalent norms, which generalizes Theorems 7.1, 7.3, 7.4:

Theorem 7.9. *Let Ω be a Nikodym domain, $|u|_1$ a seminorm on $W^{k,p}(\Omega)$; we assume that $|u|_1 \leq c|u|_{W^{k,p}(\Omega)}$. Let*

$$|\tilde{u}|_2 = \left[|u|_1^p + \sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)}\right]^{1/p},$$

and let $P \subset P_{(k-1)}$ be the set all polynomials u such that $|u|_2 = 0$. Then we get the following inequalities:

$$c_1|\tilde{u}|_{W^{k,p}(\Omega)/P} \leq |\tilde{u}|_2 \leq c_2|\tilde{u}|_{W^{k,p}(\Omega)/P}.$$

Proof. It is sufficient to prove that $W^{k,p}(\Omega)/P$ with the norm $|\tilde{u}|_2$ is complete. Let $\tilde{u}_n$ be a Cauchy sequence. According to Theorem 7.7, and since Ω is a Nikodym domain, there exists a sequence $p_n \in P_{(k-1)}$ such that $\lim_{n\to\infty}(u_n + p_n) = u$ in $W^{k,p}(\Omega)$. But $|\tilde{p}|_2$ is a norm on $P_{(k-1)}/P$, therefore $\lim_{n\to\infty} \tilde{p}_n = \tilde{p}$ in $P_{(k-1)}/P$, and consequently also in $W^{k,p}(\Omega)/P$. Thus $\lim_{n\to\infty} \tilde{u}_n = \tilde{u} - \tilde{p}$. □

Remark 7.8. If Ω is a Nikodym domain, we deduce from Theorem 7.9 the *generalized Poincaré inequality*, i.e. for $u \in W^{k,p}(\Omega)$ we have

$$|u|_{W^{k,p}(\Omega)} \leq c\big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} + \sum_{|\alpha|\leq k-1} \big|\int_\Omega D^\alpha u\,\mathrm{d}x\big|^p\big]^{1/p}.$$

Exercise 7.3. Let C be the unit disc without the interval $y = 0$, $0 \le x \le 1$. Prove that C is a Nikodym domain.

Exercise 7.4. Let Ω be a starshapped domain with respect to the origin. Prove that Ω is a Nikodym domain.

Theorem 7.10. *Let Ω be a Nikodym domain, $V \subset W^{k,p}(\Omega)$ a closed subset, and $P_V = V \cap P_{(k-1)}$. Then $(\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)})^{1/p}$ is an equivalent norm on V/P_V.*

Proof. Let $P_{(k-1)} = P_V \dotplus L$ be the direct sum with $L \cap P_V = 0$, $q_1, q_2, \ldots, q_\kappa$ a basis of L (the functions q_i are linearly independent). It is sufficient to prove that V/P_V with the norm given above is complete. Let $\tilde{u}_n$ be a Cauchy sequence. According to Theorem 7.7 and due to the Nikodym property of Ω, there exist $p_n \in P_{(k-1)}$ such that $u_n + p_n$ is a Cauchy sequence in $W^{k,p}(\Omega)$, hence $\tilde{u}_n + \tilde{p}_n$ is also a Cauchy sequence in $W^{k,p}(\Omega)/P_V$. We have:

$$p_n = a_n + b_n, \quad a_n \in P_V,\ b_n \in L,\ \tilde{b}_n = \sum_{i=1}^{\kappa} \lambda_{ni} \tilde{q}_i.$$

Let us consider:

$$\tilde{u}_n + \sum_{i=1}^{\kappa-1} \lambda_{ni} \tilde{q}_i. \tag{2.122}$$

The space $V/P_V \dotplus \cup_{i=1}^{\kappa-1} \tilde{q}_i = \tilde{W}$ is closed in $W^{k,p}(\Omega)/P_V$ and by the Hahn-Banach theorem, there exists a functional f on $W^{k,p}(\Omega)/P_V$ such that $f\tilde{v} = 0$ for $\tilde{v} \in \tilde{W}$ and $f\tilde{q}_\kappa = 1$. The sequence $\tilde{u}_n + \tilde{b}_n$ is a Cauchy sequence, therefore we have $f(\tilde{u}_n + \tilde{b}_n) = \lambda_{n\kappa}$ and $\lim_{n\to\infty} \lambda_{n\kappa} = \lambda_\kappa$. Using a recurrence process we get $\lim_{n\to\infty} \lambda_{ni} = \lambda_i$, $i = 1, 2, \ldots, \kappa$, and $\tilde{u}_n$ is a Cauchy sequence in $W^{k,p}(\Omega)/P_V$. □

Theorem 7.11. *Let Ω be a Nikodym domain, $V \subset W^{k,p}(\Omega)$ a closed subspace, and $|u|_1$ a seminorm on V; we assume $u \in V \Longrightarrow |u|_1 \le c|u|_{W^{k,p}(\Omega)}$. Let*

$$|u|_2 = \Big[|u|_1^p + \sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)}\Big]^{1/p}$$

and $P \subset P_{(k-1)} \cap V$ the space of polynomials in $P_{(k-1)} \cap V$ for which $|u|_1 = 0$. Then we have:

$$c_1 |\tilde{u}|_{V/P} \le |u|_2 \le c_2 |\tilde{u}|_{V/P}.$$

Proof. As in Theorem 7.9, it is sufficient to prove that a Cauchy sequence $\tilde{u}_n$ for the norm $|\tilde{u}|_2$ is a Cauchy sequence in V/P. With the same notations as in the previous theorem $u_n + p_n$ is a Cauchy sequence in $W^{k,p}(\Omega)$, and moreover $\tilde{u}_n + \tilde{p}_n$ is a Cauchy sequence in $W^{k,p}(\Omega)/P$; $p_n \in P_V$. But since $\tilde{p}_n$ is a Cauchy sequence in the norm $|\tilde{u}|_1$, it is also a Cauchy sequence in P_V/P. Hence $\tilde{p}_n$ is a Cauchy sequence in $W^{k,p}(\Omega)/P$. □

Chapter 3
Existence, Uniqueness and Fundamental Properties of Solutions of Boundary Value Problems

The title of this chapter is clear and specifies its content. The important references are E. Magenes, G. Stampacchia [1], J.L. Lions [1–5], S. Agmon, A. Douglis, L. Nirenberg [1].

A complementary bibliography will given in the chapter. Some results are due to the author.

For further literature cf.: S. Agmon [1, 2, 4], S. Agmon, A Douglis, L. Nirenberg [2], N. Aronszajn [2], I. Babuška, [2–4], A.V. Bicadze [1], F. E. Browder [1–7], S. Campanato [1, 2], R. Courant, D. Hilbert [1], G. Fichera [3, 4, 7, 8], D.G. De Figueiredo [1], K.O. Friedrichs, [1], L. Gårding, L. D. Kudriavcev [1, 4], P. D. Lax, A. Milgram [1], J.L. Lions [11, 13], K. Maurin [1], S. G. Mikhlin [2, 3], C. Miranda [1], S.M. Nikolskii [1], L. Nirenberg [2], P.C. Rosenbloom [1], G. Stampacchia [8, 9], S.L. Sobolev [1], M.I. Vishik [1, 2], M.I. Vishik, O.A. Ladyzhenskaya [1], H. Weyl [1], S. Zaremba [1].

We begin with some preliminary results.

3.1 The Boundary Integral, Green's Formula

3.1.1 The Boundary Integral

The theory of the boundary integral can be found in Saks' book (S. Saks [1]). We are interested in Green's formula, and therefore, it is more convenient to define the boundary integral using a partition of unity.

We start with a domain of type $\mathfrak{N}^{0,1}$ (cf. 2.1.1) and with the partition of unity of a covering of $\partial\Omega$ (cf. 2.4.1)

In 2.4.1 we defined the space $L^p(\partial\Omega)$. Let us consider a system S of charts (x_r', x_{rN}), $r = 1, 2, \ldots, m$, where the functions $a_r(x_r')$ describe the boundary $\partial\Omega$ of the domain (cf. 2.4.1).

J. Nečas et al., *Direct Methods in the Theory of Elliptic Equations*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-10455-8_3,

Lemma 1.1. *Let be $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$ and S_i, $i = 1,2$, two systems of charts. Then the corresponding two norms on $L^p(\partial\Omega)$ are equivalent. The space $L^p(\partial\Omega)$ does not depend on the system.*

Proof. Let U_{ir}, $r = 1,2,\dots,m_i$, be open sets of the system S_i. For r and s fixed, let $M = \partial\Omega \bigcap U_{1r} \bigcap U_{2s} \neq \emptyset$. Let us denote by (x'_r, x_{rN}), (x'_s, x_{sN}), the corresponding coordinates and by P_r (resp. P_s) the projection of M on the hyperplane $x_{rN} = 0$ (resp. $x_{sN} = 0$). Let T be the relation between P_r and P_s defined in 2.4.2. Using Lemma 2.3.1, we obtain the following inequality:

$$c_1 \int_{P_r} |f(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r \leq \int_{P_s} |f(x'_s, a_s(x'_s))|^p \, \mathrm{d}x'_s \leq c_2 \int_{P_r} |f(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r. \tag{3.1}$$

□

Let $\Omega \in \mathfrak{N}^{0,1}$, $u \in L^1(\partial\Omega)$. Let us define the boundary integral by:

$$\int_{\partial\Omega} u \, \mathrm{d}S = \sum_{r=1}^{m} \int_{\Delta_r} u(x'_r, a_r(x'_r)) \varphi_r(x'_r, a_r(x'_r)) \Big(1 + \sum_{i=1}^{N-1} (\frac{\partial a_r}{\partial x_{ri}})^2\Big)^{1/2} \mathrm{d}x'_r.$$[1]

Lemma 1.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, $u \in L^p(\partial\Omega)$; then*

$$c_1 \int_{\partial\Omega} |u|^p \, \mathrm{d}S \leq \sum_{r=1}^{m} \int_{\Delta_r} |u(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r \leq c_2 \int_{\partial\Omega} |u|^p \, \mathrm{d}S. \tag{3.2}$$

Proof. It is sufficient to prove the right inequality. Let U_r be a given system of charts; let be $U_{i_1}, U_{i_2}, \dots U_{i_l}$, the domains of the system U_s, $s = 1,2,\dots,m$, such that $M_k = \partial\Omega \bigcap U_r \bigcap U_{i_k} \neq \emptyset$. Let P_k (resp. Q_k) be the projection of M_k on the hyperplane $x_{rN} = 0$ (resp. $x_{i_k N} = 0$). Let $g = |u|^p$. Using Lemma 2.3.1, we get:

$$\int_{P_k} g\varphi_{i_k} \, \mathrm{d}x'_r \leq c \int_{Q_k} g\varphi_{i_k} \, \mathrm{d}x'_{i_k}. \tag{3.3}$$

We must prove:

$$\int_{\Delta_r} g\varphi_{i_k} \, \mathrm{d}x'_r \leq c_1 \int_{Q_k} g\varphi_{i_k} \, \mathrm{d}x'_{i_k} \leq c_2 \int_{\partial\Omega} g \, \mathrm{d}S,$$

we used the following property: $g\varphi_{i_k} = 0$ for $x'_r \in \Delta_r - P_k$. Moreover we have:

$$\int_{\Delta_r} g \, \mathrm{d}x'_r = \sum_{k=1}^{l} \int_{\Delta_r} g\varphi_{i_k} \, \mathrm{d}x'_r \leq c_3 \int_{\partial\Omega} g \, \mathrm{d}S.$$

□

[1] Later we shall prove that this definition does depend nor on S neither on the corresponding partition of unity.

3.1.2 Green's Formula

For $\Omega \in \mathfrak{N}^{0,1}$ we have proved, using Lemma 2.4.2, the existence of the exterior normal almost everywhere on $\partial\Omega$. Then we can formulate

Theorem 1.1. *Let $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$ where $1/p+1/q \leq (N+1)/N$, if $N > p \geq 1$, $N > q \geq 1$ with $q > 1$ if $p \geq N$ and with $p > 1$ if $q \geq N$. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i} v \,\mathrm{d}x = \int_{\partial\Omega} uvn_i \,\mathrm{d}S - \int_\Omega u \frac{\partial v}{\partial x_i}\,\mathrm{d}x,$$

where $(n_1, n_2, \ldots, n_N)$ is the exterior normal.

Proof. First, we assume $u, v \in C^\infty(\overline{\Omega})$, and set $f = uv$, $f_r = f\varphi_r$, $r = 1, 2, \ldots, m$. We have:

$$\int_\Omega \frac{\partial f_r}{\partial x_{rN}}\,\mathrm{d}x = \int_{V_r} \frac{\partial f_r}{\partial x_{rN}}\,\mathrm{d}x = \int_{\Delta_r} \mathrm{d}x'_r \int_{a_r(x'_r)}^{a_r(x'_r)+\beta} \frac{\partial f_r}{\partial x_{rN}}\,\mathrm{d}x_{rN}$$
$$= -\int_{\Delta_r} f_r(x'_r, a_r(x'_r))\,\mathrm{d}x'_r = \int_{\Delta_r} f_r(x'_r, a_r(x'_r)) n_{rN} \Big(1 + \sum_{i=1}^{N-1} \big(\frac{\partial a_r}{\partial x_{r_i}}\big)^2\Big)^{\frac{1}{2}}\,\mathrm{d}x'_r, \tag{3.4}$$

where $n_{r1}, n_{r2}, \ldots, n_{rN}$ are the components of the exterior normal in the coordinates (x'_r, x_{rN}). Let be $\alpha' < \alpha$. Using the regularizing operator we obtain $\lim_{h\to\infty} a_{rh} = a_r$ in $C^0(\overline{\Delta'_r}), W^{1,2}(\Delta'_r)$. We can choose λ_h, $\lim_{h\to 0}\lambda_h = 0$ such that $b_{rh}(x'_r) = a_{rh}(x'_r) + \lambda_h \geq a_r(x'_r)$ if $x'_r \in \overline{\Delta'_r}$. Let us denote $V'_{rh} = \{x \in \mathbb{R}^N, x'_r \in \Delta_r(\alpha'), b_{rh}(x'_r) < x_{rN} < b_{rh}(x'_r) + \beta\}$, $V'_r = \{x \in \mathbb{R}^N, x_r \in \Delta_r(\alpha'), a_r(x'_r) < x_{rN} < a_r(x'_r) + \beta\}$. If $j \leq N-1$, then for α' in a small neighborhood of α we have:

$$\int_\Omega \frac{\partial f_r}{\partial x_{rj}}\,\mathrm{d}x = \int_{V'_r} \frac{\partial f_r}{\partial x_{rj}}\,\mathrm{d}x = \lim_{h\to 0} \int_{V'_{rh}} \frac{\partial f_r}{\partial x_{rj}}\,\mathrm{d}x. \tag{3.5}$$

Denote $K' = \{y \in E_N, y = (y', y_N), y' \in \Delta(\alpha'), 0 < y_N < \beta\}$ and T_h the transformation of K' on V'_{rh} defined by:

$$x'_r = y', \quad x_{rN} = y_N + b_{rh}(y'). \tag{3.6}$$

The Jacobian of the transformation (3.6) is equal to 1. Let $g(y) = f_r(T_h(y))$; we have:

$$\frac{\partial f_r}{\partial x_{rj}} = \frac{\partial g}{\partial y_j} - \frac{\partial g}{\partial y_N}\frac{\partial a_{rh}}{\partial y_j}.$$

For h sufficiently small we obtain:

$$\int_{V'_{rh}} \frac{\partial f_r}{\partial x_{rj}}\,\mathrm{d}x = \int_{K'} \left(\frac{\partial g}{\partial y_j} - \frac{\partial g}{\partial y_N}\frac{\partial a_{rh}}{\partial y_j}\right)\mathrm{d}y = \int_{\Delta_r(\alpha')} g(y',0)\frac{\partial a_{rh}}{\partial y_j}(y')\,\mathrm{d}y'$$
$$= \int_{\Delta_r(\alpha')} f_r(x'_r, b_{rh}(x'_r))\frac{\partial a_{rh}(x'_r)}{\partial x_{rj}}(x')\,\mathrm{d}x'_r.$$

If h tends to zero, then:

$$\int_{V_r} \frac{\partial f_r}{\partial x_{rj}}\,\mathrm{d}x = \int_{\Delta_r(\alpha)} f_r(x'_r, a(x'_r))n_{rj}\left(1+\sum_{i=1}^{N-1}\left(\frac{\partial a_r}{\partial x_{ri}}\right)^2\right)^{\frac{1}{2}}\mathrm{d}x'_r. \tag{3.7}$$

But

$$\frac{\partial f_r}{\partial x_i} = \sum_{j=1}^{N}\frac{\partial f_r}{\partial x_{rj}}\frac{\partial x_{rj}}{\partial x_i},$$

where $\partial x_{rj}/\partial x_j$ are constants. Then from (3.4) and (3.7) we get:

$$\int_{\Omega} \frac{\partial f_r}{\partial x_i}\,\mathrm{d}x = \int_{\Delta_r(\alpha)} f_r(x'_r, a(x'_r))\varphi_r(x'_r, a_r(x'_r))n_i\left(1+\sum_{i=1}^{N-1}\left(\frac{\partial a_r}{\partial x_{ri}}\right)^2\right)^{\frac{1}{2}}dx'_r; \tag{3.8}$$

now by summation on r, we obtain:

$$\int_{\Omega}\frac{\partial f}{\partial x_i}\,\mathrm{d}x = \int_{\partial\Omega} f n_i\,\mathrm{d}S, \tag{3.9}$$

this proves the theorem in the case $u,v \in C^\infty(\overline{\Omega})$. Now let $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$; according to Theorem 2.3.1, there exist two sequences $u_n, v_n \in C^\infty(\overline{\Omega})$ such that $\lim_{n\to\infty} u_n = u$ in $W^{1,p}(\Omega)$, $\lim_{n\to\infty} v_n = v$ in $W^{1,q}(\Omega)$. If $q<N$, and if $1/q' = 1/q - 1/N$, then Theorem 2.3.4 implies $\lim_{n\to\infty} v_n = v$ in $L^{q'}(\Omega)$. The inequality $1/p + 1/q' \le 1$ implies due to Theorem 2.3.5 that

$$\lim_{n\to\infty}\int_{\Omega}\frac{\partial u_n}{\partial x_i}v_n\,\mathrm{d}x = \int_{\Omega}\frac{\partial u}{\partial x_i}v\,\mathrm{d}x. \tag{3.10}$$

If $q=N$, then $\lim_{n\to\infty} v_n = v$ in $L^{\frac{p}{p-1}}(\Omega)$, and (3.10) holds. If $q>N$, then $\lim_{n\to\infty} v_n = v$ in $C^0(\overline{\Omega})$, and using Theorem 2.3.8, we have (3.10). Let $q<N$, $p<N$, and define:

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N-1}\frac{p-1}{p}, \qquad \frac{1}{q^*} = \frac{1}{q} - \frac{1}{N-1}\frac{q-1}{q}.$$

Now due to Theorem 2.4.2, $\lim_{n\to\infty} u_n = u$ in $L^{p^*}(\partial\Omega)$ and $\lim_{n\to\infty} v_n = v$ in $L^{q^*}(\partial\Omega)$; but $\frac{1}{p^*}+\frac{1}{q*}\leq 1$, hence

$$\lim_{n\to\infty}\int_{\partial\Omega} u_n v_n n_i \, \mathrm{d}S = \int_{\partial\Omega} u v n_i \, \mathrm{d}S. \tag{3.11}$$

If p or q is equal to N, then we prove (3.11) using Theorem 2.4.5. If $p > N$ or $q > N$ we use Theorem 2.3.7. □

Lemma 1.3. *Let $\Omega \in \mathfrak{N}^{0,1}$. Then the definition of the boundary integral depends neither on S (i.e. on the system used to define $\partial\Omega$) nor on the corresponding partition of unity.*

Proof. Let $u \in L^1(\partial\Omega)$; we have $un_i \in L^1(\partial\Omega)$. Theorem 2.4.8 implies the existence of a sequence $v_k \in C^\infty(\overline{\Omega})$ such that $\lim_{k\to\infty} v_k = un_i$ in $L^1(\partial\Omega)$. Theorem 1.1 proves that $\int_{\partial\Omega} v_k n_i \, \mathrm{d}S$ depends neither on S nor on the partition of unity. This also holds for $\int_{\partial\Omega} un_i^2 \, \mathrm{d}S$, and hence

$$\sum_{i=1}^{N}\int_{\partial\Omega} un_i^2 \, \mathrm{d}S = \int_{\partial\Omega} u \, \mathrm{d}S.$$

□

Exercise 1.1. Let $\Omega \in \mathfrak{N}^{0,1}$. We denote by $B(x,\rho)$ the ball with centre at x, and with radius ρ. We can find $\rho > 0$ such that $\partial\Omega \bigcap B(x,\rho)$ can be represented in local coordinates by lipschitzian functions $a(x')$; we set:

$$\mathrm{d}S = \left(1+\sum_{i=1}^{N-1}\left(\frac{\partial a}{\partial x_i}\right)^2\right)^{\frac{1}{2}} \mathrm{d}x'.$$

Let us define a boundary measure independently on the local representation, and define the boundary integral, using the same method as in the definition of the general Lebesgue integral. Prove the equivalence of these two definitions.

3.2 Formulation of the Problem. Existence and Uniqueness of the Solution

3.2.1 Sesquilinear Boundary Forms

In this section, we take a more general point of view on the problem than that considered in Sects. 1.1.2, 1.1.3, without repeating the simple properties. From time to time, when necessary, one may refer to the results given in Sects. 1.1.2, 1.1.3. The boundary value problem is the same as above and the properties of the solutions are almost the same as in Chap. 1, therefore we shall not give all the proofs in full detail. The method used is due to K.O. Friedrichs [3, 4], R. Courant, D. Hilbert [3], it is related to the works of E. Magenes, G. Stampacchia [1], J.L. Lions [1–5], S. G. Mikhlin [2, 3], S. L. Sobolev [1], M. I. Vishik [1, 2], L. Nirenberg [1], P. D. Lax, A. Milgram [1], etc.

The notion of an *elliptic operator* was defined in (1.29)–(1.30).[2] With an operator A, we associate its *decomposition* - the decomposition is not unique (cf. Sect. 1.1.2). For a given operator we introduce the associated *sesquilinear form* (1.3.6).

We shall give some complementary results for boundary sesquilinear forms.

A *sesquilinear* form $a(u,v)$, defined on $W^{k,2}(\Omega)\times W^{k,2}(\Omega)$ is called a *boundary form* in the case that $a(u,v)=0$, if at least one of the functions u,v is in $W_0^{k,2}(\Omega)$.

Using the traces as defined in Sect. 2.2.4, Chap. 2, we have:

Proposition 2.1. *Let $\Omega\in\mathfrak{N}^{0,1}$, and define:*

$$a(u,v)=\int_{\partial\Omega}\sum_{i=0}^{k-1}\sum_{|\alpha|\le k-1}\overline{b}_{i\alpha}\frac{\partial^i v}{\partial n^i}D^\alpha\overline{u}\,\mathrm{d}S, \tag{3.12}$$

where $b_{i\alpha}\in L^\infty(\partial\Omega)$. Then $a(u,v)$ is a sesquilinear boundary form.

Remark 2.1. The proposition can be generalized with other conditions on $b_{i\alpha}$ using the theorems from Sects. 2.2.4, 2.2.5, Chap. 2.

Example 2.1. Let $k=1$, $N\ge 3$, $a(u,v)=\int_{\partial\Omega}\overline{b}v\overline{u}\,\mathrm{d}S$. Due to Theorem 2.4.2, it is sufficient to have $b\in L^{N-1}(\partial\Omega)$. If $N=2$, it is sufficient, due to Theorem 2.4.6, to have $b\in L^p(\partial\Omega)$, $p>1$.

Remark 2.2. If $\Omega\in\mathfrak{N}^{0,1}$, using theorems given in Sect. 2.2.3, Chap. 2, we can generalize the conditions on a_{ij} of the form $A(u,v)$.

Example 2.2. Let $k=1$,

$$A=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)+\sum_{i=1}^{N}\frac{\partial}{\partial x_i}(a_i)+\sum_{j=1}^{N}b_j\frac{\partial}{\partial x_j}+c.$$

If $N\ge 3$, using Theorem 2.3.4, it is sufficient to have $a_i,b_j\in L^N(\Omega)$, $c\in L^{N/2}(\Omega)$. If $N=2$, we can take $a_i,b_j\in L^p(\Omega)$, $p>2$, $c\in L^q(\Omega)$, $q>1$; this follows from Theorem 2.3.5. Then we have

Proposition 2.2. *Let $\Omega\in\mathfrak{N}^{0,1}$, $b_{ij}\in C^{0,1}(\overline{\Omega})$. Then the sesquilinear form*

$$a(u,v)=\int_{\partial\Omega}\sum_{i,j=1}^{N}(\overline{b}ij-\overline{b}_{ji})v\frac{\partial\overline{u}}{\partial x_j}n_i\,\mathrm{d}S \tag{3.13}$$

[2] In Sect. 7 we shall study briefly elliptic systems.

can be extended by continuity from $C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$ onto $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, as a boundary form.

Proof. For $u, v \in C^\infty(\Omega)$ we have:

$$\int_\Omega \sum_{i,j=1}^N (\overline{b}_{ij} - \overline{b}_{ji}) \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_j} \, \mathrm{d}x = \int_{\partial\Omega} \sum_{i,j=1}^N (\overline{b}_{ij} - \overline{b}_{ji}) v \frac{\partial \overline{u}}{\partial x_j} n_i \, \mathrm{d}S$$
$$- \int_\Omega \sum_{i,j=1}^N \left(\frac{\partial \overline{b}_{ij}}{\partial x_i} - \frac{\partial \overline{b}_{ji}}{\partial x_i} \right) v \frac{\partial \overline{u}}{\partial x_j} \, \mathrm{d}x - \int_{\partial\Omega} \sum_{i,j=1}^N (\overline{b}ij - \overline{b}_{ji}) v \frac{\partial^2 \overline{u}}{\partial x_j \partial x_j} \, \mathrm{d}x.$$

The last integral is zero due to the symmetry in i, j , hence the result follows. □

For the following proposition, cf. also J.L. Lions [3]:

Proposition 2.3. *Let $\Omega \in \mathfrak{N}^{1,1}$, $t = (t_1, t_2, ..., t_N)$ a tangent vector to $\partial\Omega$ with $t_j \in C^{0,1}(\partial\Omega)$. Then there exist $b_{ij} \in C^{0,1}(\overline{\Omega})$ such that $\sum_{i=1}^N (b_{ij} - b_{ji}) n_i = t_j$.*

Indeed, let us put $b_{ij} = n_i t_j$. We have $\sum_{i=1}^N (n_i t_j - n_j t_i) n_i = t_j$; as $b_{ij} \in C^{0,1}(\partial\Omega)$, using the techniques given in Theorem 2.4.8 we can extend this function to $\overline{\Omega}$ as a function from $C^{0,1}(\overline{\Omega})$.

3.2.2 Sesquilinear Boundary Forms (Continuation)

We recall the notion of *order of transversality*: to do this we use the local coordinates given in 1.2.4 to describe $\partial\Omega$. We assume $\Omega \in \mathfrak{N}^{2k}$. The boundary operator $\sum_{|\alpha| \le 2k-1} b_\alpha D^\alpha$, with $b_\alpha \in L^\infty(\partial\Omega)$, is *at most $(k-1)-$times transversal* if it can be written in the form $\sum_{|\alpha| \le 2k-1} b'_\alpha D^\alpha$ with (in the local coordinates (σ, t))

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial \sigma_1^{\alpha_1} ... \partial \sigma_{N-1}^{\alpha_{N-1}} \partial t^{\alpha_N}}, \quad \alpha_N \le k - 1.$$

Now we can formulate

Proposition 2.4. *Let $\Omega \in \mathfrak{N}^{2k,1}$, $b_{i\alpha} \in L^\infty(\partial\Omega)$, if $|\alpha| - k < 0$, $b_{i\alpha} \in C^{|\alpha|-k,1}(\partial\Omega)$ if $|\alpha| - k \ge 0$. The operators $\sum_{|\alpha| \le 2k-1-i} b_{i\alpha} D^\alpha$ are at most $(k-1)$ transversal. Hence*

$$a(u, v) = \int_{\partial\Omega} \sum_{i=0}^{k-1} \sum_{|\alpha| \le 2k-1-i} \overline{b}_{i\alpha} \frac{\partial^i v}{\partial n^i} D^\alpha \overline{u} \, \mathrm{d}S \tag{3.14}$$

is a boundary form on $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$.

Proof. As in Theorem 1.3.1. □

3.2.3 Boundary Value Problems

The *boundary value problem* considered here, is, apart from some generalizations, in some sense the same as in 1.2.6. Of course, for each particular case, the boundary value problem can be generalized in different ways: domains, coefficients, data, etc.

Let us recall the definition. There are given:

– A bounded domain $\Omega \in \mathfrak{N}^{0,1}$. (3.15a)

– A partition of $\partial\Omega$ into a collection of disjoint open sets on $\partial\Omega$, Γ_1, $\Gamma_2, \dots, \Gamma_\kappa$. (3.15b)

– A differential operator A as (1.29) with the associated sesquilinear form $A(u,v)$, cf. (1.38). (3.15c)

– A boundary sesquilinear form $a(u,v)$ as in (3.12), (3.13) or (3.14); the type depends on the regularity of $\partial\Omega$. (3.15d)

– Boundary operators B_{is}, $i = 1,2,\dots,\kappa$, $s = 1,2,\dots,\mu_i$, from 1.2.3; if $\partial\Omega$ is smooth enough almost everywhere,[3] everything is as in Chap. 1. But if only $\Omega \in \mathfrak{N}^{0,1}$, then: (3.15e)

$$B_{is} = \frac{\partial^{i_s}}{\partial n^{j_s}} - \sum_{t=1}^{k-\mu_i} c_{ist} \frac{\partial^{i_t}}{\partial n^{i_t}},$$

where $c_{ist} \in L^\infty(\partial\Omega)$. Let us denote by $V = \{v \in W^{k,2}(\Omega),\ B_{is}v = 0,$ $i = 1,2,\dots,\kappa,\ s = 1,2,\dots,\mu_i\}$, and by $\overline{V} = \{v \in W^{k,2}(\Omega), \overline{v} \in V\}$.

– A normal Banach space Q ($\overline{C_0^\infty(\Omega)} = Q$) such that $V \subset Q$ algebraically and topologically. (3.15f)

– $f \in Q'$; on Q', we define the involution by $\overline{f}v = \overline{f\overline{v}}$. (3.15g)

– Functionals $g_{it} \in \overline{V'}$, $i = 1,2,\dots,\kappa$, $t = 1,2,\dots,k-\mu_i$, such that $g_{it}v = 0$ if $\partial^{i_t}v/\partial n^{i_t} = 0$ on Γ_i. We define $\overline{g}v = \overline{g\overline{v}}$; if $\overline{V} = W_0^{k,2}$, these functionals are not needed. (3.15i)

We have, algebraically and topologically, $Q' \subset \mathscr{D}'(\Omega)$. We always consider f as a distribution and we shall write, by the notation of 2.1.1, $fv = \langle v, f\rangle$.

This notation will be used also for $g_{it} \in \overline{V'}$; then we shall write:

$$g_{it}v = \left\langle \frac{\partial^{i_t} v}{\partial n^{i_t}}, g_{it} \right\rangle_{\partial\Omega}.$$

We shall use the notation $g = \sum_{i=1}^{\kappa} \sum_{t=1}^{k-\mu_i} g_{it}$. A function $u \in W^{k,2}$ is a *solution of the boundary value problem*, if:

– $u - u_0 \in V$; (3.16a)

– for every $v \in V$, $A(u,v) + a(u,v) \equiv ((v,u)) = \langle v, \overline{f}\rangle + \overline{g}v$. (3.16b)

[3]For almost every $y \in \partial\Omega$ there exists a neighborhood of y where $\partial\Omega$ is described, in local coordinates, by a function from $C^{k,1}(\Delta)$.

According to (3.15a)–(3.15f), the *adjoint operator* A^* is given by:

$$A^* = \sum_{|i|,|j|\leq k} (-1)^{|i|} D^i(\overline{a}_{ji} D^j),$$

the associated sesquilinear form of this adjoint operator is defined by $\overline{A^*(v,u)} = A(u,v)$. If we set $a^*(v,u) = \overline{a(u,v)}$, then by definition $((v,u))^* = A^*(v,u) + a^*(v,u)$; this also defines the *adjoint problem*.

The notion of $V/P-$*ellipticity* is the same as above, cf. the definition in 1.3.2 (if $P = \{0\}$, we have V-ellipticity). We generalize immediately Theorems 1.3.1 and 1.3.2:

Theorem 2.1. *Given the boundary value problem with the sesquilinear V/P-elliptic form $((\tilde{v},\tilde{u}))$, then a necessary and sufficient condition for the existence of a solution u reads as*

$$v \in P \Longrightarrow \langle v, \overline{f} \rangle + \overline{g}v = 0. \tag{3.17}$$

The solution is determined modulo a polynomial $p \in P$; moreover we have:

$$|u|_{W^{k,2}(\Omega)} \leq c(|f|_{Q'} + |u_0|_{W^{k,2}(\Omega)} + |g|_{V'}) \tag{3.18}$$

for a well chosen solution. This choice can be made unique if we impose:

$$p \in P \Longrightarrow (p,u) = 0.^4$$

Remark 2.3. If $u \in W^{k,2}(\Omega)$, then $Au \in \mathscr{D}'(\Omega)$. Moreover, if f is in $\mathscr{D}'(\Omega)$, then (3.16b) implies, in the distribution sense:

$$\varphi \in \mathscr{D}(\Omega) \Longrightarrow A(\varphi,u) = \langle \varphi, \overline{f} \rangle = \langle \varphi, \overline{Au} \rangle \Longrightarrow Au = f.$$

The boundary conditions $B_{is}u = h_{is}$ on Γ_i, $i = 1,\ldots,\kappa$, $s = 1,2,\ldots,\mu_i$, with $h_{is} = B_{is}u_0$ are *stable conditions*, cf. 1.2.6. Formally, as in Chap. 1, we obtain *nonstable boundary conditions*, $C_{it}u = g_{it}$ on Γ_i, $t = 1,2,\ldots,\kappa$, $i = 1,2,\ldots,k-\mu_i$.

Hereafter we shall simply say that u solves the problem $Au = f$ in Ω, $B_{is}u = h_{is}$, $C_{it}u = g_{it}$ on $\partial\Omega$.

If $\kappa = 1$, $B_s = \partial^{s-1}/\partial n^{s-1}$, $s = 1,2,\ldots,k$, we have the *Dirichlet problem*. According to Theorems 2.4.10, 2.4.12, 2.4.13, 2.4.14, we have, maybe with some restrictions, $W_0^{k,2}(\Omega) = V$. If we define directly $V = W_0^{k,2}(\Omega)$, we need no hypotheses on Ω. We must, of course, verify the V-ellipticity, which is more simple if Ω is bounded. If Ω is unbounded, cf. J. Deny, J.L. Lions [1], J.L. Lions [5], L.D. Kudriavcev [1], etc.

[4] We recall: $(p,u) = \int_\Omega p\overline{u}\,\mathrm{d}x$.

The generalization of V can be considered in other problems; for instance let us consider the mixed problem with $k = 1$, $\kappa = 2$. On Γ_1, put $v = 0$, on Γ_2 we do not prescribe any stable condition. Let us consider the space $\mathscr{V} \subset C^\infty(\overline{\Omega})$ of functions equal zero on Γ_1 or in a neighborhood of Γ_1. Often we have $\overline{\mathscr{V}} = V$; on the other hand we can define V for cases sufficiently general. We have in general the following problem:

Problem 2.1. In this case we characterize Ω, B_{is}, etc. or we give general enough sufficient conditions to imply the existence of $\mathscr{V} \subset C^\infty(\overline{\Omega})$ such that $\overline{\mathscr{V}} = V$.

It is clear that we can construct closed subspaces of $W^{k,2}(\Omega)$ not of (3.15e) type. If we omit the case $\kappa = \infty$ (less interesting), we can define, for instance, if $k = 2$, $V = \{v \in W^{2,2}(\Omega),\ \partial v/\partial x_1 = 0 \text{ on } \partial\Omega)$. If $N = 2$ with Ω the unit disc, we have $\partial v/\partial x_1 = (\partial v/\partial r)\cos\varphi - (\partial v/\partial\varphi)\sin\varphi \Longrightarrow \partial v/\partial r = \tan\varphi(\partial v/\partial\varphi)$, which corresponds to the case where the coefficients of B_{is} are unbounded. Problems of this type are investigated in Minakshi-Sundaram [1].

Let now Ω be a cube with the face Γ_1 in the plane $x_3 = 0$, $\Gamma_2 = \partial\Omega - \Gamma_1, V = \{v \in W^{2,2}(\Omega), \partial v/\partial x_2 - \partial v/\partial x_1 = 0 \text{ on } \Gamma_1\}$. We can solve the boundary value problem; in this case the stable condition is characteristic on Γ_1. We cannot apply the method of M. Schechter [2, 4, 5], cf. also Chap. 4.

Remark 2.4. If $V = W_0^{k,2}(\Omega)$, $W_0^{k,2}(\Omega)$ can be taken as Q, then $Q' = W^{-k,2}(\Omega)$. Q is the smallest normal Banach space, such that $V \subset Q$ algebraically and topologically. In general it is a real problem (which is, maybe, not well posed).

Problem 2.2. We must construct for each V a normal Banach space Q, such that $V \subset Q$ algebraically and topologically with $Q \subset Q_1$ algebraically and topologically if Q_1 is an another Banach space of the same type. For this problem cf. also J.L. Lions [13].

Proposition 2.5. *Let $2k < N$ and $Q = L^q(\Omega)$, $1/q = 1/2 - k/N$. Let us put $Q' = L^p(\Omega)$, $1/p + 1/q = 1$. Then $L^q(\Omega)$ is a Banach space of the type (3.15f), with dual $L^p(\Omega)$.*

The proposition is a consequence of Theorem 2.3.6.

Proposition 2.6. *Let $2k = N$ and $Q = L^q(\Omega)$, $q > 1$, $Q' = L^p(\Omega)$, $1/p + 1/q = 1$. Then $L^q(\Omega)$ is a Banach space of the type (3.15f), with dual $L^p(\Omega)$.*

This is a direct consequence of Theorem 2.3.7.

Proposition 2.7. *Let $2k > N$ and l an integer such that $l < k - N/2 \le l + 1$. Let us put*

$$\mu = k - \frac{1}{2}N - l, \text{ if } k - \frac{1}{2}N - l < 1,$$

$$\mu < 1, \text{ if } k - \frac{1}{2}N - l = 1.$$

Let $Q = C_0^{l,\mu}(\overline{\Omega})$ where $C_0^{l,\mu}(\overline{\Omega}) = \overline{C_0^\infty(\Omega)}$. Then Q is a Banach space of the type (3.15f).

In fact, it is sufficient to apply Theorem 2.3.9.

Example 2.3. Let $2k > N$. Then if $f \in Q'$, we can take $\langle v, f\rangle = v(x_0) = \delta_{x_0} v$, $x_0 \in \overline{\Omega}$, where δ_{x_0} is the Dirac measure concentrated to the point x_0. The solution of the homogeneous problem is the Green kernel. (We will come back to these questions in Chap. 4.)

In fact, it is sufficient to use Theorem 2.3.9.

Proposition 2.8. *Let $2k < N$, $\Lambda \subset \partial\Omega_1$ with $\Omega_1 \in \mathfrak{N}^{0,1}$, $\Omega_1 \subset \Omega$, $\Lambda \subset \Omega$. Let $Z \in [W_0^{k,2}(\Omega) \to L^q(\Lambda)]$ such that $u \in C^\infty(\overline{\Omega}) \Longrightarrow Zu = u$. We define Q as the closure of $W^{k,2}(\Omega)$ with respect to the norm $|u|_{L^2(\Omega)} + |Zu|_{L^q(\Lambda)}$, where $1/q = 1/2 - [1/(N-1)].[(2k-1)/2]$. Then Q is a Banach space of the type (3.15f). If $f \in Q'$, we have $\langle v, f\rangle = \int_\Omega v f_1 \,\mathrm{d}x + \int_\Lambda v g_1 \,\mathrm{d}S$, with $f_1 \in L^2(\Omega)$, $g_1 \in L^p(\Lambda)$, $1/p + 1/q = 1$, and conversely.*

Proof. By Theorem 2.4.7, Q satisfies the corresponding conditions: Indeed, if $v \in Q$, there exists a sequence $v_n \in W^{k,2}(\Omega)$ such that $\lim_{n\to\infty} v_n = v$ in Q. We can also construct a new sequence $w_n \in C_0^\infty(\Omega)$, such that $\lim_{n\to\infty} |v_n - w_n|_Q = 0$, so Q is normal. Let $Q_1 = \{v \in Q, |v|_{L^q(\Lambda)} = 0\}$, $Q_2 = \{v \in Q, |v|_{L^2(\Omega)} = 0\}$. We can obviously identify Q_1 and $L^2(\Omega)$, Q_2 and $L^q(\Lambda)$. We can write $Q = Q_1 \dot{+} Q_2$ and $v = v_1 + v_2$, $v_1 \in Q_1$, $v_2 \in Q_2$ are uniquely defined. Now we have:

$$\langle v, f\rangle = \langle v_1, f\rangle + \langle v_2, f\rangle = \int_\Omega v_1 f_1 \,\mathrm{d}x + \int_\Lambda v_2 g_1 \,\mathrm{d}S.$$

□

Remark 2.5. In some sense, in Proposition 2.8, $\int_\Lambda v_2 g_1 \,\mathrm{d}S$ represents a measure concentrated on Λ.

Example 2.4. Define

$$Q = \{v \in L^2(\Omega), \int_\Omega \sum_{i=1}^N |\frac{\partial v}{\partial x_i}|^2 \rho \,\mathrm{d}x < \infty\},$$

where $\rho(x)$ is the distance between x and $\partial\Omega$. If $\Omega \in \mathfrak{N}^{0,1}$, then we have $\overline{C_0^\infty(\Omega)} = Q$; cf. also E.T. Poulsen [1]; Q is a space as defined in (3.15f).

Example 2.5. Let $Q = W^{1/2,2}(\Omega)$. We have $\overline{C_0^\infty(\Omega)} = W^{1/2,2}(\Omega)$, if $\Omega \in \mathfrak{N}^{0,1}$, cf. J. Nečas [9], and E. Magenes, J.L. Lions [4]. According to (3.15f), Q is a normal space.

3.2.4 Remarks

Proposition 2.9. *Let $2k < N$ and q, p as in Proposition 2.8, where we replace k by $k - i_t$. Let $g \in L^p(\Gamma_i)$. Then we can take g_{it} defined by:*

$$\left\langle \frac{\partial^{i_t} v}{\partial n^{i_t}}, g_{it} \right\rangle_{\partial\Omega} = \int_{\Gamma_i} \frac{\partial^{i_t} v}{\partial n^{i_t}} g \, \mathrm{d}S.$$

This proposition is a consequence of Theorem 2.4.7.

If $\Omega \in \mathfrak{N}^{k-1,1}$, we denote by $W^{-k+i+1/2,2}(\partial\Omega)$ the dual of $W^{k-i-1/2,2}(\partial\Omega)$, $i = 0, 1, \ldots, k-1$.

Due to Theorem 2.5.5, we have:

Proposition 2.10. *Let $\Omega \in \mathfrak{N}^{k-1,1}$, $\kappa = 1$, $g \in W^{-k+i_t+1/2,2}(\partial\Omega)$. Then we can take for g_{it} in (3.15i):*

$$\left\langle \frac{\partial^{i_t} v}{\partial n^{i_t}}, g_{it} \right\rangle_{\partial\Omega} = \left\langle \frac{\partial^{i_t} v}{\partial n^{i_t}}, g \right\rangle_{\partial\Omega}.$$

We have proved the existence of the solution of the boundary value problem associated with the differential form $((v,u))$. It is easy to see that the proof can be adapted to the case of integrodifferential equations, with very general boundary conditions.

We can immediately make some generalizations, for instance: Let

$$V = \{v \in L^2(\Omega), \triangle v \in L^2(\Omega)\}.$$

We endow the Hilbert space V with the scalar product $[v,u] = (v,u) + (\triangle v, \triangle u)$ and set $((v,u)) = [v,u]$. Let be $Q = L^2(\Omega)$. We have $V \subset Q$ algebraically and topologically, Q is normal.

Let $f \in L^2(\Omega)$; we look for $u \in V$ such that $v \in V \Longrightarrow ((v,u)) = (f,v)$. Formally this corresponds to the problem $\triangle^2 u + u = f$ in Ω, $\triangle u = 0$ and $(\partial/\partial n)\triangle u = 0$ on $\partial\Omega$; cf. Example 1.2.19a.

We can generalize this problem in the following direction: Let us define the spaces $H(A,\Omega)$, where $A = \{A_1, A_2, \ldots, A_\nu\}$ is a system of linear differential operators with constant coefficients; $H(A,\Omega)$ is the space of functions $u \in L^2(\Omega)$ such that $A_i u \in L^2(\Omega)$, $i = 1, 2, \ldots, \nu$. On $H(A,\Omega)$ we introduce the scalar product $[v,u] = (v,u) + \sum_{i=1}^{\nu} (A_i v, A_i u)$; cf J.L. Lions [1, 2, 4] and also Sect. 7 in this chapter. In this space we can consider boundary value problems for the operators $A = \sum_{i,j=1}^{\nu} A_i^*(g_{ij} A_i)$, where $g_{ij} \in L^\infty(\Omega)$, and A_i^* is defined by $\langle \varphi, A_i^* f\rangle = \langle \overline{A_i \varphi}, f\rangle$, $\varphi \in C_0^\infty(\Omega)$, $f \in L^2(\Omega)$; cf. E. Magenes, G. Stampacchia [1] and the papers by S.M. Nikolskii [1, 3] and M. Pagni [1].

The boundary value problem can be generalized in another direction: let V be a Hilbert space, such that $C_0^\infty(\Omega) \subset V \subset W^{k,2}(\Omega)$ algebraically and topologically, V is not necessarily closed in $W^{k,2}(\Omega)$. Here we give an example (the details can be found in J.L. Lions [4]):

Example 2.6. Let $\Omega = \mathbb{R}_+^N$, $V = \{v \in W^{1,2}(\mathbb{R}_+^N), v(x',0) \in W^{1,2}(\mathbb{R}^{N-1})\}$; in this space we introduce the scalar product defined by:

$$[v,u] = \int_{\mathbb{R}_+^N} \Big(\sum_{i=1}^N \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} + v\overline{u} \Big) \,\mathrm{d}x + \int_{\mathbb{R}^{N-1}} \Big(\sum_{i=1}^{N-1} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \Big) \,\mathrm{d}x'.$$

Let $f \in L^2(\mathbb{R}_+^N)$; we look for $u \in V$ such that for all $v \in V$: $[v,u] = (v,f)$. The function u solves formally the problem $-\triangle u + u = f$ in $\mathbb{R}_+^N$, $\partial u/\partial n - \triangle_{x'} u = 0$ for $x_N = 0$, where $\triangle_{x'} = \sum_{i=1}^{N-1} \partial^2/\partial x_i^2$.

Another generalization due to the author [7, 6], and A. Kufner [3], will be considered in Chap. 6; cf. also M.I. Vishik [4], H. Morel [1].

Now we give another example strongly connected with Example 2.6 and with a remark given above:

Example 2.7. Let $\Omega \in \mathfrak{N}^{0,1}$, $N \geq 3$. We look for the solution of the problem $-\triangle u + u = f$, $f \in L^2(\Omega)$ in Ω, $\partial u/\partial n + au = 0$ on $\partial\Omega$ with $a \geq 0$, $\int_{\partial\Omega} |a|^N \,\mathrm{d}S = \infty$, $\int_{\partial\Omega} |a| \,\mathrm{d}S < \infty$. Let $V = \overline{C^\infty(\Omega)}$ with the norm defined by:

$$\int_\Omega \Big(\sum_{i=1}^N \Big| \frac{\partial v}{\partial x_i} \Big|^2 + |v|^2 \Big) \,\mathrm{d}x + \int_{\partial\Omega} a|v|^2 \,\mathrm{d}S.$$

It follows from Theorem 2.4.2 that in general we have $V \neq W^{1,2}(\Omega)$.

3.2.5 The Differential Operators

So far the given operator A, together with its decomposition, could be considered as a mapping from $[W^{k,2}(\Omega) \to \mathscr{D}'(\Omega)]$. We can define the boundary value problems in another form (very close to our framework), cf. J.L. Lions [1, 2, 4]; we will consider only the homogenous case (i.e. the case where $h_{is} = g_{it} = 0$ on $\partial\Omega$).

Let $V, Q, ((v,u))$ be given as in 3.2.3. Let $D(A)$ be the space of $u \in V$ such that the linear form in v, $((v,u))$, is continuous in the topology induced by Q.

Proposition 2.11. *There exists one and only one linear operator A, in general unbounded, from $D(A)$ into Q', such that*

$$v \in V \Longrightarrow ((v,u)) = \langle v, \overline{Au} \rangle. \tag{3.19}$$

Proof. Clearly, $u \in D(A)$ implies that $((v,u))$ is continuous with respect to v on V with the topology of Q; as V is dense in Q, $((v,u))$ can be extended by continuity to a linear form on Q.

□

Moreover, we justify the notation for the operator in Proposition 2.11 by the following:

Proposition 2.12. *Let be $u \in D(A)$. Then, in the sense of distributions,*

$$Au = \sum_{|i|,|j| \le k} (-1)^{|i|} D^i (a_{ij} D^j u)$$

Proof. We have for all $v \in C_0^\infty(\Omega) \subset V$:

$$\left\langle v, \sum_{|i|,|j| \le k} (-1)^{|i|} D^i (\overline{a}_{ij} D^j \overline{u}) \right\rangle = A(v,u) = ((v,u)) = \langle v, \overline{Au} \rangle.$$

□

Theorem 2.2. *Given V, Q, $((v,u))$ as in 3.2.3, $((v,u))$ V-elliptic, then A maps $D(A)$ onto Q', A is one-to-one; if $f \in Q'$, then*

$$|A^{-1} f|_{W^{k,2}(\Omega)} \le c|f|_{Q'} \tag{3.20}$$

Proof. It is clear that we have $Au = 0 \Longrightarrow ((u,u)) = \langle u, \overline{Au} \rangle = 0 \Longrightarrow u = 0$, using the V-ellipticity. Let $f \in Q'$. According to Theorem 2.1 we have the existence of $u \in V$ such that $v \in V \Longrightarrow ((v,u)) = \langle v, \overline{f} \rangle$. The V-ellipticity implies $|u|^2_{W^{k,2}(\Omega)} \le c_1|((u,u))| = c_1|\langle u, \overline{f} \rangle| \le c_1|f|_{Q'}|u|_Q \le c_2|f|_{Q'}|u|_{W^{k,2}(\Omega)}$; thus (3.20). □

The operator A^{-1} is called the *Green operator*.

Theorem 2.3. *The hypotheses are the same as in the previous theorem. Then $D(A)$ is dense in V and Q, and A is a closed operator.*

Proof. By contradiction: let us assume $\overline{D(A)} \neq V$, Z is a mapping from $[V \to V]$, defined for $v,u \in V$ by $((v,u)) = (v, Zu)_k$, cf. Lemma 1.3.1; Z is one-to-one and onto. But the hypotheses imply $Z(\overline{D(A)}) \neq V \Longrightarrow$ there exists $v \in V$, $v \neq 0$, such that $0 = (v, Zu)_k = ((v,u))$ if $u \in D(A)$. Then $0 = ((v,u)) = \langle v, \overline{Au} \rangle$ if $u \in D(A)$. But we have $A(D(A)) = Q' \Longrightarrow v = 0$, contradiction.

$D(A)$ is dense in V, V is dense in Q, hence $D(A)$ is dense in Q.

Let $\lim_{n\to\infty} u_n = u$ in V, $u_n \in D(A)$, $\lim_{n\to\infty} Au_n = f$ in Q'. But as $f \in Q'$, there exists $u^* \in D(A)$ such that $Au^* = f$. Using (3.20), $\lim_{n\to\infty} u_n = u^*$ in V, so $u = u^*$.

□

Corollary 2.1. *If $D(A)$ is equipped with the norm*

$$\left(|u|_V^2 + |Au|_{Q'}^2\right)^{1/2}, \tag{3.21}$$

then A is an isomorphism of $D(A)$ onto Q'.

Remark 2.6. If $V = W_0^{k,2}(\Omega) = Q$, then A is a continuous operator from V onto $W^{-k,2}(\Omega)$ and the norm (3.21) is equivalent to $|u|_{W^{k,2}(\Omega)}$.

Theorem 2.4. *Let $((v,u))$ be V-elliptic and $Q = L^2(\Omega)$. Then the operator A^*, defined in Proposition 2.11, for $((v,u))^*$, is the adjoint operator of A. If $((v,u))$ is an hermitian form, A is selfadjoint.*

Proof. If A_1 is the adjoint operator of A, we have $D(A^*) \subset D(A_1)$. Indeed, let $u \in D(A^*)$, then for $v \in D(A)$, $((v,u))^* = (v, A^*u) = \overline{((u,v))} = \overline{(u,Av)} = (Av,u)$; the result is a consequence of the density of $D(A)$ in $L^2(\Omega)$.

Let $u \in D(A_1)$. By Theorem 2.2, there exists precisely one $u^* \in D(A^*)$, such that $A^*u^* = A_1u$. Then we obtain for $v \in D(A)$ that $(Au,v) = (v,A_1u) = (v,A^*u^*) = ((v,u^*))^* = \overline{((u^*,v))} = \overline{(u^*,Av)} = (Av,u^*)$. But $A(D(A)) = L^2(\Omega)$, so $u = u^*$, and $u \in D(A^*)$; then $D(A_1) = D(A^*)$, we have proved $u \in D(A^*)$, this implies $A^*u = A_1u$. □

Remark 2.7. Using the same ideas we can formulate the nonhomogenous problem in the following form (cf. for instance E. Magenes, G. Stampacchia [1], R. Courant, D. Hilbert [1]): given a subspace of distributions, say K, such that $D(A) \subset K$ and that A can be extended to $K \to Q'$. Given $f \in Q', h \in K$, we look for $u \in K$ such that $u - h \in D(A)$, $Au = f$. The space K is obtained by the process used in Theorem 2.1, ($P = \{0\}$), as the set of solutions for all $f \in Q'$, $u_0 \in W^{k,2}(\Omega)$, $g_{it} \in \overline{V'}$. It is also the point of view adopted by the cited authors; nevertheless, in Chap. 4, we shall generalize the boundary conditions, in the classes of h_{is}, g_{it}, cf. G. Stampacchia [10], [11], G. Fichera [6, 7], G. Cimmino [1], B. Pini [1, 2]. For regular domains, which is the case in all the cited works, we shall give more important generalizations as in M.I. Vishik, S.L. Sobolev [1], J.L. Lions, E. Magenes [1–3, 5] in Chap. 4. Chaps. 5, 6 and a part of Chap. 7 will be devoted to direct methods in nonregular domains for generalized boundary conditions.

Exercise 2.1. Let Ω be a domain in $\mathbb{R}^3$ such that the origin is a point of Ω. Let us consider the operator $Au = -\triangle u + u/r^\alpha$. Find the upper bound of α such that Theorem 2.1 can be applied, cf. Remark 2.2 and Example 2.2. After that, use the ideas of Example 2.7 for other values of α.

Exercise 2.2. Let Ω be the half disc in $\mathbb{R}^2$: $x_1^2 + x_2^2 < 1$, $x_2 > 0$, and $l = (l_1, l_2)$ a nontangent vector. At the vertices of $\partial\Omega, l$ is rotated by $+\pi/2$. Applying formally Lemma 2.4 and constructing (3.14) we arrive at the problem $-\Delta u = f$ in Ω, $\frac{\partial u}{\partial l} = g$ on $\partial\Omega$.

Exercise 2.3. Let Ω be as in the previous exercise, $V = \{v \in W^{1,2}(\Omega), v = 0$ for $|x_1| \le 1, x_2 = 0\}$. Let $\mathscr{V} \subset C^\infty(\overline{\Omega})$ be the space of functions equal to 0 if $0 \le x_2 < \varepsilon$, ε depending on the function. Prove that $\overline{\mathscr{V}} = V$.

Remark 2.8. If the operator A in Proposition 2.11 is selfadjoint, we obtain again the selfadjoint extension of Friedrichs, given for a symmetric operator $\tilde{A}$ defined on a domain $\mathscr{V} \subset V$ dense in V; cf. F. Riesz, B. Sz. Nagy [1]. Using the spectral decomposition, we can define A^λ, $0 \le \lambda \le 1$, and $D(A^\lambda)$. We obtain $D(A^{1/2}) = V$, $D(A^0) = L^2(\Omega)$. If Ω and the coefficients are sufficiently smooth, $D(A)$ is a closed subspace in $W^{2k,2}(\Omega)$; cf. also Chap. 4. If $V = W^{k,2}(\Omega)$, we have for $0 \le \lambda \le 1/2$, and with hypotheses weak enough $D(A^\lambda) = W^{2k\lambda,2}(\Omega)$, cf. J.L. Lions [12], T. Kato [1].

We have not investigated the problems in the case that Ω is unbounded. For a general investigation cf. L. D. Kudriavcev [1]. The fundamental question is the V-ellipticity, if it is satisfied, we can proceed as in this section. It is necessary to choose "good" Q, g_{it}. For instance we can take $Q = \{u, \int_\Omega |u|^2 p \,\mathrm{d}x < \infty\}$ where p is a weight with a reasonably "good" behaviour at infinity.

3.3 The Fredholm Alternative

In this section we shall give some direct generalizations of results obtained in Sect. 1.1.6, Chap. 1; we do not recall the basic properties of the spectral theory for the case of a hermitian form $((v,u))$. The literature can be found in J.L. Lions [1].

A boundary value problem is given by the V-elliptic sesquilinear form $((v,u))$. Let $N \subset \overline{V'}$ be the subspace of nonstable conditions, that is to say the closure in V' of the set $g = \sum_{i=1}^{\kappa} \sum_{t=1}^{k-\mu_i} g_{it}$, cf. (3.15i). We define the Green operator $G \in [Q' \times W^{k,2}(\Omega) \times N \to W^{k,2}(\Omega)]$, taking $u = G(f, u_0, g)$, where u is the solution of the problem.

Let $Q \subset L^2(\Omega) \subset Q'$. We write $G(f,0,0) = Gf$.

According to Proposition 1.5.1, G is a compact mapping from V to V (by definition, $\Omega \in \mathfrak{N}^{0,1}$).

Proposition 3.1. *Let $(f, u_0, g) \in Q' \times W^{k,2}(\Omega) \times N$. Then the boundary value problem has a solution u for the operator $A - \lambda$ if and only if*

$$u - \lambda G u = G(f, u_0, g). \tag{3.22}$$

Proof. Indeed, we must have $u - u_0 \in V$, and for every $v \in V$

$$((v,u)) = \overline{\lambda}(v,u) + \langle v, f\rangle + \overline{g}v \Longrightarrow u = G(\lambda u + f, u_0, g) = \lambda G u + G(f, u_0, g).$$

□

We use in $W^{k,2}(\Omega)$ the scalar product (1.2.3), with the notation $(v,u)_k$.

By Theorem 1.1.4, the operator G is compact from $W^{k,2}(\Omega)$ into $W^{k,2}(\Omega)$, hence (cf. F. Riesz, B.S. Nagy [1]) we have:

Proposition 3.2. *The equation (3.22) has for every (f,u_0,g) a unique solution, except for a denumerable set of values of λ, which are eigenvalues. If λ is such an eigenvalue, there exist functions $u_1,u_2,\dots,u_\nu \in V$, which form a linearly independent basis of eigenfunctions. If G^* is adjoint of G (with respect to the scalar product $(v,u)_k$ in V), we have:*

$$w - \overline{\lambda} G^* w = 0 \tag{3.23}$$

for $w_1,w_2,\dots,w_\nu$ elements of the basis of eigenfunctions. (3.22) has a solution if and only if

$$(w_i, -u_0 + G(f+\lambda u_0, u_0, g)) = 0, \quad i = 1,2,\dots,\nu. \tag{3.24}$$

In this case, the solution of (3.22) is unique modulo a linear combination of eigenfunctions $u_i, i = 1,2,\dots,\nu$.

Indeed: The equation (3.22) can be written as $u - u_0 - \lambda G(u-u_0) = -u_0 + G(f+\lambda u_0,u_0,g)$; $u - u_0$ is a solution of (3.22), the right hand side of (3.22), $-u_0 + G(f+\lambda u_0,u_0,g) \in V$ hence (3.24). □

Lemma 3.1. *Let G be the Green operator associated with $((v,u))$ and G_* the Green operator associated with $((v,u))^*$. Let Z be the one-to-one operator $V \to V$ such that for $v,u \in V$, $((v,u)) = (v,Zu)_k$. If Z^* is the adjoint of Z, we have:*

$$G_* = Z^{*-1} G^* Z^*. \tag{3.25}$$

Proof. Due to Lemma 1.3.1, Z exists and is unique. For all $v,u \in V$, we have $((G^*u,v)) = \overline{((v,G_*u))^*} = (u,v) = (Z^*G_*u,v)_k$; on the other hand $((u,Gv)) = (u,v) = (u,ZGv)_k$, hence $(ZG)^* = Z^*G_*$. □

Theorem 3.1. *Suppose we are given a boundary value problem with the V-elliptic sesquilinear form $((v,u))$. Then for all (f,u_0,g) and for the operator $A-\lambda$, the problem has an unique solution, except for an at most denumerable set of eigenvalues λ. If λ is an eigenvalue, there exist eigenfunctions $u_1,u_2,\dots,u_\nu$, associated with λ. The adjoint problem is given by the sesquilinear form $((v,u))^* - \lambda(v,u)$, corresponding to the operator $A^* - \overline{\lambda}$, $\overline{\lambda}$ is the eigenvalue for the adjoint problem; to $\overline{\lambda}$ correspond the eigenfunctions $v_1,v_2,\dots,v_\nu$, a basis of the space generated by the eigenfunctions. The original problem has a solution if and only if*

$$\langle v_i, \overline{f} \rangle + \overline{g} v_i - ((v_i,u_0)) + \overline{\lambda}(v_i,u_0) = 0, \quad i = 1,2,\dots,\nu. \tag{3.26}$$

In this case the solution is defined uniquely modulo a linear combination of eigen functions.

Proof. Let λ be an eigenvalue. According to Propositions 3.1, 3.2 and Lemma 3.1, using the same notation as before, we denote $v_i = Z^{*-1}w_i$ and obtain a basis of eigenfunctions of the problem

$$v - \overline{\lambda} G_* v = 0. \tag{3.27}$$

From (3.24) we can write:

$$(w_i, -u_0 + G(f + \lambda u_0, u_0, g))_k = (Z^* v_i, -u_0 + G(f + \lambda u_0, u_0, g))_k$$
$$= ((v_i, -u_0 + G(f + \lambda u_0, u_0, g))) = \langle v_i, \overline{f} \rangle + \overline{g} v_i - ((v_i, u_0)) + \overline{\lambda}(v_i, u_0).$$

□

3.3.1 Remarks

Corollary 3.1. *For the given boundary value problem, we assume that* $((v,u)) + \lambda(v,u)$ *is* V*-elliptic. Then for the original problem, we have the Fredholm alternative.*

Proposition 3.3. *Let* $P \subset P_{(k-1)} \cap V$, $((\tilde{v}, \tilde{u}))$ *a sesquilinear form on* $W^{k,2}/P \times W^{k,2}/P$. *For* $\tilde{v} \in V/P$, *suppose*

$$\text{Re } ((\tilde{v}, \tilde{v})) \geq c|\tilde{v}|^2_{W^{k,2}(\Omega)/P}. \tag{3.28}$$

Then $((\tilde{v}, \tilde{u}))$ *is* V/P*-elliptic, and for* $\lambda > 0$,

$$v \in V \Longrightarrow \text{Re } ((v,v)) + \lambda(v,v) \geq c|v|^2_{W^{k,2}(\Omega)}.$$

Remark 3.1. If (3.28) holds, we have Theorem 1.1 with $P \neq \{0\}$, using this theorem with $((v,u)) + \lambda(v,u)$ and Theorem 3.1 with "$-\lambda$".

Remark 3.2. Let B be a linear differential operator, $B \in [W^{k,2}(\Omega) \to Q']$, assume GB compact, $V \to V$, and assume $Q \subset L^2(\Omega) \subset Q'$. For a given $f, u_0, g, ((v,u))$ the Fredholm alternative is true for $Au = f + \lambda Bu$. The problem is the same as above for the equation:

$$u - \lambda GBu = G(f, u_0, g). \tag{3.29}$$

Examples are given in S.G. Mikhlin [2].

Exercise 3.1. Let $A = -\triangle$, $V = W_0^{k,2}(\Omega)$, $B = \text{div}$. Prove that GB is compact, $V \to V$.

3.4 The V-ellipticity

3.4.1 Coercivity

In this section, we shall give necessary and sufficient conditions implying the V-ellipticity. For references, cf. S.G. Mikhlin [3], E. Magenes, G. Stampacchia [1], J.L. Lions [1, 2, 4], N. Aronszajn [2], S. Agmon [1, 4], L. Gårding [1], M.I. Vishik [1], M.Schechter [1], P.C. Rosenbloom [1], L. Nirenberg [1], K. Yosida [1].

Following N. Aronszajn, we shall say that the sesquilinear form $((v,u))$ on $W^{k,2}(\Omega)\times W^{k,2}(\Omega)$ is *V-coercive*, if there exist $\lambda\geq 0, c>0$ such that for all $v\in V$, where V is the space defined in (3.15e),

$$\mathrm{Re}\,((v,v))+\lambda(v,v)\geq c|v|^2_{W^{k,2}(\Omega)}. \tag{3.30}$$

If $\lambda=0$ is admissible in (3.30), then $((v,u))$ is called *strongly V-coercive* (cf. M. Troisi [1]).

Theorem 4.1. *Given V, $A(v,u)$, suppose that for all $v\in V$ we have*

$$\mathrm{Re}\int_\Omega \sum_{|i|=|j|=k}\overline{a}_{ij}D^i v D^j\overline{v}\,\mathrm{d}x+\lambda_0\int_\Omega v\overline{v}\,\mathrm{d}x\geq c_1|v|^2_{W^{k,2}(\Omega)}.$$

Let us assume that the identity imbedding $W^{k,2}(\Omega)\to L^2(\Omega)$ is compact.[5] *Then if λ_0 is large enough, the form $A(v,u)$ is V-coercive.*

Proof. Let us denote

$$A_k(v,u)=\int_\Omega \sum_{|i|=|j|=k}\overline{a}_{ij}D^i v D^j\overline{u}\,\mathrm{d}x.$$

Using Lemma 2.6.1 and the classical inequality

$$2ab\leq \varepsilon^2a^2+(1/\varepsilon^2)b^2, \tag{3.30bis}$$

valid for $a\geq 0, b\geq 0, \varepsilon>0$, we can find $\lambda(c_1/2)$ such that

$$|\mathrm{Re}\,A(v,v)-\mathrm{Re}\,A_k(v,v)|\leq (c_1/2)|v|^2_{W^{k,2}(\Omega)}+\lambda(c_1/2)|v|^2_{L^2(\Omega)}.$$

We obtain:

$$\mathrm{Re}\,A(v,v)+\lambda(v,v)\geq \mathrm{Re}\,A_k(v,v)|+\lambda(v,v)-|\mathrm{Re}\,A(v,v)-\mathrm{Re}\,A_k(v,v)|,$$

[5] Just for this section V can be any closed space such that $W^{k,2}_0(\Omega)\subset V\subset W^{k,2}(\Omega)$.

and the result holds with $\lambda = \lambda_0 + \lambda(c_1/2)$. □

We immediately have:

Proposition 4.1. *Strong V-coercivity implies V-ellipticity.*

Proposition 4.2. *Let $A(v,u)$ be V-elliptic with*

$$|A(v,v)| \geq c_1 |v|^2_{W^{k,2}(\Omega)}. \tag{3.31}$$

Let $\Omega \in \mathfrak{N}^{0,1}$, and let $a(v,u)$ be a sesquilinear boundary form, such that for all $v \in V$, $|a(v,v)| \leq (c_1 - \varepsilon)|v|^2_{W^{k,2}(\Omega)}$, $0 < \varepsilon < c_1$. Then $((v,u))$ is V-elliptic.

Proposition 4.3. *Let $A(v,u)$ be strongly V-coercive and* $\operatorname{Re} a(v,v) \geq 0$ *for $v \in V$. Then $((v,v))$ is V-elliptic.*

Lemma 4.1. *Let B_1, B_2, Q be three Banach spaces, $B_1 \subset B_2$ algebraically and topologically, B_1 reflexive. Let Z be a compact mapping: $B_1 \to Q$. Then for every $\varepsilon > 0$, there exists a $\lambda(\varepsilon)$ such that for all $u \in B_1 \Longrightarrow |Zu|_Q \leq \varepsilon |u|_{B_1} + \lambda(\varepsilon)|u|_{B_2}$.*

Proof. By contradiction, let us assume that there exist $\varepsilon > 0$ and a sequence $u_n \in B_1, |u_n|_{B_1} = 1$ such that

$$|Zu_n|_Q > \varepsilon + n|u_n|_{B_1}. \tag{3.32}$$

Due to the property of reflexivity of B_1, there exists a subsequence of the sequence u_n denoted again u_n such that $\lim_{n\to\infty} u_n = u$ weakly in B_1, and by (3.32) $\lim_{n\to\infty} u_n = 0$ strongly in B_2. Hence $\lim_{n\to\infty} u_n = 0$ weakly in B_1, which implies $\lim_{n\to\infty} Zu_n = Zu = 0$ in Q, and this is a contradiction to (3.32). □

Theorem 4.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, $A(v,u)$ V-coercive, $a(v,u)$ a form given by (3.12). Then $((v,u))$ is V-coercive.*

Proof. We use Lemma 4.1, where $B_1 = W^{k,2}(\Omega)$, $B_2 = L^2(\Omega)$, $Q = L^2(\partial\Omega)$. Due to Theorem 2.6.2 and by Lemma 4.1 with $Z = D^\alpha$, $|\alpha| \leq k-1$, we obtain:

$$|a(v,v)| \leq \varepsilon |v|^2_{W^{k,2}(\Omega)} + \lambda_2(\varepsilon)|v|^2_{L^2(\Omega)}. \tag{3.33}$$

□

Remark 4.1. If $a(v,v)$ is as in (3.14), then the conclusion of Proposition 4.2 is true if $\Omega \in \mathfrak{N}^{2k,1}$ and the sum

$$\sum_{i=0}^{k-1} \sum_{|\alpha| \leq 2k-1-i} |b_{i\alpha}|_{C^{|\alpha|-k,1}(\partial\Omega)}$$

is sufficiently small. Nevertheless we have:

Theorem 4.3. *Let $\Omega \in \mathfrak{N}^{0,1}$, b_{ij} real functions in $C^{0,1}(\overline{\Omega})$, and*

$$a(v,u) = \int_{\partial\Omega} \sum_{i,j=1}^{N} (b_{ij} - b_{ji}) v \frac{\partial \overline{u}}{\partial x_j} n_i \, \mathrm{d}S.$$

Let $k = 1$, $A(v,u)$ V-coercive. Then $((v,u))$ is also V-coercive.

Proof. For $v \in C^\infty(\overline{\Omega})$ we have:

$$\begin{aligned} 2\mathrm{Re}\, a(v,v) &= \int_{\partial\Omega} \sum_{i,j=1}^{N} (b_{ij} - b_{ji}) v \frac{\partial \overline{v}}{\partial x_j} n_i \, \mathrm{d}S + \int_{\partial\Omega} \sum_{i,j=1}^{N} (b_{ij} - b_{ji}) \overline{v} \frac{\partial v}{\partial x_j} n_i \, \mathrm{d}S \\ &= \int_{\Omega} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (b_{ij} - b_{ji}) v \frac{\partial \overline{v}}{\partial x_j} \, \mathrm{d}x + \int_{\Omega} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (b_{ij} - b_{ji}) \overline{v} \frac{\partial v}{\partial x_j} \, \mathrm{d}x, \end{aligned}$$

and the result follows from this equality and from Theorem 4.1. □

Remark 4.2. If $\Omega \in \mathfrak{N}^{1,1}$, then Proposition 2.3 implies that all forms of the following type:

$$\int_{\partial\Omega} \sum_{i=1}^{N} v \frac{\partial \overline{u}}{\partial x_i} t_i \, \mathrm{d}S,$$

such that $\sum_{i=1}^{N} t_i n_i = 0$, $t_i \in C^{0,1}(\partial\Omega)$ are admissible due to Theorem 4.3 (cf. also J.L. Lions [3]). We solve the oblique derivative problem in the general case. The problem is the Neumann problem associated with the decomposition of the operator

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (a_i) + \sum_{j=1}^{N} b_j \frac{\partial}{\partial x_j} + c$$

in the following form:

$$\begin{aligned} A = &-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left((a_{ij} + b_{ij} - b_{ji}) \frac{\partial}{\partial x_j} \right) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (a_i) \\ &+ \sum_{i,j=1}^{N} \frac{\partial (b_{ij} - b_{ji})}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{j=1}^{N} b_j \frac{\partial}{\partial x_j} + c. \end{aligned}$$

Theorem 4.4. *Let $\Omega \in \mathfrak{N}^{2k,1}$, $a(v,u)$ given as in (3.14). Let us assume $b_{i\alpha} = 0$ for $|\alpha| = 2k - 1 - i$, $A(v,u)$ V-coercive. Then $((v,u))$ is V-coercive.*

Proof. If $k = 1$, we have $|\alpha| = 0$ and are in the hypotheses of Theorem 4.2. If $k \geq 2$, we follow the proof of Theorem 1.3.1. We have $i + |\alpha| - k \leq k - 2$ and use again Theorem 4.2. □

3.4.2 The Aronszajn Theorem

Proposition 4.4. *Let* $P \subset P_{(k-1)} \cap V$*,* Ω *bounded,* $((v,u))$ V/P*-elliptic. Then* $((v,u))$ *is* $W_0^{k,2}(\Omega)$*-elliptic.*

Proof. We have:

$$|((v,v))| \geq c \sum_{|\alpha|=k} |D^\alpha v|^2_{L^2(\Omega)}, \quad c > 0$$

for $v \in C_0^\infty(\Omega)$. By Theorem 1.1.1 we get the result. □

We recall that the operator A defined in Ω is said to be *elliptic at the point* x if for $\xi \in \mathbb{R}^N, \xi \neq 0$,

$$\sum_{|i|=|j|=k} \overline{a_{ij}}(x)\xi^{i+j} \neq 0, \quad (\xi^{i+j} = \xi_1^{i_1+j_1} \dots \xi_N^{i_N+j_N}). \tag{3.34}$$

The operator A is said to be *elliptic* in Ω, if it is elliptic almost everywhere in Ω.

The operator A is *uniformly elliptic* in Ω, if almost everywhere in Ω

$$\left| \sum_{|i|=|j|=k} \overline{a_{ij}}(x)\xi^{i+j} \right| \geq c\,|\xi|^{2k}, \quad c > 0. \tag{3.35}$$

Concerning the V-ellipticity, condition (3.34) is necessary (cf. also N. Aronszajn [2]).

Theorem 4.5. *Let* Ω *be bounded,* $x_0 \in \overline{\Omega}$ *a point of continuity of the coefficients* a_{ij}*,* $|i| = |j| = k$*. Then, if* $((v,u))$ *is* $W_0^{k,2}(\Omega)$*-elliptic, the operator* A *is elliptic at* x_0*.*

Proof. By contradiction: let us assume that there exists a vector $\xi \in \mathbb{R}^N$ such that

$$|\xi| = 1, \qquad \sum_{|i|=|j|=k} \overline{a_{ij}}(x)\xi^i\xi^j = 0.$$

For $v \in C_0^\infty(\Omega)$ we have:

$$|((v,v))| \geq c_1\,|v|^2_{W^{k,2}(\Omega)}, \quad c_1 > 0; \tag{3.35a}$$

to simplify the computations, here we take the norm:

$$|v|^2_{W^{k,2}(\Omega)} = \int_\Omega \sum_{|i|=k} \frac{|i|!}{i_1!i_2!\cdots i_N!} \left|D^i v\right|^2 \mathrm{d}x.$$

We can find a cube $C \subset \overline{C} \subset \Omega$, with center y in Ω, the faces parallel or orthogonal to ξ, meas $C = (2\delta)^N$ such that

$$x \in \overline{C} \Longrightarrow |\sum_{|i|=|j|=k} \overline{a}_{ij}(x)\xi^i\xi^j| < c_1/2.$$

Let $\delta > \delta' \geq (2/3)^{1/(N-1)}\delta$, C' the cube with center y "parallel" to C, meas $C' = (2\delta')^{2N}$, $\varphi \in C_0^\infty(C)$, $0 \leq \varphi(x) \leq 1$, $\varphi(x) = 1$ for $x \in C'$. Let us denote

$$u_\varepsilon(x) = \frac{\varphi(x)}{\sum_{i=1}^N (x_i - y_i)\xi_i - \mathrm{i}\varepsilon}, \quad \varepsilon > 0.$$

We have $u_\varepsilon \in C_0^\infty(\Omega)$ and

$$\begin{aligned}
\left|\int_\Omega \sum_{|i|,|j|\leq k} \overline{a}_{ij} D^i u_\varepsilon D^j \overline{u_\varepsilon}\,\mathrm{d}x\right| &\leq \left|(k!)^2 \int_C \sum_{|i|,|j|=k} \overline{a}_{ij}\xi^{i+j} |\sum_{i=1}^N (x_i - y_i)\xi_i - \mathrm{i}\,\varepsilon|^{-2(k+1)}\,\mathrm{d}x\right| \\
&\quad + c_2 \int_C \sum_{l=2}^{2k+1} \left|\sum_{i=1}^N (x_i - y_i)\xi_i - \mathrm{i}\,\varepsilon\right|^{-l} \mathrm{d}x \\
&\leq \frac{c_1}{2}(k!)^2 \int_C [(\sum_{i=1}^N (x_i - y_i)\xi_i)^2 + \varepsilon^2]^{-k-1}\,\mathrm{d}x \\
&\quad + c_2 \int_C \sum_{l=2}^{2k+1} [(\sum_{i=1}^N (x_i - y_i)\xi_i)^2 + \varepsilon^2]^{-l/2}\,\mathrm{d}x \\
&\leq \frac{c_1(k!)^2(2\delta)^{N-1}}{2\varepsilon^{2k+1}} \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \frac{\mathrm{d}s}{(1+s^2)^{k+1}} + \frac{c_3}{\varepsilon^{2k}}.
\end{aligned} \tag{3.36}$$

Now we have:

$$\begin{aligned}
|u_\varepsilon|^2_{W^{k,2}(\Omega)} &\geq \int_{C'} \sum_{|i|=k} \frac{k!}{i_1!i_2!\ldots i_N!} |D^i u_\varepsilon|^2\,\mathrm{d}x \\
&= (k!)^2 \sum_{|i|=k} \frac{k!}{i!}\xi^{2i} \int_{C'} \frac{dx}{[(\sum_{i=1}^N (x_i - y_i)\xi_i)^2 + \varepsilon^2]^{k+1}} \\
&= \frac{(k!)^2(2\delta')^{N-1}}{\varepsilon^{2k+1}} \int_{-\delta'/\varepsilon}^{\delta'/\varepsilon} \frac{ds}{(1+s^2)^{1+k}}.
\end{aligned} \tag{3.37}$$

Using (3.35a)–(3.37), we obtain:

$$C_1 \frac{(k!)^2(2\delta')^{N-1}}{\varepsilon^{2k+1}} \int_{-\delta'/\varepsilon}^{\delta'/\varepsilon} \frac{\mathrm{d}s}{(1+s^2)^{1+k}} \leq C_1 \frac{(k!)^2(2\delta)^{N-1}}{2\varepsilon^{2k+1}} \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \frac{\mathrm{d}s}{(1+s^2)^{1+k}} + \frac{c_3}{\varepsilon^{2k}},$$

and this is impossible if ε is sufficiently small. □

Corollary 4.1. *Let Ω be bounded, $a_{ij} \in C(\overline{\Omega})$, $|i| = |j| = k$, and $((v,u))$ $W_0^{k,2}(\Omega)$-elliptic. Then (3.35) holds.*

3.4.3 Strongly Elliptic Operators

According to M. I. Vishik, [1], L. Nirenberg [1], L.Gårding [1], an operator A is called *strongly elliptic at $x \in \Omega$,* if for all $\xi \in \mathbb{R}^N$:

$$\operatorname{Re} \sum_{|i|=|j|=k} \overline{a}_{ij}(x)\xi^i\xi^j \geq c(x)|\xi|^{2k}, \quad c(x) > 0. \tag{3.38}$$

The operator is called *uniformly strongly elliptic in Ω*, if almost everywhere:

$$\operatorname{Re} \sum_{|i|=|j|=k} \overline{a}_{ij}(x)\xi^i\xi^j \geq c|\xi|^{2k}, \quad c > 0. \tag{3.39}$$

The following theorem is due to L. Gårding [1], cf. also M.I. Vishik [1], K. Yosida [1].

Theorem 4.6. *Let Ω be bounded, $a_{ij} \in C(\overline{\Omega})$, $|i| = |j| = k$, A uniformly strongly elliptic. Then $A(v,u)$ is $W_0^{k,2}(\Omega)-$coercive.*

Proof. First we assume the coefficients a_{ij} constant, equal zero if $|i| + |j| < 2k$. Let $\varphi \in C_0^\infty(\Omega)$. We have:

$$\begin{aligned}\operatorname{Re} \int_\Omega \sum_{|i|=|j|=k} \overline{a}_{ij} D^i\varphi D^j\overline{\varphi}\,\mathrm{d}x &= \operatorname{Re} \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \sum_{|i|=|j|=k} \overline{a}_{ij}\xi^{i+j}|\hat{\varphi}|^2\,\mathrm{d}\xi \\ \geq \frac{c_1}{(2\pi)^N} \int_{\mathbb{R}^N} |\xi|^{2k}|\hat{\varphi}|^2\,\mathrm{d}\xi &= c_1 \int_\Omega \sum_{|i|=k} \frac{k!}{i!}|D^i\varphi|^2\,\mathrm{d}x,\end{aligned} \tag{3.40}$$

where

$$\hat{\varphi}(\xi) = \int_{\mathbb{R}^N} \varphi(x)e^{-\mathrm{i}(x,\xi)}\,\mathrm{d}x;$$

using the properties of Parseval's equality, (3.40) implies the conclusion in the particular case of constant coefficients.

Let us now consider, for $\rho > 0$, the function:

$$\omega(\rho) = \max_{|i|=|j|=k} \Big(\max_{\substack{x,y\in\overline{\Omega} \\ |x-y|\leq\rho}} |a_{ij}(x) - a_{ij}(y)| \Big)$$

and let $h_r \in C_0^\infty(\mathbb{R}^N)$ be given real functions such that

$$x \in \overline{\Omega} \Longrightarrow \sum_{r=1}^{R} h_r^2(x) = 1$$

with support in a set of diameter not bigger than ρ. If we denote $u_r = uh_r$, we have:

$$\begin{aligned}
&\operatorname{Re}\int_\Omega \sum_{|i|,|j|\le k} \overline{a}_{ij} D^i u D^j \overline{u}\,\mathrm{d}x \equiv \operatorname{Re} A(u,u) \\
&= \operatorname{Re} \sum_{|i|,|j|\le k} \sum_{r=1}^{R} \int_\Omega \overline{a}_{ij} D^i u_r D^j \overline{u}_r \,\mathrm{d}x + \operatorname{Re} \sum_{\substack{|i|,|j|\le k \\ |i|+|j|<2k}} \int_\Omega \overline{b}_{ij} D^i u D^j \overline{u}\,\mathrm{d}x,
\end{aligned} \tag{3.41}$$

where b_{ij} are measurable and bounded.

We denote:

$$A_k(v,u) = \int_\Omega \sum_{|i|=|j|=k} \overline{a}_{ij} D^i v D^j \overline{u}\,\mathrm{d}x,$$

and obtain:

$$\operatorname{Re} A(u,u) \ge \sum_{r=1}^{R} \operatorname{Re} A_k(u_r,u_r) - c_2 |u|_{W^{k-1,2}(\Omega)} |u|_{W^{k,2}(\Omega)}. \tag{3.42}$$

Now

$$\begin{aligned}
\operatorname{Re} A_k(u_r,u_r) = \operatorname{Re} &\int_\Omega \sum_{|i|=|j|=k} \overline{a}_{ij}({}_r x)) D^i u_r D^j \overline{u}_r \,\mathrm{d}x \\
&+ \operatorname{Re} \int_\Omega \sum_{|i|=|j|=k} (\overline{a}_{ij}(x) - \overline{a}_{ij}({}_r x)) D^i u_r D^j \overline{u}_r \,\mathrm{d}x,
\end{aligned} \tag{3.43}$$

where ${}_r x$ is in supp h_r. Due to (3.40)

$$\operatorname{Re} A_k(u_r,u_r) \ge \alpha |u_r|_k^2 - c_3 \omega(\rho) |u_r|_k^2, \quad \alpha > 0. \tag{3.44}$$

But

$$\sum_{r=1}^{R} |u_r|_k^2 \ge \beta |u|_k^2, \quad \beta > 0, \tag{3.45}$$

and by (3.42), (3.43), (3.45) we obtain for ρ sufficiently small,

$$\operatorname{Re} A(u,u) \ge \gamma |u|_k^2 - c_5 |u|_{k-1}^2, \quad \gamma > 0. \tag{3.46}$$

Using Lemma 2.6.1 with $B_1 = W_0^{k,2}(\Omega)$, $B_2 = W_0^{k-1,2}(\Omega)$, $B_3 = L^2(\Omega)$, the result follows from (3.46). □

We have also the converse theorem (for a second order operator see J.L. Lions [4]):

Theorem 4.7. *If Ω is a bounded domain and the coefficients a_{ij}, $|i| = |j| = k$, of the operator A are in $C(\overline{\Omega})$, and if $A(v,u)$ is $W_0^{k,2}(\Omega)$-coercive, then A is uniformly strongly elliptic.*

Proof. It is clear that it is sufficient to consider the operator

$$A_0 = \sum_{|i|=|j|=k} (-1)^{|i|} D^i (a_{ij}(x_0) D^j),$$

in a ball $B(x_0, r)$, with center $x_0 \in \overline{\Omega}$, and radius $r = r(x_0) > 0$; indeed if we choose $r'(x_0) \le r(x_0)$ small enough, the strong ellipticity is proved in the balls $B(x_0, r'(x_0))$, and, by Borel's theorem, the result follows.

Without loss of generality we choose x_0 as the origin of our space, since the property is invariant by translation. The existence of $r(x_0)$ and the coerciveness of the form $A_0(v,u)$ in $B(x_0,r)$ are consequences of the continuity of the coefficients a_{ij}, $|i| = |j| = k$, and of Lemma 2.6.1.

We claim that $A_0(v,u)$ is coercive in $\mathbb{R}^N$. To prove this let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and denote $\varphi_\mu(x) = \varphi(x/\mu)$. If μ is sufficiently small, $0 < \mu \le 1$, we have $\varphi_\mu \in C_0^\infty(B_r)$, $B_r = B(x_0, r)$, and thus

$$\begin{aligned}
&\operatorname{Re} \int_{B_r} \sum_{|i|=|j|=k} \overline{a}_{ij}(0) D^i \varphi_\mu D^j \overline{\varphi_\mu}\, \mathrm{d}x + \lambda \int_{B_r} |\varphi_\mu|^2\, \mathrm{d}x \\
&= \operatorname{Re} \mu^{N-2k} \int_{B_{r/\mu}} \sum_{|i|,|j|=k} \overline{a}_{ij}(0) D^i \varphi D^j \overline{\varphi}\, \mathrm{d}y + \lambda \mu^N \int_{B_r/\mu} |\varphi|^2\, \mathrm{d}y \\
&\ge c_1 \mu^{N-2k} \int_{B_{r/\mu}} \sum_{|i|=k} |D^i \varphi|^2\, \mathrm{d}y + c_1 \mu^N \int_{B_{r/\mu}} |\varphi|^2\, \mathrm{d}y, \quad y = x/\mu.
\end{aligned}$$

Then we get the inequality:

$$\begin{aligned}
&\operatorname{Re} \int_{\mathbb{R}^N} \sum_{|i|,|j|=k} \overline{a}_{ij}(0) D^i \varphi D^j \overline{\varphi}\, \mathrm{d}y + (\lambda+1) \int_{\mathbb{R}^N} |\varphi|^2\, \mathrm{d}y \\
&\ge c_1 \int_{\mathbb{R}^N} \sum_{|i|=k} |D^i \varphi|^2\, \mathrm{d}y + \int_{\mathbb{R}^N} |\varphi|^2\, \mathrm{d}y.
\end{aligned}$$

Using the Fourier transform, we obtain:

$$\begin{aligned}
&\operatorname{Re} \int_{\mathbb{R}^N} \sum_{|i|=|j|=k} \overline{a}_{ij}(0) D^i \varphi D^j \overline{\varphi}\, \mathrm{d}y + (\lambda+1) \int_{\mathbb{R}^N} |\varphi|^2\, \mathrm{d}y \\
&= \operatorname{Re} \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \sum_{|i|=|j|=k} \overline{a}_{ij}(0) \xi^{i+j} |\hat{\varphi}|^2\, \mathrm{d}\xi + \frac{(\lambda+1)}{(2\pi)^N} \int_{\mathbb{R}^N} |\hat{\varphi}|^2\, \mathrm{d}\xi \\
&= \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} (\operatorname{Re} \sum_{|i|,|j|=k} \overline{a}_{ij}(0) \xi^{i+j} + (\lambda+1)) |\hat{\varphi}|^2\, \mathrm{d}\xi \\
&\ge \frac{c_2}{(2\pi)^N} \int_{\mathbb{R}^N} (1 + |\xi|^2)^k |\hat{\varphi}|^2\, \mathrm{d}\xi.
\end{aligned} \tag{3.47}$$

Now we can take $\hat{\varphi}$ with arbitrarily small compact support: $\varphi \in W^{k,2}(\Omega)$; then using the property of density $\overline{C_0^\infty(\mathbb{R}^N)} = W^{k,2}(\mathbb{R}^N)$ – this is Proposition 2.2.6 - and (3.47), the result follows. □

Remark 4.3. If the coefficients $a_{ij}, |i|, |j| = k$ are real, (3.39) is equivalent to (3.35).

We have a stronger theorem:

Theorem 4.8. *Let $k = 1$, A uniformly elliptic, a_{ij}, $i, j = 1, 2, \ldots, N$, real. Then $A(v,u)$ is $W^{1,2}(\Omega)$-coercive.*

Proof. Let $\zeta_i = \xi_i + \mathrm{i}\,\eta_i$, $\xi, \eta \in \mathbb{R}^N$. We have:

$$\operatorname{Re} \sum_{i,j=1}^{N} a_{ij}\zeta_i\overline{\zeta}_j = \sum_{i,j=1}^{N} (a_{ij}\xi_i\xi_j + a_{ij}\eta_i\eta_j) \geq c \sum_{i=1}^{N} |\zeta_i|^2.$$

Hence

$$u \in W^{1,2}(\Omega) \Longrightarrow \operatorname{Re} \int_\Omega \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \overline{u}}{\partial x_j} dx \geq \int_\Omega \sum_{i=1^N} \Big|\frac{\partial u}{\partial x_i}\Big|^2 \mathrm{d}x,$$

and the result follows from (3.30). □

Remark 4.4. It is important to know whether a sesquilinear form $((v,u))$ is strongly coercive; moreover it is possible to have $((v,u))$ coercive with $\lambda < 0$: Indeed, let us assume $((v,u))$ hermitian, V-elliptic, $((v,v)) \geq 0$ (Ω smooth enough, for instance $\Omega \in \mathfrak{N}^0$). By Theorem 1.5.1, it is necessary and sufficient for the V-ellipticity of the form $((v,u)) + \lambda(v,u)$ to have $-\lambda_1 < \lambda$, where λ_1 is the first eigenvalue of the problem.

We recall the fact observed in Example 1.3.7; in other problems, as the Neumann problem, in general the strong ellipticity is not a sufficient condition.

3.4.4 Algebraic Conditions for the $W^{k,2}(\Omega)$-ellipticity

In the previous parts, we considered the algebraic conditions for the $W_0^{k,2}(\Omega)$-ellipticity, now we consider the more general case: the $W^{k,2}(\Omega)$-ellipticity. The case of "coercivity" conditions for "intermediary" problems, for domains with smooth boundary was investigated by S. Agmon [1]; in the case of formally positive forms, it is possible to find the first steps in N. Aronszajn [2], followed by M. Schechter [1, 3] and others. Briefly, let $A_i v$, $i = 1, 2, \ldots, h$, be operators defined by the following formula:

$$A_i v = \sum_{|\alpha| \leq k} a_{i\alpha} D^\alpha v, \quad a_{i\alpha} \in L^\infty(\Omega).$$

A *formally positive form* is:

$$\int_\Omega \sum_{i=1}^{h} |A_i v|^2 \, \mathrm{d}x.$$

In the last section of this chapter we shall study formally positive forms for systems of equations and we shall obtain new, simple enough, algebraic conditions for $W^{k,2}(\Omega)$-coercivity of these forms, for domains $\Omega \in \mathfrak{N}^{0,1}$.

We have:

Proposition 4.5. *Let Ω be a bounded domain, A an operator with constant coefficients which are equal to zero if $|i|+|j|<2k$. Then a necessary condition implying the $W^{k,2}(\Omega)/P_{(k-1)}$-ellipticity is*

$$|\sum_{|i|=|j|=k} \overline{a}_{ij}\zeta_i\overline{\zeta_j}| \geq c \sum_{|i|=k} |\zeta_i|^2, \quad c>0, \tag{3.48}$$

where ζ_i are arbitrary complex numbers.

Proof. By contradiction: let us assume that there exist ζ_i, $\sum_{|i|=k}|\zeta_i|^2 = 1$, such that $\sum_{|i|=|j|=k} \overline{a}_{ij}\zeta_i\overline{\zeta}_j = 0$.

Let

$$p(x) = \sum_{|i|=k} \frac{1}{i!}\zeta_i x^i \quad (i! = i_1!i_2!\dots i_N!). \tag{3.49}$$

We have:

$$|A(p,p)| = \mathrm{meas}(\Omega)| \sum_{|i|=|j|=k} \overline{a}_{ij}\zeta_i\overline{\zeta_j}| = 0,$$

on the other hand,

$$\int_\Omega \sum_{|i|=k} |D^i p|^2 \, \mathrm{d}x = \mu(\Omega) \neq 0,$$

giving a contradiction. □

Proposition 4.6. *Let Ω be a Nikodym open set, A such that $a_{ij}=0$ for $|i|+|j|<2k$. Let us assume that for complex numbers ζ_i, $|i|=k$*

$$\mathrm{Re} \sum_{|i|=|j|=k} \overline{a}_{ij}(x)\zeta_i\overline{\zeta_j} \geq c \sum_{|i|=k} |\zeta_i|^2, \quad c>0, \tag{3.50}$$

almost everywhere in Ω. Then $A(\tilde{v}, \tilde{u})$ is $W^{k,2}(\Omega)/P_{(k-1)}$-elliptic.

Proof. Indeed, if $u \in W^{k,2}(\Omega)$, we have:

$$\mathrm{Re}\, A(u,u) = \mathrm{Re} \int_\Omega \sum_{|i|=|j|=k} \overline{a}_{ij} D^i u D^j \overline{u} \, \mathrm{d}x \geq c \int_\Omega \sum_{|i|=k} |D^i u|^2 \, \mathrm{d}x. \tag{3.50bis}$$

Then the result follows from Theorem 2.7.7. □

We obtain immediately the converse proposition:

Proposition 4.7. *Let Ω be bounded, A given with constant coefficients a_{ij} for $|i| = |j| = k$, equal zero for $|i| + |j| < 2k$. Then if*

$$\operatorname{Re} A(\tilde{v}, \tilde{v}) \geq c_1 |v|^2_{W^{k,2}(\Omega)/P_{(k-1)}},$$

inequality (3.50 bis) holds.

Indeed: it suffices to choose p as in (3.49) for $v \in \tilde{v}$ the polynomial p as in (3.49). □

Proposition 4.8. *Let $\Omega \in \mathfrak{N}^0$, A satisfying (3.50). Then $A(v,u)$ is $W^{k,2}(\Omega)$-coercive.*

This results follows from (3.50) and Lemma 2.6.1.

For a more general theorem using condition (3.50) cf. Theorem 1.3.3.

Remark 4.5. Condition (1.4.3) in Theorem 1.3.3 is satisfied if for complex numbers ζ_i, $|i| \leq k$, we have almost everywhere:

$$\operatorname{Re} \sum_{\substack{|i|,|j| \leq k \\ |i|+|j|<2k}} \overline{a}_{ij} \zeta_i \overline{\zeta_j} \geq 0.$$

Remark 4.6. It is clear that if A is an operator as in (1.29), the conditions of type (3.38), (3.50), etc. can hold for aA, $|a| = 1$, a a complex number. Instead of A, we look at aA.

Exercise 4.1. Let $\Omega \in \mathfrak{N}^{0,1}$, V be given by (3.15e), X_i, $i = 1,2,\ldots,R$, open sets such that $\overline{\Omega} \subset \cup_{i=1}^R X_i$. Assume that for every such covering, there exists a partition of unity associated with the covering:

$$x \in \overline{\Omega} \Longrightarrow \sum_i^R \varphi_i(x) = 1,$$

with the property: $v \in V \Longrightarrow \varphi_i v \in V$, $i = 1,2,\ldots,R$. (This is true, if for instance $V = W_0^{k,2}(\Omega)$ or $V = W^{k,2}(\Omega)$.) The sesquilinear form $((v,u)) + \lambda(v,u)$ is called *locally elliptic (coercive)*, if for every $y \in \overline{\Omega}$, there exists a neighborhood $S(y)$ such that $((v,u)) + \lambda(v,u)$ is V-elliptic for v with compact support in $S(y) \cap \overline{\Omega}$. Prove that a necessary and sufficient condition for the V-ellipticity of $((v,u)) + \lambda(v,u), \lambda$ sufficiently large, is the local V-ellipticity.

Remark 4.7. It is possible to study with the same method (we have done this in $\mathbb{R}^N_+$) the same type of problems for unbounded domains; we obtain estimates of type (3.56). Cf. also S. Agmon, A. Douglis, L. Nirenberg [1].

3.5 The V-ellipticity of Forms $\int_\Omega Av A\bar{u} \mathrm{d}x$

3.5.1 Definition

In this section we adopt a point of view taken by M. Schechter [2, 4, 5]. We only consider the questions concerning the V-ellipticity. The problem of existence for $Au = f$ will be discussed in Chap. 4.

Let us consider an operator of order $2k$ of the following type:

$$Au = \sum_{|i| \le 2k} a_i D^i u; \tag{3.51}$$

we assume the coefficients a_i measurable and bounded. We have:

$$A^* u = \sum_{|i| \le 2k} (-1)^{|i|} D^i (\overline{a}_i u). \tag{3.52}$$

Let us consider the operator:

$$A^* A u = \sum_{|i|,|j| \le 2k} (-1)^{|i|} D^i (\overline{a}_i a_j D^j u) \tag{3.53}$$

and the associated sesquilinear form:

$$A^* A(v,u) = \int_\Omega \sum_{|i|,|j| \le 2k} (a_i \overline{a}_j D^i v D^j \overline{u}) dx = \int_\Omega Av \overline{Au} \, \mathrm{d}x.$$

In this section we consider only $\Omega \in \mathfrak{N}^{2k,1}$; this restriction of regularity for domains is very natural.

Let us consider boundary operators

$$B_s u = \frac{\partial^{j_s} u}{\partial n^{j_s}} - \sum_{|\alpha| \le j_s} c_{s\alpha} D^\alpha u,$$

of order $j_s \le 2k-1$, with local representation 1.3.2; we assume $0 \le j_s \le 2k-1$, $s = 1,2,\ldots,k$, $0 \le i_t \le 2k-1$, $t = 1,2,\ldots,k$, V given by the conditions $B_s v = 0$ on $\partial\Omega$, $s = 1,2,\ldots,k$. We consider the problem of coercivity corresponding to the sesquilinear form $\int_\Omega Av\overline{Au} \, \mathrm{d}x$ following the ideas in S. Agmon, A. Douglis, L. Nirenberg [1], F. Browder [3–5]; we shall simplify their method using the Fourier transform. Let us observe that our method works, using the multiplier theory (cf. N. Marcinkiewicz [1], S.G. Mikhlin [4], B. Malgrange [2]), in the case of *a priori* estimates

$$\int_\Omega |Av|^p \, \mathrm{d}x \ge c|v|^p_{W^{2k,p}(\Omega)} - \lambda |v|^p_{L^p(\Omega)}, \quad 1 < p < \infty. \tag{3.54}$$

With the same approach, we can probably obtain the inequality

$$|Av|_{C^{2k,\mu}(\overline{\Omega})} \geq c|v|_{C^{2k,\mu}(\overline{\Omega})} - \lambda|v|_{C^0(\overline{\Omega})}, \quad 0<\mu<1. \tag{3.55}$$

Using the following inequality:

$$\int_\Omega |Av|^2\,\mathrm{d}x \geq c|v|^2_{W^{k,2}(\Omega)} - \lambda|v|^2_{L^2(\Omega)}, \tag{3.56}$$

we shall study in Chap. 4 the existence of a function $u \in V$ such that for $f \in L^2$, $Au = f$. We obtain the same result as M. Schechter [2, 4], implying the fact that A is an isomorphism of V onto $L^2(\Omega)$ if the adjoint problem $A^*v = 0, B^*v = 0$ has only the trivial solution.

3.5.2 The Fundamental Solution

We give a simple construction of the "fundamental solution":

Theorem 5.1. *Let $C = (0,1)^N$, $l \geq 0$ an integer, $f \in W^{l,2}(C)$, $f = \partial f/\partial n = \ldots = \partial^{l-1}f/\partial n^{l-1} = 0$ on ∂C except the face in the plane $x_N = 0$. Let us denote by V the space of such functions and let $Au = \sum_{|i|=2k} a_i D^i u$ be an elliptic operator with constant coefficients. Then there exists $R \in [V \to W^{2k+l,2}(\mathbb{R}^N)]$ such that, in $\mathbb{R}^N_+$, the solution $u = Rf$ satisfies:*

$$Au = f \tag{3.57}$$

and

$$|u|_{W^{2k+1,2}(\mathbb{R}^N)} \leq c_1|f|_{L^2(C)}. \tag{3.58}$$

Let $m \geq 0$ be an integer; then the transformation R can be constructed in such a way that $u(x)x^\beta, |\beta| \leq m$, is in $W^{2k+l,2}(\mathbb{R}^N)$ and satisfies the following inequality:

$$|ux^\beta|_{W^{2k+l,2}(\mathbb{R}^N)} \leq c_2|f|_{W^{l,2}(C)}. \tag{3.59}$$

Proof. Let the numbers λ_r, $r = 1,2,\ldots,2k+l+m$, be solutions of the following linear system with determinant $\neq 0$:

$$\sum_{r=1}^{2k+l+m} \lambda_r(-r)^h = 1, \quad h = -l, -l+1, \ldots 2k+m-1. \tag{3.60}$$

We define f in $\mathbb{R}^N$ by

$$f(x', x_N) = \begin{cases} f(x', x_N), & x_N > 0, \\ -\sum_{r=1}^{2k+l+m} (\lambda_r/r) f(x', -x_N/r), & x_N < 0. \end{cases} \tag{3.61}$$

It is easy to see, as in Theorem 2.3.9, that we have constructed an extension of $f : V \to W^{l,2}(\mathbb{R}^N)$, such that $\operatorname{supp} f \subset \overline{P}$, $P = \{0 < x_i < 1, i = 1,2,\dots,N-1, -2k-l-m < x_N < 1\}$. Now let us put

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{(-1)^k \sum_{|i|=2k} a_i \xi^i}. \tag{3.62}$$

We have, for $|\beta| \leq m$,

$$|D^\beta \hat{u}|_{L^2(\mathbb{R}^N)} \leq c_3 |f|_{L^2(C)}. \tag{3.63}$$

Indeed, let us consider first $|\xi| \leq 1$. If $|\alpha| \leq m$, we get:

$$\begin{aligned}
|D^\alpha \hat{f}(\xi)| &= \left| \int_{\mathbb{R}^N_+} e^{-\mathrm{i}(x,\xi)} x^\alpha f(x)\,\mathrm{d}x - \int_{\mathbb{R}^N_-} x^\alpha \sum_{r=1}^{2k+l+m} \frac{\lambda_r}{r} f(x', -\frac{x_N}{r}) e^{-\mathrm{i}(x,\xi)}\,\mathrm{d}x \right| \\
&= \left| \int_{\mathbb{R}^N_+} x^\alpha f(x) e^{(-ix',\xi')} \left(e^{-\mathrm{i}x_N\xi_N} - \sum_{r=1}^{2k+l+m} (-r)^{\alpha_N} \lambda_r e^{irx_N\xi_N} \right) \mathrm{d}x \right| \\
&\leq \int_C |f(x)|\,\mathrm{d}x |\xi|^{2k+m-\alpha_N} \sup_{0<x_N<1} \frac{|e^{-\mathrm{i}x_N\xi_N} - \sum_{r=1}^{2k+l+m} (-r)^{\alpha_N} \lambda_r e^{irx_N\xi_N}|}{|\xi|^{2k+m-\alpha_N}} \\
&\leq |f|_{L^2(C)|} |\xi|^{2k+m-\alpha_N} \sup_{|\eta|<1} \frac{|e^{-\mathrm{i}\eta} - \sum_{r=1}^{2k+l+m} (-r)^{\alpha_N} \lambda_r e^{\mathrm{i}r\eta}|}{|\eta|^{2k+m-\alpha_N}} \\
&\leq c_4 |f|_{L^2(C)} |\xi|^{2k+m-\alpha_N},
\end{aligned} \tag{3.64}$$

the last inequality results from (3.61).

By ellipticity of A, it follows that

$$|\sum_{|i|=2k} a_i \xi^i| \geq c|\xi|^{2k}, \tag{3.65}$$

then, if $|\xi| \leq 1, |\beta| \leq m$:

$$|D^\alpha \hat{u}(\xi)| \leq c_5 |f|_{L^2(C)}; \tag{3.66}$$

and if $|\xi| > 1$, we have obviously

$$|D^\alpha \hat{u}(\xi)| \leq c_6 \sum_{|\alpha| \leq |\beta|} |D^\alpha \hat{f}(\xi)|. \tag{3.67}$$

Now using (3.66), (3.67), we obtain (3.63). We have for $x^\alpha u \in W^{2k+l,2}(\mathbb{R}^N)$, with $|\alpha| \leq m$,

$$|x^\alpha u|_{W^{2k+l,2}(\mathbb{R}^N)} \leq c_7 |f|_{W^{l,2}(C)}. \tag{3.68}$$

Starting from

$$|f|_{W_0^{l,2}(P)} \leq c_8 |f|_{W^{l,2}(C)}, \tag{3.69}$$

if $|\alpha| \leq m$, due to Lemma 2.3.5, we have:

$$\int_{\mathbb{R}^N} |D^\alpha \hat{f}(\xi)|^2 (1+|\xi|^2)^l \,\mathrm{d}\xi \leq c_9 |f|_{W^{l,2}(C)}; \tag{3.70}$$

finally (3.62), (3.63), (3.70) imply the inequality (3.68). We obviously have (3.57); (3.58) is a consequence of (3.68) if $l = 0$. □

3.5.3 A Lemma

Let A be an elliptic operator with constant coefficients:

$$Au = \sum_{|i|=2k} a_i D^i u.$$

We denote,

$$P(\xi', \tau) = \sum_{|i|=2k} a_i \xi'^{i'} \tau^{i_N}.$$

We shall assume that for every $\xi' \neq 0$, the polynomial $P(\xi', \tau)$ has exactly

$$k \text{ roots with positive imaginary part, } \tau_i(\xi'),\ i = 1, 2, \ldots, k. \tag{3.71}$$

We define:

$$M(\xi', \tau) = \prod_{i=1}^{k} (\tau - \tau_i(\xi')) = \sum_{p=0}^{k} a_p(\xi') \tau^{k-p},$$

and

$$M_j(\xi', \tau) = \sum_{p=0}^{j} a_p(\xi') \tau^{j-p}, \quad j = 0, 1, \ldots, k-1.$$

We have (cf. S. Agmon, A. Douglis, L. Nirenberg [1]):

Lemma 5.1. *Let γ be a closed, rectifiable, Jordan curve such that all roots of the polynomial P, τ_i, $i = 1, 2, \ldots, k$, with positive imaginary part are inside of the domain with boundary γ in $\mathbb{R}^2_+$; then*

$$\frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{M_{k-1-j}(\xi', \tau)}{M(\xi', \tau)} \tau^i \,\mathrm{d}\tau = \delta_{ij}, \quad 0 \leq i, j \leq k-1. \tag{3.72}$$

Proof. Let $j \geq i$; in this case the polynomial $M_{k-1-j}(\xi', \tau)\tau^i$ has the degree $k-1-j-i \leq k-1$. Let C_n be the circle with center at the origin of the complex plane and with radius n sufficiently large. By the Cauchy theorem, we have:

$$\frac{1}{2\pi \mathrm{i}}\int_{\gamma}\frac{M_{k-1-j}(\xi',\tau)}{M(\xi',\tau)}\tau^{i}\,\mathrm{d}\tau = \frac{1}{2\pi \mathrm{i}}\int_{C_n}\frac{M_{k-1-j}(\xi',\tau)}{M(\xi',\tau)}\tau^{i}\,\mathrm{d}\tau$$
$$= \lim_{n\to\infty}\frac{1}{2\pi \mathrm{i}}\int_{C_n}\frac{M_{k-1-j}(\xi',\tau)}{M(\xi',\tau)}\tau^{i}\,\mathrm{d}\tau = \lim_{n\to\infty}\frac{1}{2\pi \mathrm{i}}\int_{C_n}\frac{a_0(\xi')\tau^{k-1-j+i}}{a_0(\xi')\tau^{k}}\,d\tau = \delta_{ij}.$$

Let $j<i$; in this case,

$$\frac{M_{k-1-j}(\xi',\tau)}{M(\xi',\tau)}\tau^{i} = Q(\xi',\tau)+\frac{Z(\xi',\tau)}{M(\xi',\tau)},$$

where $Q(\xi',\tau)$ is a polynomial of degree $i+j-1, Z(\xi,\tau)$ is the polynomial of order $\leq i-1$, the remainder from the division; then by the same approach as used above, we have:

$$\frac{1}{2\pi \mathrm{i}}\int_{\gamma}\frac{Z(\xi',\tau)}{M(\xi',\tau)}\,\mathrm{d}\tau = 0.$$

□

3.5.4 A priori Estimates in $\mathbb{R}^N_+$

With the notation given in 3.5.3, let $B_s = \sum_{|\alpha|=m_s} b_{s\alpha}D^{\alpha}$ be boundary operators (on $\partial\mathbb{R}^N_+$) with constant coefficients, $s=1,2,\dots.k$, $m_s \leq 2k-1$. In general we do not assume B_s is in the canonical form (1.34a). We define $\alpha = \alpha'+\alpha''$, $\alpha'' = (0,0,\dots,0,\alpha_N)$, and the polynomials:

$$B_s(\xi',\tau) = \sum_{|\alpha|=m_s} b_{s\alpha}\xi'^{\alpha'}\tau^{\alpha_N}, \quad \xi'\neq 0, \tag{3.73}$$

$$\text{linearly independent modulo } M(\xi',\tau), \quad \xi'\neq 0^{6} \tag{3.74}$$

We have the following:

Lemma 5.2. *The operators B_s satisfy (3.74) if and only if the determinant*

$$\det b_{st}(\xi')\neq 0, \tag{3.75}$$

where

$$B'_s(\xi',\tau) = \sum_{t=1}^{k} b_{st}(\xi')\tau^{t-1}$$

is the remainder after the division of $B(\xi',\tau)$ by $M(\xi',\tau)$.

[6] This means that there do not exist constants c_s, $s=1,2,\dots,k$, $\sum|c_s|\neq 0$, and a polynomial $Q(\xi',\tau)$ such that $\sum_{s=1}^{k} c_s B_s(\xi',\tau) = Q(\xi',\tau)M(\xi',\tau)$.

Lemma 5.3. *Let $[b^{st}(\xi')]$ be the inverse matrix of $[b_{st}(\xi')]$, and let us set:*

$$N_j(\xi',\tau) = \sum_{i=1}^{k} b^{ij}(\xi')M_{k-i}(\xi',\tau).$$

Let γ be as in Lemma 5.1; then we have:

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{N_j(\xi',\tau)B_i(\xi',\tau)}{M(\xi',\tau)}\,\mathrm{d}\tau = \delta_{ij}. \tag{3.76}$$

Proof. We have,

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{N_j(\xi',\tau)B_i(\xi',\tau)}{M(\xi',\tau)}\,\mathrm{d}\tau \equiv \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{N_jB_i'}{M}\,d\tau = \sum_{s,t=1}^{k} b^{sj}b_{it}\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{M_{k-s}\tau^{t-1}}{M}\,d\tau$$

$$= \sum_{s,t=1}^{k} b^{sj}b_{it}\delta_{st} = \sum_{s=1}^{k} b^{sj}b_{is} = \delta_{ij}.$$

□

Theorem 5.2. *Let A, B_s be operators satisfying (3.71), (3.74), $x^\alpha u \in W^{2k+1,2}(\mathbb{R}^N_+)$, $|\alpha| \le m$, $m > N$, $Au = 0$ almost everywhere in $\mathbb{R}^N_+$. We denote by $G_s(x') = (B_s u)(x',0)$,*

$$\hat{G}_s(\xi') = \int_{\mathbb{R}^{N-1}} G_s(x')e^{-\mathrm{i}(x',\xi')}\,\mathrm{d}x'.$$

Then

$$|u|^2_{W^{2k,2}(\mathbb{R}^N_+)} \le c\sum_{s=1}^{k}\int_{\mathbb{R}^{N-1}} |\hat{G}(\xi')|^2(1+|\xi'|)^{2k-m_s-1/2}\,\mathrm{d}\xi' + \lambda|u|^2_{L^2(\mathbb{R}^N_+)}. \tag{3.77}$$

Proof. First of all we have $u \in W^{2k+1,1}(\mathbb{R}^N_+)$; this follows from the inequality

$$\int_{\mathbb{R}^N_+} |D^\alpha u|\,\mathrm{d}x \le \left(\int_{\mathbb{R}^N_+} |D^\alpha u|^2(1+|x|^2)^{m/2}\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^N}(1+|x|^2)^{-m/2}\,\mathrm{d}x\right)^{1/2}$$

where $|\alpha| \le 2k$. If $0 \le x_N < \infty, |\alpha| \le 2k$, we have the following inequalities which can be proved as in (1.8):

$$\begin{aligned}\int_{\mathbb{R}^{N-1}} |D^\alpha u(x',x_N)|\,\mathrm{d}x' \le c_1\int_{\mathbb{R}^{N-1}}\int_{x_N}^{x_N+1} |D^\alpha u(x',y)|\,\mathrm{d}x'\,\mathrm{d}y \\ + c_1\int_{\mathbb{R}^{N-1}}\int_{x_N}^{x_N+1}\left|\frac{\partial}{\partial x_N}D^\alpha u(x',y)\right|\,\mathrm{d}x'\,\mathrm{d}y.\end{aligned} \tag{3.78}$$

This inequality implies in the case $|\alpha| \le 2k$ that

$$\lim_{x_N\to\infty}\int_{\mathbb{R}^{N-1}} |D^\alpha u(x',x_N)|\,\mathrm{d}x' = 0. \tag{3.79}$$

We have also for $x_N \ge 0$ (if $x_N = 0$, then we take $h > 0$):

$$\lim_{h\to 0}\int_{\mathbb{R}^{N-1}} |D^\alpha u(x',x_N+h) - D^\alpha u(x',x_N)|\,\mathrm{d}x' = 0, \quad |\alpha| \le 2k. \tag{3.80}$$

Let us denote

$$U(\xi',x) = \int_{\mathbb{R}^{N-1}} e^{-\mathrm{i}(\xi',x')} u(x',x_N)dx'.$$

We have for $t = 1,2,\ldots,2k$, $x_N \ge 0$:

$$\frac{d^t U}{dx_N^t} = \int_{\mathbb{R}^{N-1}} e^{-\mathrm{i}(\xi',x')}\frac{\partial^t u(x',x_N)}{\partial x_N^t}\,\mathrm{d}x'.$$

To obtain this equality, it is sufficient to prove that (we use Proposition 2.2.6)

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{\partial^{t-1}u(x',x_N+h)}{\partial x_N^{t-1}} - \frac{\partial^{t-1}u(x',x_N)}{\partial x_N^{t-1}}\right) = \frac{\partial^t u(x',x_N)}{\partial x_N^t} \text{ in } L^1(\mathbb{R}^{N-1}).$$

Now we have, as $h \to 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}}\left|\frac{1}{h}\left(\frac{\partial^{t-1}u(x',x_N+h)}{\partial x_N^{t-1}} - \frac{\partial^{t-1}u(x',x_N)}{\partial x_N^{t-1}}\right) - \frac{\partial^t u(x',x_N)}{\partial x_N^t}\right|\mathrm{d}x' \\
&= \int_{\mathbb{R}^{N-1}}\left|\frac{1}{h}\int_{x_N}^{x_N+h}\left(\frac{\partial^t u(x',\tau)}{\partial x_N^t} - \frac{\partial^t u(x',x_N)}{\partial x_N^t}\right)\mathrm{d}\tau\right|\mathrm{d}x' \\
&\le \frac{1}{|h|}\int_{\mathbb{R}^{N-1}}\mathrm{d}x'\int_{x_N}^{x_N+h}\mathrm{d}\tau\int_{x_N}^{\tau}\left|\frac{\partial^{t+1}u(x',\sigma)}{\partial x_N^{t+1}}\right|\mathrm{d}\sigma \\
&= \frac{1}{|h|}\int_{\mathbb{R}^{N-1}}dx'\int_{x_N}^{x_N+h}\left|\frac{\partial^{t+1}u(x',\sigma)}{\partial x_N^{t+1}}\right|(x_N+h-\sigma)\,\mathrm{d}\sigma \\
&\le \int_{\mathbb{R}^{N-1}}\mathrm{d}x'\int_{x_N}^{x_N+h}\left|\frac{\partial^{t+1}u(x',\sigma)}{\partial x_N^{t+1}}\right|\mathrm{d}\sigma \to 0.
\end{aligned}$$

Now $U(\xi',x_N)$ is the solution of the following ordinary differential equation with constant coefficients:

$$\sum_{|i|=2k}(\mathrm{i})^{|i'|}\xi'^{i'}a_i\frac{d^{i_N}U}{dx_N^{i_N}} = 0, \quad x_N \ge 0. \tag{3.81}$$

If $x_N = 0$, then by (3.80) it follows:

$$\hat{G}_s(\xi') = \sum_{|\alpha|=m_s} b_{s\alpha}(\mathrm{i})^{|\alpha'|}\xi'^{\alpha'}\frac{d^{\alpha_N}U}{dx_N^{\alpha_N}}(\xi',0). \tag{3.82}$$

For each ξ', $U(\xi',x_n)$ and all its derivatives of order $\leq 2k$, are continuous functions for $x_N \geq 0$ and by (3.79), we obtain:

$$\lim_{x_N\to\infty}\frac{d^iU}{d^ix_N}(\xi',x_N), \quad i = 1,2,\dots,2k. \tag{3.83}$$

Now we claim that the solution of the problem (3.81)–(3.83) (for $i = 0$) with $\xi' \neq 0$, is unique. Indeed: let us consider the characteristic polynomial associated with (3.81):

$$\sum_{|i|=2k}(\mathrm{i})^{|i'|}\xi'^{i'}a_i\sigma^{i_N},$$

and put $\sigma = \mathrm{i}\,\tau$. We obtain $(-1)^kP(\xi',\tau)$ as in 3.5.3. Let $V(\xi',x_N)$ be a solution of the homogeneous problem; $V(\xi',x_N)$ is a linear combination of functions of the type $e^{\mathrm{i}\,\tau x_N}H(x_N)$, where H is a polynomial of degree equal to one less then the multiplicity of the root τ^* of the characteristic polynomial. According to (3.83), we see that only the roots with positive imaginary part are admissible. Let γ be a curve as in Lemma 5.1; we can find a polynomial Q of the variable τ of degree $\leq k-1$ such that

$$V(\xi',x_N) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{Q(\xi',\tau)}{M(\xi',\tau)}e^{\mathrm{i}\,\tau x_N}\,\mathrm{d}\tau,$$

where $M(\xi',\tau)$ is as in 3.5.3. Using (3.82), we obtain:

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{Q(\xi',\tau)B_s(\xi',\tau)}{M(\xi',\tau)}e^{\mathrm{i}\,\tau x_N}\,d\tau = 0 \quad s = 1,2,\dots,k.$$

According to (3.75) this is equivalent to

$$\frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{Q(\xi',\tau)B_s(\xi',\tau)}{M(\xi',\tau)}d\tau = 0 \Longrightarrow \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{Q(\xi',\tau)\tau^{t-1}}{M(\xi',\tau)}\,\mathrm{d}\tau = 0, \quad t = 1,2,\dots,k.$$

If $t = 1$, using the same approach as in the proof of Lemma 5.1, we find that $Q(\xi',\tau)$ has degree $\leq k-2$. Now for $t = 2,3,\dots,k$ it is easy to see that $Q(\xi',\tau) \equiv 0$.

Using the previous results, we obtain:

$$U(\xi',x_N) = \frac{1}{2\pi\mathrm{i}}\sum_{s=1}^{k}(\mathrm{i})^{-m_s}\hat{G}_s(\xi')\int_\gamma \frac{N_s(\xi',\tau)}{M(\xi',\tau)}e^{\mathrm{i}\,\tau x_N}\,\mathrm{d}\tau. \tag{3.84}$$

If we have (3.81), (3.83), it is sufficient to verify (3.82); using Lemma 5.3, we obtain:

$$\sum_{|\alpha|=m_t} b_{t\alpha}(\mathrm{i})^{|\alpha'|}\xi'^{\alpha'}\frac{d^{\alpha_N}U}{dx_N^{\alpha_N}} = \frac{1}{2\pi\mathrm{i}}\sum_{s=1}^{k}(\mathrm{i})^{-m_s+m_t}\hat{G}_s(\xi')\int_\gamma \frac{N_s(\xi',\tau)B_t(\xi',\tau)}{M(\xi',\tau)}\,\mathrm{d}\tau$$
$$= \hat{G}_t(\xi'),$$

and (3.84) follows.

For simplicity let γ_R be the boundary of the rectangle $-R < \mathrm{Re}\,\tau < R$, $\varepsilon < \mathrm{Im}\tau < R$. We can take R sufficiently large and $\varepsilon > 0$ sufficiently small such that all the roots of the polynomial P with positive imaginary part are inside of the rectangle with the condition $|\xi'| = 1$; this is possible because the roots $\tau_i(\xi')$ are continuous functions.[7] It is clear that, for all $\xi \neq 0$, we have:

$$P(\xi',\tau) = |\xi'|^{2k} P\left(\frac{\xi'}{|\xi'|},\frac{\tau}{|\xi'|}\right), \tag{3.85}$$

$$M(\xi',\tau) = |\xi'|^{k} M\left(\frac{\xi'}{|\xi'|},\frac{\tau}{|\xi'|}\right), \tag{3.86}$$

$$M_j(\xi',\tau) = |\xi'|^{j} M_j\left(\frac{\xi'}{|\xi'|},\frac{\tau}{|\xi'|}\right), \tag{3.87}$$

$$B_j(\xi',\tau) = |\xi'|^{m_j} B_j\left(\frac{\xi'}{|\xi'|},\frac{\tau}{|\xi'|}\right), \tag{3.88}$$

$$b_{ji}(\xi') = |\xi'|^{m_j-i+1} b_{ji}\left(\frac{\xi'}{|\xi'|}\right), \tag{3.89}$$

$$b^{ji}(\xi') = |\xi'|^{j-1-m_i} b_{ji}\left(\frac{\xi'}{|\xi'|}\right), \tag{3.90}$$

$$N_j(\xi',\tau) = |\xi'|^{k-1-m_j} N_j\left(\frac{\xi'}{|\xi'|},\frac{\tau}{|\xi'|}\right). \tag{3.91}$$

Using the definitions (3.85)–(3.91), we obtain from (3.84):

$$U(\xi',x_N) = \frac{1}{2\pi\mathrm{i}}\sum_{s=1}^{k}(\mathrm{i})^{-m_s}|\xi'|^{-m_s}\hat{G}_s(\xi')\int_{\gamma_R}\frac{N_s(\xi'/|\xi'|,\mu)}{M(\xi'/|\xi'|,\mu)}e^{\mathrm{i}|\xi'|\mu x_N}\,\mathrm{d}\mu. \tag{3.92}$$

[7] More specifically: Let ξ' be a given point in $\mathbb{R}^{N-1}$, $\tau_1(\xi'),\tau_2(\xi'),\dots\tau_k(\xi'),\tau_{k+1}(\xi'),\dots,\tau_{2k}(\xi')$ the roots of the polynomial $P(\xi',\tau)$ (this property holds for all polynomials continuously depending on the parameters). Let $\varepsilon > 0$, then there exists $\delta > 0$ such that the roots $\tau_1(\eta'),\tau_2(\eta'),\dots,\tau_{2k}(\eta')$ are in the set $\cup_{i=1}^{2k}B(\tau_i(\xi'),\varepsilon)$ for $|\xi'-\eta'| < \delta$ where $B(\tau_i(\xi'),\varepsilon)$ are the discs with center $\tau_i(\xi')$ and radius ε. This result can be proved by contradiction.

Let $|\alpha| = 2k$ and let us consider $D^\alpha u$; we have:

$$\int_{\mathbb{R}^{N-1}} e^{-\mathrm{i}(x',\xi')} D^\alpha u(x',x_N)\,\mathrm{d}x' = (\mathrm{i})^{|\alpha'|}\xi'^{\alpha'} \frac{d^{\alpha_N}U}{dx_N^{\alpha_N}}(\xi',x_N),$$

then using (3.83), we obtain:

$$(\mathrm{i})^{|\alpha'|}\xi'^{\alpha'} \frac{d^{\alpha_N}U}{dx_N^{\alpha_N}}(\xi',x_N)$$

$$= (-1)^k \xi'^{\alpha'} |\,\xi'|^{\alpha_N} \sum_{s=1}^{k} (\mathrm{i})^{-m_s} |\xi'|^{-m_s} \hat{G}_s(\xi') \frac{1}{2\pi\mathrm{i}} \int_{\gamma_R} \frac{N_s(\xi'/|\xi'|,\mu)}{M(\xi'/|\xi'|,\mu)} e^{i|\xi'|\mu x_N} d\mu. \tag{3.93}$$

Starting from (3.93) we deduce:

$$\int_{\mathbb{R}^N_+} e^{-\mathrm{i}(x,\xi)} D^\alpha u(x',x_N)\,\mathrm{d}x \equiv H_\alpha(\xi)$$

$$= (-1)^{k-1} \xi'^{\alpha'} |\,\xi'|^{\alpha_N - m_s} \sum_{s=1}^{k} (\mathrm{i})^{-m_s} \hat{G}_s(\xi') \frac{1}{2\pi\mathrm{i}} \int_{\gamma_R} \frac{N_s(\xi'/|\xi'|,\mu)\mu^{\alpha_N}}{M(\xi'/|\xi'|,\mu)} \frac{d\mu}{\mathrm{i}\,|\xi'|\mu - \mathrm{i}\,\xi^N}. \tag{3.94}$$

Denoting $\mu = \mu_1 + \mathrm{i}\,\mu_2$, we have:

$$||\xi'|\mu - \xi_N|^2 \geq \frac{\mu_2^2}{2}|\xi'|^2 + \frac{\mu_2^2/2}{\mu_2^2/2 + \mu_1^2}\xi_N^2 \geq \frac{\varepsilon^2}{2}|\xi'|^2 + \frac{\varepsilon^2/2}{\varepsilon^2/2 + R^2}\xi_N^2 \geq c_2(|\xi'|^2 + \xi_N^2).$$

But we have also the estimate:

$$\left| \frac{N_s(\xi'/|\xi'|,\mu)\mu^{\alpha_N}}{M(\xi'/|\xi'|,\mu)} \right| \leq c_3. \tag{3.95}$$

Using (3.94), we get:

$$|H_\alpha(\xi)| \leq c_4 \frac{|\xi'|^{2k}}{(|\xi'|^2 + \xi_N^2)^{1/2}} \sum_{s=1}^{k} |\xi'|^{-m_s} |\hat{G}_s(\xi')|. \tag{3.96}$$

Now we get:

$$|H_\alpha|_{L^2(\mathbb{R}^N)} \leq c_5 \sum_{s=1}^{k} \int_{\mathbb{R}^N} \frac{|\xi'|^{4k-2m_s}}{|\xi'|^2 + \xi_N^2} |\hat{G}_s(\xi')|^2\,\mathrm{d}\xi$$

$$= c_5 \sum_{s=1}^{k} \int_{\mathbb{R}^N} \frac{|\xi'|^{4k-2m_s}(1+|\xi'|^2)^{-2k+m_s+1/2}}{|\xi'|^2 + \xi_N^2} |\hat{G}_s(\xi')|^2 (1+|\xi'|^2)^{2k-m_s-1/2}\,\mathrm{d}\xi$$

$$\leq c_5 \sum_{s=1}^{k} \int_{\mathbb{R}^N} |\hat{G}_s(\xi')|^2 (1+|\xi'|^2)^{2k-m_s-1/2} \mathrm{d}\xi' \int_{-\infty}^{\infty} \frac{|\xi'|}{|\xi'|^2+\xi_N^2} d\xi_N,$$

then, using the Parseval equality,

$$\sum_{|\alpha|=2k} |D^\alpha u|_{L^2(\mathbb{R}^N_+)} \leq c_6 \sum_{s=1}^{k} \int_{\mathbb{R}^N} |\hat{G}_s(\xi')|^2 (1+|\xi'|^2)^{2k-m_s-1/2} \mathrm{d}\xi';$$

to finish, it suffices to prove for $u \in W^{h,2}(\mathbb{R}^N)$ the inequality:

$$|u|^2_{W^{h,2}(\mathbb{R}^N)} \leq c_7 (|u|^2_{L^2(\mathbb{R}^N_+)} + \sum_{|\alpha|=h} |D^\alpha u|^2_{L^2(\mathbb{R}^N_+)}). \tag{3.97}$$

To do this, we use Theorem 2.3.9; we extend the function u from $\mathbb{R}^N_+$ to $\mathbb{R}^N$ in such a way such that

$$|u|^2_{L^2(\mathbb{R}^N)} + \sum_{|\alpha|=h} |D^\alpha u|^2_{L^2(\mathbb{R}^N)} \leq c_8 (|u|^2_{L^2(\mathbb{R}^N_+)} + \sum_{|\alpha|=h} |D^\alpha u|^2_{L^2(\mathbb{R}^N_+)});$$

using the Fourier transform, we have without difficulty:

$$|u|^2_{W^{h,2}(\mathbb{R}^N)} \leq c_9 (|u|^2_{L^2(\mathbb{R}^N)} + \sum_{|\alpha|=h} |D^\alpha u|^2_{L^2(\mathbb{R}^N)}).$$

□

3.5.5 Properly Elliptic Operators

Let Ω be a bounded domain with a sufficiently smooth boundary, and $A = \sum_{|i|\leq 2k} a_i D^i$ an operator with coefficients measurable and bounded in $\overline{\Omega}$ if $|i| \leq 2k-1$ and continuous if $|i| = 2k$. This operator is called *properly elliptic in* $x \in \overline{\Omega}$, if for any pair of linearly independent vectors $\xi \neq 0$, $\eta \neq 0$, the polynomial of the variable τ given by

$$P_{\xi,\eta}(\tau) = \sum_{|i|=2k} a_i(x)(\xi+\tau\eta)^i \tag{3.98}$$

has exactly k roots with strictly positive imaginary part.

Remark 5.1. A properly elliptic operator is elliptic: Indeed, if along with the given vector η, we also consider the vector $-\eta$, the polynomial $P_{\xi,-\eta}(\tau)$ has exactly k roots with strictly negative imaginary part, which implies that $\tau = 0$ is not a root.

Remark 5.2. If $N > 2$, an elliptic operator is properly elliptic. Indeed, let $\tau_1, \tau_2, \ldots, \tau_\mu$, be the roots with positive imaginary part. The polynomial $P_{\xi,-\eta}(\tau) = \sum_{|i|=2k} a_i(x)(\xi-\tau\eta)^i$ has exactly μ roots with negative imaginary part, i.e. $-\tau_1, -\tau_2, \dots, -\tau_\mu$. For every $\alpha \in]0,1[$ we can find a vector of length $|\eta|$, say $\eta_{\overline{\alpha}}$, such that $\eta_{\overline{0}} = \eta, \eta_{\overline{1}} = -\eta, \eta_{\overline{\alpha}}$ depending continuously on α, ξ, $\eta_{\bar{\alpha}}$ are independent for any α. The polynomial (3.98), $P_{\xi,\eta}(\tau)$, has precisely μ roots with negative imaginary part, hence $\mu = k$.

Let $B_s = \sum_{|\alpha|\le m_s} b_{s\alpha} D^\alpha$ be boundary operators with sufficiently smooth coefficients, $s = 1,2,\dots,k$, of order $m_s \le 2k-1$. We say that A, B_s satisfy *the covering conditions at the point* $x \in \partial\Omega$[8] where A is properly elliptic, if for all vectors $\xi \neq 0$ tangent at $x \in \partial\Omega$ and $\eta \neq 0$ normal at $x \in \partial\Omega$ the polynomials

$$\sum_{|\alpha|=m_s} b_{s\alpha}(\xi + \tau\eta)^\alpha \tag{3.99}$$

are linearly independent modulo

$$M(\xi,\tau) \equiv \prod_{i=1}^{k} (\tau - \tau_i),$$

where τ_i are the roots with positive imaginary part of the polynomial defined in (3.98).

If (A, B_s) satisfy the covering conditions for all $x \in \partial\Omega$, we say that (A, B_s) satisfy the *covering conditions.*

Theorem 5.3. *Let $\Omega \in \mathfrak{N}^{4k-2,1}$, $A = \sum_{|i|\le 2k} a_i D^i$ an operator with $a_i \in L^\infty(\Omega)$, and moreover $a_i \in C^0(\overline{\Omega})$ if $|i| = 2k$. We assume A uniformly elliptic in $\overline{\Omega}$.*

Let $B_s = \sum_{|\alpha|\le m_s} b_{s\alpha} D^\alpha$ be boundary operators of order $m_s \le 2k-1$, $s = 1,2,\dots,k$, with coefficients $b_{s\alpha} \in C^{2k-1-m_s,1}(\partial\Omega)$. Finally we assume that (A, B_s) satisfy the covering conditions. Then there exists $\lambda \ge 0$ such that for all $u \in W^{2k,2}(\Omega)$, we have the inequality

$$|u|_{W^{2k,2}(\Omega)} \le c\Big(\sum_{s=1}^{k} |B_s u|_{W^{2k-m_s-1/2,2}(\partial\Omega)} + |Au|_{L^2(\Omega)}\Big) + \lambda |u|_{L^2(\Omega)}. \tag{3.100}$$

Proof. We use the domains G_r, $r = 1,2,\dots,M+1$, as in 1.2.4, such that $G_{M+1} \subset \Omega$, $\partial\Omega \subset \cup_{r=1}^{M} G_r$, $\overline{\Omega} \subset \cup_{r=1}^{M+1} G_r$; we use also the mapping (1.4.7). Without loss of generality, we choose $\nu_N = -1$ at the point $x = 0$. Let us put $u_r = u\psi_r$. First consider u_{M+1}, the support of u_{M+1} is in G_{M+1}; let us denote $B(x_{\bar{i}}, \rho)$ the ball with center $x_{\bar{i}}$ and radius ρ. Let $v \in W_0^{2k,2}(B(x_{\bar{i}}, \rho))$ and let us denote $A' = \sum_{|i|=2k} a_i D^i$, $A'_{x_{\bar{j}}} = \sum_{|i|=2k} a_i(x_{\bar{j}}) D^i$. Using the Fourier transform, we obtain:

[8] Cf. S. Agmon, A. Douglis, L. Nirenberg [1]. In the Russian literature these conditions are called Shapiro-Lopatinskii conditions.

$$\widehat{A'_{x_{\bar{j}}}v}(\xi) = (-1)^k \Big(\sum_{|i|=2k} a_i(x_{\bar{j}})\xi^i \Big) \hat{v}(\xi),$$

hence

$$\hat{v}(\xi)) = \frac{(-1)^k \widehat{A'_{x_{\bar{j}}}v}(\xi)}{\sum_{|i|=2k} a_i(x_{\bar{j}})\xi^i}.$$

Let $|\alpha| = 2k$; we have $(\widehat{D^\alpha v})(\xi) = (-1)^k \xi^\alpha \hat{v}(\xi)$, hence according to the uniform ellipticity and Parseval equality, it follows:

$$\sum_{|\alpha|=2k} |D^\alpha v|_{L^2(\Omega)} \le c_1 \left| A'_{x_{\bar{j}}} v \right|_{L^2(\Omega)}. \tag{3.101}$$

We have:

$$\left| A'_{x_{\bar{j}}} v - A'v \right|_{L^2(\Omega)} \le \omega_j(\rho) \sum_{|\alpha|=2k} |D^\alpha v|_{L^2(\Omega)},$$

where

$$\omega_j(\rho) = \max_{|i|=2k} \big(\max_{x \in \overline{B(x_{\bar{j}}, \rho)}} |a_i(x_{\bar{j}}) - a_i(x)| \big).$$

We can choose ρ sufficiently small such that $\omega_j(\rho) < 1/2$ uniformly with respect to $x_{\bar{j}} \in \overline{G_{M+1}}$; then (3.101) implies:

$$\sum_{|\alpha|=2k} |D^\alpha v|_{L^2(\Omega)} \le \frac{c_1}{2} |A'v|_{L^2(\Omega)}. \tag{3.102}$$

Suppose ρ is a fixed number, $x_{\bar{j}}$, $\bar{j} = 1, 2, \ldots, l$, points in $\overline{G_{M+1}}$ such that the following inclusion occurs, $\overline{G_{M+1}} \subset \cup_{j=1}^{l} B(x_{\bar{j}}, \rho)$, and let $h_j \in C_0^\infty(B(x_{\bar{j}}, \rho))$ such that $x \in \overline{G_{M+1}} \Longrightarrow \sum_{j=1}^{l} h_j^2(x) = 1$. Let us denote $w_j = u_{M+1} h_j$; using (3.102), we obtain:

$$|u_{M+1}|_{W^{2k,2}(\Omega)} \le c_2 \sum_{j=1}^{l} |w_j|_{W^{2k,2}(\Omega)} \le c_3 \big(|u_{M+1}|_{W^{2k-1,2}(\Omega)} + \sum_{j=1}^{l} |A'w_j|_{L^2(\Omega)} \big)$$

$$\le c_4 \big(|u_{M+1}|_{W^{2k-1,2}(\Omega)} + |A'u_{M+1}|_{L^2(\Omega)} \big) \le c_5 \big(|u_{M+1}|_{W^{2k-1,2}(\Omega)} + |Au_{M+1}|_{L^2(\Omega)} \big),$$

hence, by Lemma 2.6.1, we can find λ_1 sufficiently large such that

$$|u_{M+1}|_{W^{2k,2}(\Omega)} \le c_6 |Au_{M+1}|_{L^2(\Omega)} + \lambda_1 |u|_{L^2(\Omega)}. \tag{3.103}$$

Let us consider now u_r, $r = 1, 2, \ldots, M$. The mapping (1.35) transforms $W^{2k,2}(\Omega \cap G_r)$ onto $W^{2k,2}(K_+)$, where $K_+ = \{(\sigma, t), |\sigma_i| < \alpha, i = 1, 2, \ldots, N-1, 0 < t < \delta\}$. Using the local coordinates (σ, t) the operator A can be written as $A_r = \sum_{|i| \le 2k} a_{ri} D^i$. We do not restrict the generality of computations if we assume α

and δ small, such that $v \in W^{2k,2}(K_+)$ implies:

$$\left|A'_r v - A'_{r0} v\right|_{L^2(K_+)} \leq \frac{1}{2} \sum_{|\alpha| \leq 2k} |D^\alpha v|_{L^2(K_+)};$$

here $A'_{r0} = \sum_{|i|=2k} a_{ri}(0) D^i$; the previous inequality is a direct consequence of the continuity of the coefficients $a_{ri}, |i| = 2k$. The boundary operator B_s can be written in local coordinates in the form: $B_{rs} = \sum_{|\alpha| \leq m_s} b_{rs\alpha} D^\alpha$, $b_{rs\alpha} \in C^{2k-1-m_s,1}(\overline{\Delta})$, where $\Delta = (-\alpha, \alpha)^{N-1}$. We denote $B'_{rs} = \sum_{|\alpha| = m_s} b_{rs\alpha} D^\alpha$; it is necessary to choose α, δ sufficiently small to have the estimates given later; we will return to this particular point. For simplicity we denote u_r the function u_r after the mapping (1.35); the support of u_r is in $K_+ \cup \Delta$, cf. 1.2.4. Now we can find a sequence $u_{r,n}$, $n = 1, 2, \ldots$, $\lim_{n \to \infty} u_{r,n} = u_r$ in $W^{2k,2}(K_+), u_{r,n} \in C^\infty(\overline{K_+})$ with support in $K_+ \bigcup \Lambda$.

It is clear that A'_{r0} is an elliptic operator: this is a simple consequence of the fact that, at the point $\sigma = 0$, $t = 0$, the mapping (1.35) is the identity mapping. A being properly elliptic, the condition (3.71) holds for A'_{r0}.

Denoting as above τ_i the roots of the polynomial associated with A'_{r0}, we can find $\varepsilon > 0$ such that $\operatorname{Im} \tau_i \geq \varepsilon$, uniformly with respect to $x \in \partial\Omega$, the origin of coordinates in our case.

Let $B'_{rs0} = \sum_{|\alpha| = m_s} b_{rs\alpha}(0) D^\alpha$. We have (3.75), moreover, if $|\xi'| = 1$, the absolute value of this determinant is bigger than a positive constant, uniformly with respect to $x \in \partial\Omega$.

Let $f_{r,n} = A'_{r0} u_{r,n}$. We use Theorem 5.1 with $l = 1$, $m > N$; we can assume without loss of generality $K_+ \subset C$ defined in Theorem 5.1. Let $v_{r,n} \in W^{2k,2}(\mathbb{R}^N)$ as in Theorem 5.1 such that $A'_{r0} v_{r,n} = f_{r,n}$. Then

$$|v_{r,n}|_{W^{2k,2}(\mathbb{R}^N)} \leq c_7 |f_{r,n}|_{L^2(K_+)}. \tag{3.104}$$

The function $(u_{r,n} - v_{r,n})$ satisfies the hypotheses of Theorem 5.1, hence

$$\begin{aligned} |u_{rn}|_{W^{2k,2}(\mathbb{R}^N_+)} \leq c_8 \sum_{s=1}^{k} \Big(\int_{\mathbb{R}^{N-1}} |(\widehat{B'_{rs0} u_{r,n}})(\xi')|^2 (1 + |\xi'|^2)^{2k - m_s - 1/2)} \,\mathrm{d}\xi' \Big)^{1/2} \\ + c_8 \sum_{s=1}^{k} \Big(\int_{\mathbb{R}^{N-1}} |(\widehat{B'_{rs0} v_{r,n}})(\xi')|^2 (1 + |\xi'|^2)^{2k - m_s - 1/2)} \,\mathrm{d}\xi' \Big)^{1/2} \\ + c_8 (|f_{r,n}|_{L^2(K_+)} + |u_{r,n}|_{L^2(\mathbb{R}^N_+)}). \end{aligned} \tag{3.105}$$

Let us denote $\Delta = (-\alpha, \alpha)^{N-1}$ as in 1.2.4 and assume

$$\Delta \subset (-1/2, 1/2)^{N-1} \subset (-1, 1)^{N-1} \equiv \Delta_1.$$

Using Theorem 2.5.1 and Lemma 2.5.6, we can estimate:

$$\left(\int_{\mathbb{R}^{N-1}} |(\widehat{B'_{rs_0}u_{r,n}})(\xi')|^2(1+|\xi'|^2)^{2k-m_s-1/2)}\,\mathrm{d}\xi'\right)^{1/2}$$
$$\leq c_{10}\left|B'_{rs_0}u_{r,n}\right|_{W^{2k-m_s-1/2,2}(\Delta_1)}. \tag{3.106}$$

Now (3.104)–(3.106) imply:

$$|u_{r,n}|_{W^{2k,2}(\mathbb{R}^N_+)} \leq c_{11}\left(\sum_{s=1}^{k}|B'_{rs0}u_{r,n}|_{W^{2k-m_s-1/2,2}(\Delta_1)} + |f_{r,n}|_{L^2(K_+)} + |u_{r,n}|_{L^2(\mathbb{R}^N_+)}\right),$$

hence, for $n \to \infty$,

$$|u_r|_{W^{2k,2}(\mathbb{R}^N_+)} \leq c_{11}\left(\sum_{s=1}^{k}\left|B'_{rs_0}u_r\right|_{W^{2k-m_s-1/2}(\Delta_1)} + |f_r|_{L^2(K_+)} + |u_r|_{L^2(\mathbb{R}^N_+)}\right). \tag{3.107}$$

We observe that c_{11} does not depend on α, δ; hence we can take α, δ sufficiently small such that for $v \in W^{2k,2}(\mathbb{R}^N_+)$ with support in $K_+ \cup \Lambda$, we have:

$$c_{11}\left|B'_{rs}v - B'_{rs_0}v\right|_{W^{2k-m_s-1/2,2}(\Delta_1)} \leq (1/2k)|v|_{W^{2k,2}(\mathbb{R}^N_+)}. \tag{3.108}$$

It follows, according to (3.107) and (3.108) that

$$|u_r|_{W^{2k,2}(\mathbb{R}^N_+)} \leq 2c_{11}(\sum_{s=1}^{k}\left|B'_{rs_0}u_r\right|_{W^{2k-m_s-1/2,2}(\Delta_1)} + |f_r|_{L^2(K_+)} + |u_r|_{L^2(\mathbb{R}^N_+)}). \tag{3.109}$$

Then, by Theorem 2.5.4 and Lemma 2.6.1:

$$|u_r|_{W^{2k,2}(\mathbb{R}^N_+)} \leq c_{12}(\sum_{s=1}^{k}|B_{rs}u_r|_{W^{2k-m_s-1/2,2}(\Delta_1)} + |f_r|_{L^2(K_+)} + |u_r|_{L^2(\mathbb{R}^N_+)}). \tag{3.110}$$

Now we use again the representation by local coordinates and get:

$$\begin{aligned}|u_r|_{W^{2k,2}(G_r)} &\leq c_{13}(\sum_{s=1}^{k}\left|B'_su_r\right|_{W^{2k-m_s-1/2,2}(\partial\Omega)} + |f_r|_{L^2(G_{r+})} + |u_r|_{L^2(G_{r+})})\\ &\leq c_{14}(\sum_{s=1}^{k}\left|B'_su\right|_{W^{2k-m_s-1/2,2}(\partial\Omega)} + |f|_{L^2(\Omega)} + |u|_{L^2(\Omega)});\end{aligned} \tag{3.111}$$

here $G_{r+} = G_r \cap \Omega$; c_{14} does not depend on r. Finally we can write:

$$|u|_{W^{2k,2}(\Omega)} \le c_{15} \sum_{r=1}^{M+1} |u_r|_{W^{2k,2}(\Omega)},$$

and (3.103) and (3.111) are proved. □

Corollary 5.1. *Under the hypotheses of Theorem 5.3, let* $u \in W^{2k,2}(\Omega)$ *be such that* $B_s u = 0$ *on* $\partial\Omega$, $s = 1,2,\ldots,k$. *Then (3.56) occurs.*

Remark 5.3. It is possible to consider with the same method (we have done this in $\mathbb{R}^N_+$) the same type of problems for unbounded domains; we obtain estimates of the type (3.56). Cf. also S. Agmon, A. Douglis, L. Nirenberg [1].

3.5.6 Properly Elliptic Operators (Continuation)

Using an adapted method of S. Agmon, A. Douglis, L. Nirenberg [1], we obtain:

Theorem 5.4. *We assume the hypotheses as in the previous theorem, except that the covering conditions are not satisfied at a point* $x_0 \in \partial\Omega$, *where* A *is a properly elliptic operator on* $\partial\Omega$, *uniformly elliptic in* $\overline{\Omega}$. *Then (3.100) does not hold for any constant* c.

Proof. Without loss of generality, we can assume that the particular point x_0 is the origin of one coordinate system and that the tangent plane to this point is the hyperplane $x_N = 0$. Let $|\xi'| = 1$ be the exceptional vector, and introduce the operators

$$A'_0 = \sum_{|i|=2k} a_i(0)D^i, \quad B'_{s0} = \sum_{|\alpha|=m_s} b_{s\alpha}(0)D^\alpha,$$

and the polynomial

$$B'_{s0}(\xi',\tau) = \sum_{|\alpha|=m_s} b_{s\alpha}(0)\xi'^{\alpha'}\tau^{\alpha_N}.$$

Then we can find a nontrivial solution of the equation

$$\sum_{|i|=2k} (\mathrm{i})^{|i'|}\xi'^{i'} a_i \frac{d^{i_N}U}{dx_N^{i_N}} = 0, \tag{3.112}$$

which tends to zero as $x_N \to \infty$ and satisfies the boundary conditions:

$$\sum_{|\alpha|=m_s} b_{s\alpha}(0)(\mathrm{i})^{|\alpha'|}\xi'^{\alpha'} \frac{d^{\alpha_N}U}{dx_N^{\alpha_N}}(\xi',0) = 0, \quad s = 1,2,\ldots,k. \tag{3.113}$$

Using the approach which gives the uniqueness of the solution of the problem (3.81)–(3.83), we construct such a solution in the following form:

$$v(x_N) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{Q(\xi', \tau)}{M(\xi', \tau)} e^{\mathrm{i}\,\tau x_N} d\tau, \tag{3.114}$$

$Q(\xi', \tau) \neq 0$ is a polynomial of degree $\leq k-1$. Let T be the mapping (1.4.7) of K_+ onto G_{1+} (we assume that the origin is in G_1, cf. 1.2.4), $G_{1+} = G_1 \cap \Omega$. Let us set $s = (\sigma, t)$ and

$$u_\lambda(s) = \lambda^{-2k+1} e^{\mathrm{i}\,\lambda(\xi', \sigma)} v(\lambda t). \tag{3.115}$$

Let $\varphi \in C_0^\infty(K)$, $K = \{s = (\sigma, t), \sigma \in \Delta, |t| < \delta\}$, $\varphi(s) = 1$ for $|\sigma_i| < (1/2)\alpha$, $i = 1, 2, \ldots, N-1$, $0 < t < (1/2)\delta$, $0 \leq \varphi(s) \leq 1$, $\omega_\lambda(s) = \varphi(s) u_\lambda(s) = \omega_\lambda(T^{-1}(x)) = z_\lambda(x)$; $z_\lambda \in W^{2k,2}(\Omega)$. Let us assume that inequality (3.100) occurs with c_1. It is possible to find α and δ sufficiently small (cf. K_+) such that

$$\sum_{s=1}^{k} |B_s z_\lambda|_{W^{2k-\mu_s-1/2,2}(\partial\Omega)} \leq \varepsilon |z_\lambda|_{W^{2k,2}(\Omega)} + \lambda_1(\varepsilon) |z_\lambda|_{W^{2k-1,2}(\Omega)}, \tag{3.116}$$

$$|A z_\lambda|_{L^2(\Omega)}) \leq \varepsilon |z_\lambda|_{W^{2k,2}(\Omega)} + \lambda_2(\varepsilon) |z_\lambda|_{W^{2k-1,2}(\Omega)}, \tag{3.117}$$

with $(c_1 + c_2)\varepsilon < 1/2$. Then using (3.100), (3.116), (3.117):

$$(1/2) |z_\lambda|_{W^{2k,2}(\Omega)} \leq c_1(\lambda_1(\varepsilon) + \lambda_2(\varepsilon)) |z_\lambda|_{W^{2k-1}(\Omega)} \tag{3.118}$$

Now, taking into account the fact that $v(x_N)$ is a linear combination of functions of the type $H(x_N) e^{\mathrm{i}\,\tau_j x_N}$, τ_j a root with positive imaginary part of the polynomial in consideration, we obtain:

$$\lim_{\lambda \to \infty} |z_\lambda|_{W^{2k,2}(\Omega)} = c_2 > 0, \quad \lim_{\lambda \to \infty} |z_\lambda|_{W^{2k-1,2}(\Omega)} = 0.$$

But this is a contradiction to (3.118) □

3.6 Continuous Dependence on the Data

3.6.1 Dependence on the Coefficients

The notion of a well posed problem in the sense of Hadamard concerns only a part of the data, in our case the set f, u_0, g. Inequality (3.18) gives the property of continuity with respect to these data. It remains to consider the continuous dependence on the coefficients of the sesquilinear form $((v, u))$, on the coefficients of the boundary operators B_i, and on the domains Ω. All these questions are not completely solved in the most general setting and many open interesting and fundamental problems must be considered. The problems concerning the dependence on coefficients can be found in the works of S.G. Mikhlin [2], J.L. Lions [1], J. Nečas [14]; concerning the dependence on domains cf. I. Babuška [2–4], J. Deny, J.L. Lions [1], J. Nečas [2, 8], A. Kufner [1], J. Kautsky [1]. In these works only partial results are obtained, for instance the domain Ω is fixed and the investigations concern only the simultaneous dependence on coefficients, etc. We shall give some theorems with general dependence.

Now we begin by a generalization of a theorem of J.L. Lions [1]:

Theorem 6.1. *Suppose we are given* $\Omega, V, Q, ((v,u))$ *V-elliptic,* $((v,u))_n$, $n = 1,2,\ldots$, *a sequence of sesquilinear forms such that for all* $v,u \in W^{2k,2}(\Omega)$*:*

$$\lim_{n\to\infty}\Big(\sup_{|v|_k\le 1,|u|_k\le 1}|((v,u))-((v,u))_n|\Big) \equiv \lim_{n\to\infty}\varepsilon_n = 0; \tag{3.119}$$

let G_n*, resp.* G*, be the associated Green operators from* $[Q'\times W^{k,2}(\Omega)\times N \to W^{k,2}(\Omega)]$ *(cf. 3.1). Then* $\lim_{n\to\infty} G_n = G$ *in* $[Q'\times W^{k,2}(\Omega)\times N \to W^{k,2}(\Omega)]$.

Proof. It is clear that $((v,u))_n$ is V-elliptic if n is sufficiently large, moreover, $v \in V$ implies:

$$|((v,v))_n| \ge c_1|v|_k^2, \tag{3.120}$$

where c_1 does not depend on n. Let be $u_n = G_n(f,u_0,g)$, $u = G(f,u_0,g)$. For all $v \in V$, we have $((v,u_n-u))_n = ((v,u)) - ((v,u))_n$; then for $v = u_n - u$ we obtain:

$$\begin{aligned}|((u_n-u,u_n-u))_n| &= |((u_n u,u)) - ((u_n-u,u))_n| \le \varepsilon_n|u_n-u|_k|u|_k\\ &\le \varepsilon_n|u_n-u|_k|G|(|f|_{Q'}+|u_0|_k+|g|_{V'}).\end{aligned}$$

Hence using (3.120), it follows that

$$|u_n-u|_k \le \varepsilon_n\frac{|G|}{c_1}(|f|_{Q'}+|u_0|_{W^{k,2}(\Omega)}+|g|_{V'}).$$

□

Obviously we have

Proposition 6.1. *Let* $a_n(v,u)$ *be given forms as in (3.12), let condition (3.119) be satisfied if:*

$$((v,u))_n = \int_\Omega \sum_{|i|,|j|\le k}\overline{a}_{ij,n}D^i vD^j\overline{u}\,\mathrm{d}x + \int_{\partial\Omega}\sum_{i=0}^{k-1}\overline{b}_{i\alpha,n}\frac{\partial^i v}{\partial n^i}D^\alpha\overline{u}\,\mathrm{d}S,$$

where

$$\lim_{n\to\infty} a_{ij,n} = a_{ij} \quad in\ L^\infty(\Omega), \tag{3.121}$$

$$\lim_{n\to\infty} b_{i\alpha,n} = b_{i\alpha} \quad in\ L^\infty(\partial\Omega). \tag{3.122}$$

If $a_n(v,u)$ is given as in (3.14), $\Omega \in \mathfrak{N}^{2k,1}$, it is sufficient to replace (3.122) by:

$$\lim_{n\to\infty} b_{i\alpha,n} = b_{i\alpha} \quad \text{in } C^{|\alpha|-k,1}(\partial\Omega), \quad |\alpha| \le k.$$
$$\lim_{n\to\infty} b_{i\alpha,n} = b_{i\alpha} \quad \text{in } L^{\infty}(\partial\Omega), \quad |\alpha| < k. \tag{3.123}$$

Remark 6.1. If $u_0 = 0$ in Theorem 6.1, it is sufficient to have (3.119) for $v,u \in V$. This remark will be used often in the sequel.

Remark 6.2. With the same hypotheses as in Theorem 6.1, if $\lim_{n\to\infty} f_n = f$ in Q', $\lim_{n\to\infty} u_{0,n} = u_0$ in $W^{k,2}(\Omega)$, $\lim_{n\to\infty} g_n = g$ in $\overline{V}'$, then

$$|u_n - u|_k \le c_1(|f_n - f|_{Q'} + |u_{0,n} - u_0|_{W^{k,2}(\Omega)} + |g_n - g|_{V'}) + c_2\varepsilon_n(|f|_{Q'} + |u_0|_{W^{k,2}(\Omega)} + |g|_{V'}). \tag{3.124}$$

3.6.2 Dependence on the Coefficients (Continuation)

Theorem 6.2. *Let $\Omega, V, Q, ((v,u))$, and $((v,u))_n, n = 1,2,\ldots$, a sequence of sesquilinear forms be given such that for all $v \in V$:*

$$|((v,v))| \ge \alpha |v|_k^2, \quad |((v,v))_n| \ge \alpha |v|_k^2. \tag{3.125}$$

Let $u \in W^{k,2}(\Omega)$; we assume:

$$\lim_{n\to\infty} \big(\sup_{|v|_k \le 1} |((v,u)) - ((v,u))_n| \big) \equiv \lim_{n\to\infty} \varepsilon_n(u) = 0, \tag{3.126a}$$

$$|((v,u))_n| \le \beta |v|_k |u|_k. \tag{3.126b}$$

Let $\lim_{n\to\infty} f_n = f$ in Q', $\lim_{n\to\infty} u_{0,n} = u_0$ in $W^{k,2}(\Omega)$, $\lim_{n\to\infty} g_n = g$ in $\overline{V}'$ with u_n, resp. u, the solutions of the corresponding problems. Then $\lim_{n\to\infty} u_n = u$ in $W^{k,2}(\Omega)$.

Proof. For $v \in V$, we have:

$$\begin{aligned} ((v, u_n - u + u_0 - u_{0,n}))_n &= ((v,u_n))_n - ((v,u)) + ((v,u)) \\ &\quad - ((v,u))_n + ((v, u_0 - u_{0,n}))_n \\ &= \langle v, \overline{f}_n \rangle - \langle v, \overline{f} \rangle + \overline{g}_n v - \overline{g} v + ((v,u)) \\ &\quad - ((v,u))_n + ((v, u_0 - u_{0,n}))_n. \end{aligned}$$

Using (3.125), (3.126a), (3.126b), we obtain:

$$|((v, u_n - u + u_0 - u_{0,n}))| \leq$$
$$\leq |f_n - f|_{Q'}|v|_k + |g_n - g|_{V'}|v|_k + \varepsilon_n(u)|v|_k + \beta|v|_k|u_0 - u_{0,n}|_k. \tag{3.127}$$

Let us choose $v = u_n - u + u_0 - u_{0,n}$, then (3.127) and (3.125) give:

$$|u_n - u + u_0 - u_{0,n}|_k \leq \frac{1}{\alpha}(|f_n - f|_{Q'} + |g_n - g|_{V'} + \varepsilon_n(u) + \beta|u_0 - u_{0,n}|_k). \tag{3.128}$$

□

Proposition 6.2. *Let $a_n(v,u)$ be given by (3.12), $\Omega \in \mathfrak{N}^{0,1}$, then conditions (3.125), (3.126a), (3.126b) hold if*

$$\lim_{n\to\infty} a_{ij,n} = a_{ij} \quad \textit{in measure on } \partial\Omega, \tag{3.129}$$

$$|a_{ij,n}| \leq c_1, \tag{3.130}$$

$$\lim_{n\to\infty} b_{i\alpha,n} = b_{i\alpha} \quad \textit{in measure on } \partial\Omega, \tag{3.131}$$

$$|b_{i\alpha,n}| \leq c_2. \tag{3.132}$$

If the sequence $a_n(v,u)$ is given as in (3.14), $\Omega \in \mathfrak{N}^{2k,1}$, it is sufficient to replace (3.131), (3.132) for $|\alpha| \geq k$ by:

$$\lim_{n\to\infty} b_{i\alpha,n} = b_{i\alpha} \quad \textit{in } C^{|\alpha|-k}(\partial\Omega), \tag{3.133a}$$

$$\lim_{n\to\infty} \frac{\partial^{|\alpha|-k+1} b_{i\alpha,n}}{\partial\sigma_1^{i_1}\dots\partial\sigma_{N-1}^{i_{N-1}}} = \frac{\partial^{|\alpha|-k+1} b_{i\alpha}}{\partial\sigma_1^{i_1}\dots\partial\sigma_{N-1}^{i_{N-1}}} \quad \textit{in measure on } \partial\Omega, \tag{3.133b}$$

$$|b_{i\alpha,n}|_{C^{|\alpha|-k,1}(\partial\Omega)} \leq c_3. \tag{3.134}$$

Proof. We have:

$$A(v,u) - A_n(v,u) = \int_\Omega \sum_{|i|,|j|\leq k} (\overline{a}_{ij} - \overline{a}_{ij,n}) D^i v D^j \overline{u}\, \mathrm{d}x.$$

Let $\varepsilon > 0$, and

$$M_n = \{x \in \Omega,\ \max_{i,j} |a_{ij}(x) - a_{ij,n}(x)| \geq \varepsilon\}.$$

We have $\lim_{n\to\infty}(\mathrm{meas} M_n) = 0$, hence

$$|A(v,u) - A_n(v,u)| \leq \varepsilon \int_{\Omega - M_n} \sum_{|i|,|j|\leq k} |D^i v||D^j u| dx + 2c_2 \int_{M_n} \sum_{|i|,|j|\leq k} |D^i v||D^j u| dx$$
$$\leq c_4 \varepsilon |v|_{W^{k,2}(\Omega)} |u|_{W^{k,2}(\Omega)} + c_5 |v|_{W^{k,2}(\Omega)} |u|_{W^{k,2}(M_n)}.$$

Then we have (3.126a) for $A_n(v,u)$, $A(v,u)$. For given $a_n(v,u)$, $a(v,u)$ satisfying (3.12), the situation is the same; if (3.14) is verified, the proof is as in Theorem 1.2.1. □

Remark 6.3. Theorems 6.1, 6.2 can be modified without difficulty to the case of V/P-ellipticity.

We are interested in the following particular case:

3.6.3 The Singular Case

Theorem 6.3. *Given $\Omega, V, Q, P \subset V \cap P_{(k-1)}$, $((v,u))$ V/P-elliptic, $((v,u))_n$ V-elliptic, $n = 1,2,\ldots$, a sequence of sesquilinear forms such that for all $v \in V$:*

$$|((\tilde{v},\tilde{u}))| \le c_1 |\tilde{v}|_{W^{k,2}(\Omega)/P} |\tilde{u}|_{W^{k,2}(\Omega)/P}, \quad |((v,v))_n| \ge c_1 |\tilde{v}|^2_{W^{k,2}/P}, \tag{3.135}$$

$$|((v,v))_n| \ge \alpha_n |v|^2_{W^{k,2}(\Omega)}, \quad \alpha_n > 0 \quad (\lim_{n\to\infty} \alpha_n = 0 \textit{ is possible}). \tag{3.136}$$

For all $v, u \in W^{k,2}(\Omega)$ we assume

$$|((v,u))_n| \le c_2 |((v,v))_n|^{1/2} |u|_k, \tag{3.137}$$

$$\lim_{n\to\infty} \Big(\sup_{|((v,v))_n|^{1/2} \le 1, |u|_k \le 1} |((v,u)) - ((v,u))_n| \Big) \equiv \lim_{n\to\infty} \varepsilon_n = 0, \tag{3.138a}$$

either

$$\lim_{n\to\infty} \Big(\sup_{|((v,v))_n|^{1/2} \le 1} |((v,u)) - ((v,u))_n| \Big) \equiv \lim_{n\to\infty} \varepsilon_n(u) = 0, \tag{3.138b}$$

Let $\lim_{n\to\infty} f_n = f$ *in* Q', $\lim_{n\to\infty} g_n = g$ *in* $\overline{V}'$, $\lim_{n\to\infty} u_{0,n} = u_0$ *in* $W^{k,2}(\Omega)$. *Then* $v \in P \Longrightarrow \langle v, \overline{f} \rangle + \overline{g} v = 0 = \langle v, \overline{f}_n \rangle + \overline{g}_n v$.

Let u_n, u be the solutions of the corresponding problems. Then with hypothesis (3.138a) we have:

$$|\tilde{u}_n - \tilde{u}|_{W^{k,2}(\Omega)/P} \le c(|f_n - f|_{Q'} + |u_{0,n} - u_0|_k + |g_n - g|_{V'}) + \varepsilon_n(|f|_{Q'} + |u_0|_k + |g|_{V'}),$$

with hypothesis (3.138b) we have $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}$ *in* $W^{k,2}(\Omega)/P$.

Proof. We choose $u \in \tilde{u}$ such that $|u|_k \le 2|\tilde{u}|_{W^{k,2}(\Omega)/P}$; we have:

$$\begin{aligned} ((v, u_n - u + u_0 - u_{0,n}))_n &= ((v,u_n))_n - ((v,u)) + ((v,u)) \\ &\quad - ((v,u))_n + ((v, u_0 - u_{0,n}))_n \\ &= \langle v, \overline{f}_n \rangle + \overline{g}_n v - \langle v, \overline{f} \rangle - \overline{g} v + ((v,u)) - ((v,u))_n + ((v, u_0 - u_{0,n}))_n, \end{aligned}$$

then

$$|((v,u_n-u+u_0-u_{0,n}))_n| \le c_3|f_n-f|_{Q'}|((v,v))|^{1/2}+c_3|g_n-g|_{V'}|((v,v))_n|^{1/2} \\ +\varepsilon_n(u)|((v,v))_n|^{1/2}+c_2|((v,v))_n|^{1/2}|u_0-u_{0,n}|_k. \tag{3.139}$$

If we choose $v=u_n-u+u_0-u_{0,n}$, then using (3.135) and (3.139), we obtain:

$$|\tilde{u}_n-\tilde{u}+\tilde{u}_0-\tilde{u}_{0,n}|_{W^{k,2}(\Omega)/P} \le c_4(|f_n-f|_{Q'}+|g_n-g|_{V'} \\ +\varepsilon_n(u)+|u_0-u_{0,n}|_{W^{k,2}(\Omega)}). \tag{3.140}$$

□

Example 6.1. Let $\Omega \in \mathfrak{N}^0$, $V=W^{1,2}(\Omega)$, $Q=L^2(\Omega)$, $A_n=-\triangle+b_n, b_n\ge 0$, $b_n \ne 0$, $\lim_{n\to\infty} b_n=0$ in $L^\infty(\Omega)$, $a_n(v,u)=a(v,u)\equiv 0$, $g_n=g=0$. All hypotheses of Theorem 6.3 are satisfied, $P=P_{(0)}$, (3.135) is clear, (3.136) is proved using Theorem 1.4.3, (3.137) is a consequence of the Schwarz inequality. Since $((v,u))_n$ is hermitian, we have for (3.138a):

$$|((v,v))_n|^{1/2}=\Big(\int_\Omega \sum_{i=1}^N \Big|\frac{\partial v}{\partial x_i}\Big|^2\,\mathrm{d}x+\int_\Omega b_n|v|^2\,\mathrm{d}x\Big)^{1/2}.$$

Then

$$|((v,v))_n|^{1/2}\le 1 \Longrightarrow \int_\Omega b_n|v|^2dx \le 1.$$

We obtain:

$$|((v,u))-((v,u))_n|=\Big|\int_\Omega b_n v\bar{u}\,\mathrm{d}x\Big| \le \Big(\int_\Omega b_n|v|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_\Omega b_n|u|^2\,\mathrm{d}x\Big)^{1/2} \\ \le |b_n|^{1/2}_{L^\infty(\Omega)}|u|_k.$$

If $N\ge 3$, $\Omega\in\mathfrak{N}^{0,1}$, then it is sufficient to have $\lim_{n\to\infty}|b_n|_{L^{N/2}(\Omega)}=0$; this follows from Theorem 2.3.4.

The following theorem is more or less a theoretical example; the modifications depend on the particular cases.

Theorem 6.4. *Fix* $\Omega, V, Q, P \subset P_{(k-1)}$, $((v,u))_1$, $((v,u))_{1,n}$, $((v,u))_2$, $((v,u))_{2,n}$ *sesquilinear forms on* $W^{k,2}(\Omega)\times W^{k,2}(\Omega)$*; we assume:* $\lim_{n\to\infty}((v,u))_{i,n}=((v,u))_i$, $i=1,2$, $v,u\in W^{k,2}(\Omega)$ *in sense of (3.119), with* ε_n*; we assume* $((\tilde{v},\tilde{u}))_1$, $((\tilde{v},\tilde{u}))_{1,n}$, *to be sesquilinear forms on* $W^{k,2}(\Omega)/P\times W^{k,2}(\Omega)/P$*, satisfying for all* $v\in V$*:* $\mathrm{Re}\,((\tilde{v},\tilde{v}))_{1,n}\ge c_1|\tilde{v}|^2_{V/P}$, $\mathrm{Re}\,((\tilde{v},\tilde{v}))_{2,n}\ge c_1|\tilde{v}|^2_{V/P}$*; if* $v\in V$*, we assume also:* $\mathrm{Re}\,((v,v))_1+\mathrm{Re}\,((v,v))_2\ge c_2|v|^2_{W^{k,2}(\Omega)}$. *Let* $\lim_{n\to\infty} f_n=f$ *in* Q', $\lim_{n\to\infty} g_n=g$ *in* $\overline{V}'$, $\lim_{n\to\infty} u_{0,n}=u_0$ *in* $W^{k,2}(\Omega)$*. If* $v\in P$*, then* $\langle v,\overline{f}\rangle+\overline{g}v=\langle v,\overline{f}_n\rangle+\overline{g}_n v=0.$

Let $\lambda_n > 0$, $n = 1,2,\ldots$, $\lim_{n\to\infty}\lambda_n = 0, u$ *the solution of the problem* $u - u_0 \in V$, $((v,u))_1 = \langle v, \overline{f}\rangle + \overline{g}v$, $v \in V$, *and* u_n *the solution of the problem* $u_n - u_{0,n} \in V$, $((v,u))_{1,n} + \lambda_n((v,u))_{2,n} = \langle v, \overline{f}_n\rangle + \overline{g}_n v, v \in V$. *Then there exists a unique* $u \in \tilde{u}$ *such that*

$$|u_n - u|_k \le c(|f_n - f|_{Q'} + |u_{0,n} - u_0|_{W^{k,2}(\Omega)} + |g_n - g|_{V'}) + \varepsilon_n(|f|_{Q'} + |u_0|_k + |g|_{V'})).$$

Proof. For all $v \in V$ and for n sufficiently large, we have:

$$\begin{aligned}\text{Re}\,((v,v))_{1,n} + \lambda_n \text{Re}\,((v,v))_{2,n} &\ge (1-\lambda_n)\text{Re}\,((v,v))_{1,n} + \lambda_n \text{Re}\,((v,v))_{1,n} \\ &+ \lambda_n \text{Re}\,((v,v))_{2,n} \ge \lambda_n (c_1/2)|v|_k^2.\end{aligned}$$

Now we can find $u \in \tilde{u}$, such that $v \in P \Longrightarrow ((v,u))_2 = 0, u$ unique. For $v \in P$ we have $((v,u))_{1,n} + \lambda_n((v,u_n))_{2,n} = 0$; hence for $v \in P$ we have $((v,u_n))_{2,n} = 0$. Let us put $((v,u))_n = ((v,u))_{1,n} + \lambda_n((v,u))_{2,n}$. Now we have for $v \in V$:

$$\begin{aligned}|((v, u_n - u + u_0 - u_{0,n}))_n| &\le c(|f_n - f|_{Q'}|v|_k + |g_n - g|_{V'}|v|_k \\ &+ \varepsilon_n |v|_k |u|_k + |u_0 - u_{0,n}|_k |v|_k).\end{aligned} \tag{3.141}$$

Let $v_1, v_2, \ldots, v_\kappa$ be a basis in the space P. If n is sufficiently large, $v \in V$, then

$$|\tilde{v}|_{V/P}^2 + \sum_{i=1}^{\kappa} |((v,v_i))_{2,n}|^2 \ge c_3 |v|_k^2. \tag{3.142}$$

In the previous inequality, if we replace $((v,u))_{2,n}$ by $((v,u))_2$, we obtain:

$$(|\tilde{v}|_{V/P}^2 + \sum_{i=1}^{\kappa} |((v,v_i))_2|^2)^{1/2},$$

this is a norm and V is complete with respect to this norm; the result follows from Banach's theorem. If n is sufficiently large, we obtain (3.142).

In the inequality (3.141) we can take $v = u_n - u + u_0 - u_{0,n}$; then

$$\begin{aligned}c_1 |\tilde{u}_n - \tilde{u} + \tilde{u}_0 + \tilde{u}_{0,n}|_{V/P}^2 &\le |((u_n - u + u_0 - u_{0,n}, u_n - u + u_0 - u_{0,n}))_n))_{1,n} \\ &\le c(|f_n - f|_{Q'}|u_n - u + u_0 - u_{0,n}|_k + |g_n - g|_{V'}|u_n - u + u_0 - u_{0,n}|_k \\ &+ \varepsilon_n |u_n - u + u_0 - u_{0,n}|_k |u|_k + |u_n - u + u_0 - u_{0,n}|_k |u_0 - u_{0n}|_k \\ &+ \lambda_n |u_n - u + u_0 - u_{0,n}|_k^2).\end{aligned} \tag{3.143}$$

Now in the first and in the last terms of the previous inequalities we add the following term:

$$\sum_{i=1}^{\kappa} |((u_n - u + u_0 - u_{0,n}, v_i))_{2,n}|^2,$$

then we get:

$$(c_4 - \lambda_n)|u_n - u + u_0 - u_{0,n}| \le |f_n - f|_{Q'} + \varepsilon_n |u|_k + |u_0 - u_{0,n}|_k + |g_n - g|_{V'}$$
$$+ (\sum_{i=1}^{\kappa} |((v_i, u_n - u + u_0 - u_{0,n}))_{2,n})|^2)^{1/2} \frac{((\sum_{i=1}^{\kappa} |((v_i, u_n - u + u_0 - u_{0,n}))_{2,n}|^2)^{1/2}}{|u_n - u + u_0 - u_{0,n}|_k}. \tag{3.144}$$

But we have:

$$(\sum_{i=1}^{\kappa} |((v_i, u_n - u + u_0 - u_{0,n}))_{2,n}|^2)^{1/2} \le c_5 |u_n - u + u_0 - u_{0,n}|_k,$$

on the other hand,

$$|((u_n - u + u_0 - u_{0,n}, v_i))_{2,n}| = |-((u_n, v_i))_{2,n} - ((u, v_i))_2 + ((v_i, u_0 - u_{0,n}))_{2,n}|$$
$$\le c_6(\varepsilon_n |u|_k + |u_0 - u_{0,n}|_k).$$

Then, using (3.144), the theorem is proved. □

Exercise 6.1. Using a simple modification of Theorem 6.4, where the conditions (3.126a), (3.126b) replace condition (3.119), prove that $\lim_{n\to\infty} u_n = u$ in $W^{k,2}(\Omega)$.

Example 6.2. Let $\Omega \in \mathfrak{N}^0$, $V = W^{2,2}(\Omega)$, $Q = L^2(\Omega)$, $P = P_{(1)}$, $A_k = \triangle^2$, $A_{k,n} = \triangle^2$, $B_n = b_n$, $\lim_{n\to\infty} b_n = b = B$ in $L^\infty(\Omega)$; $b \ge 0$, $b \ne 0$, $A = \triangle^2 + b$. Put $f = f_n \in L^2(\Omega)$, $g_n = g = 0$ and assume:

$$\int_\Omega \overline{f} \, \mathrm{d}x = \int_\Omega x_i \overline{f} \, \mathrm{d}x = 0, \quad i = 1, 2, \ldots, N.$$

Then if $\triangle^2 u_n + \lambda_n b_n u_n = f$ resp. $\triangle^2 u = f$ in Ω, with boundary conditions generated by:

$$\int_\Omega \sum_{|i|=2} \frac{2}{i!} D^i v D^i \overline{u}_n \, \mathrm{d}x + \int_\Omega \lambda_n \overline{b}_n v \overline{u}_n \, \mathrm{d}x = \int_\Omega v \overline{f} \, \mathrm{d}x, \qquad v \in W^{2,2}(\Omega),$$

resp.

$$\int_\Omega \sum_{|i|=2} \frac{2}{i!} D^i v D^i \overline{u} \, \mathrm{d}x = \int_\Omega v \overline{f} \, \mathrm{d}x, \qquad v \in W^{2,2}(\Omega),$$

we can apply Theorem 6.4.

Remark 6.4. Let O be a domain in the Gauss plane G and let $((v,u))_\lambda$ be a sesquilinear form defined for $\lambda \in O$. The spaces V, Q are given, the sesquilinear form is V-elliptic, uniformly for all $\lambda \in O$. For $\lambda \in O$ we define $f_\lambda \in Q'$, $g_\lambda \in V'$, $u_{0\lambda} \in W^{k,2}(\Omega)$; moreover we assume that for all $v \in V$, the functions of variable $\lambda, \overline{\langle v, \overline{f}_\lambda \rangle}$, $\overline{\overline{g}_\lambda v}$, $\overline{((v, u_{0\lambda}))_\lambda}$ are analytic in O. Then $((v, u(\lambda)))_\lambda$, where $u(\lambda)$ is the solution of the corresponding problem, is also analytic in O.

Example 6.3. Let $((v,u))_1$, $((v,u))_2$ be two sesquilinear forms and let us assume that for $v \in V$, we have $c_1|v|_k^2 \leq \text{Re}\,((v,v))_i \leq c_2|v|_k^2$, $\text{Im}((v,v))_i = 0$, $i = 1,2$. Let us put $((v,u))_\lambda = ((v,u))_1(1-\overline{\lambda}) + ((v,u))_2\overline{\lambda}$.

Then we can take for O the set defined by:

$$\frac{c_1}{c_1 - c_2} + \varepsilon \leq \text{Re}\,\lambda \leq \frac{c_2}{c_2 - c_1} - \varepsilon, \quad \varepsilon > 0.$$

3.6.4 Dependence on the Space V

In this section we consider the dependence of the solution on the space V.

Let $V_n, V \subset W^{k,2}(\Omega)$ be closed spaces. We say that $\lim_{n\to\infty} V_n = V$ *uniformly*, if:

$$\lim_{n\to\infty} \Big(\sup_{\substack{|v|_k \leq 1 \\ v \in V_n}} \big(\inf_{w \in V} |v - w|_k \big) \Big) \equiv \lim_{n\to\infty} \varepsilon_n = 0, \tag{3.145}$$

$$\lim_{n\to\infty} \Big(\sup_{\substack{|w|_k \leq 1 \\ w \in V}} \big(\inf_{v \in V_n} |v - w|_k \big) \Big) \equiv \lim_{n\to\infty} \varepsilon_n' = 0. \tag{3.146}$$

Theorem 6.5. *Let $V_n, V \subset W^{k,2}(\Omega)$, $\lim_{n\to\infty} V_n = V$ uniformly. Let us assume the existence of a closed subspace $W \subset W^{k,2}(\Omega)$ such that $V_n \subset W, V \subset W$ algebraically and topologically. Let $W \subset Q$ algebraically and topologically. Suppose we are given a sesquilinear form $((v,u))$, W-elliptic, and $\lim_{n\to\infty} f_n = f$ in Q', $\lim_{n\to\infty} g_n = g$ in $\overline{W}'$, $\lim_{n\to\infty} u_{0,n} = u_0$ in $W^{k,2}(\Omega)$. Let u_n (resp. u) be the solutions of the corresponding problems. Then the following inequality holds:*

$$|u_n - u|_k \leq \frac{c_1}{1 - c_1\varepsilon_n}\Big(|u_0 - u_{0,n}|_k + (1+\varepsilon_n)(1+\varepsilon_n')(|f - f_n|_{Q'} + |g - g_n|_{W'}) + \varepsilon_n'(1+\varepsilon_n)(|f|_{Q'} + |g|_{W'}) + \varepsilon_n'(1+\varepsilon_n)(|f_n|_{Q'} + |u_{0,n}|_k + |g_n|_{W'})\Big). \tag{3.147}$$

Proof. We can write:

$$\begin{aligned} &((u - u_n + u_{0,n} - u_0, u - u_n + u_{0,n} - u_0)) \\ &= ((u - u_n + u_{0,n} - u_0, u - u_n)) + ((u - u_n + u_{0,n} - u_0, u_{0,n} - u_0)). \end{aligned} \tag{3.148}$$

Then we have:

$$((u-u_n+u_{0,n}-u_0,u-u_n)) = ((h_n,u-u_n)) + ((u-u_n+u_{0,n}-u_0-h_n,u-u_n)),$$

with $h_n \in V$ chosen to minimize $|u-u_n+u_{0,n}-u_0-h_n|_k$.

Now due to (3.145), we have:

$$|u-u_n+u_{0,n}-u_0-h_n|_k \leq \varepsilon_n |u-u_n+u_{0,n}-u_0|_k. \tag{3.149}$$

Let $h'_n \in V_n$ be chosen to minimize $|h_n-h'_n|_k$. Then (3.146), (3.149) imply:

$$|h_n-h'_n|_k \leq \varepsilon'_n(1+\varepsilon_n)|u-u_n+u_{0,n}-u_0|_k, \tag{3.150}$$

hence

$$\begin{aligned} ((h_n,u-u_n)) &= \langle h_n,\overline{f}\rangle + \overline{g}h_n - ((h_n,u_n)) \\ &= \langle h_n,\overline{f}\rangle + \overline{g}h_n - \langle h'_n,\overline{f}_n\rangle - \overline{g}_n h'_n + ((h'_n-h_n,u_n)). \end{aligned} \tag{3.151}$$

Now, using the previous formulas (3.148)–(3.151), we obtain:

$$\begin{aligned} &|u-u_n+u_{0,n}-u_0|_k^2 \leq c_1\big(|u_{0,n}-u_0|_k|u-u_n+u_{0,n}-u_0|_k \\ &\quad + \varepsilon_n|u-u_n+u_{0,n}-u_0|_k|u-u_n|_k + (1+\varepsilon_n)(1+\varepsilon'_n)|f-f_n|_{Q'}|u-u_n+u_{0,n}-u_0|_k \\ &\quad (1+\varepsilon_n)(1+\varepsilon'_n)|g-g_n|_{W'}|u-u_n \\ &\quad + u_{0,n}-u_0|_k + \varepsilon'_n(1+\varepsilon_n)(|f|_{Q'}+|g|_{W'})|u-u_n+u_{0,n}-u_0|_k \\ &\quad + \varepsilon'_n(1+\varepsilon_n)|u-u_n+u_{0,n}-u_0|_k|u_n|_k\big), \end{aligned}$$

and finally we have the estimate:

$$\begin{aligned} |u-u_n|_k \leq c_2\big(&|u_{0,n}-u_0|_k + \varepsilon_n|u-u_n|_k + (1+\varepsilon_n)(1+\varepsilon'_n)(|f_n-f|_{Q'}+|g_n-g|_{W'}) \\ &+ \varepsilon'_n(1+\varepsilon_n)(|f|_{Q'}+|g|_{W'}) + \varepsilon'_n(1+\varepsilon_n)(|f_n|_{Q'}+|g_n|_{W'}+|u_{0,n}|_k)\big). \end{aligned}$$

Hence we get (3.147). □

Exercise 6.2. We replace (3.146) by:

$$\lim_{n\to\infty}\big(\inf_{v\in V_n}|v-w|_k\big) \equiv \lim_{n\to\infty}\varepsilon''_n = 0. \tag{3.151 bis}$$

Prove that $\lim_{n\to\infty} u_n = u$ in $W^{k,2}(\Omega)$.

Hint: Start by proving weak convergence.

Remark 6.5. Condition (3.145) is trivially satisfied if $V_n \subset V$, $n = 1,2,\ldots$.

3.6.5 Dependence on the Space V (Continuation)

Let M be a subset of some vector space; we denote by $[M]$ the vector space generated by M. Instead of (3.145) we can introduce another condition:

$$V = \cap_{n=1}^{\infty} \overline{[\cup_{i=n}^{\infty} V_i]}. \tag{3.152}$$

Then we have

Theorem 6.6. *Let $V_n, V \subset W^{k,2}(\Omega)$, $\lim_{n\to\infty} V_n = V$ in the sense defined in (3.151 bis), (3.152). Denote $W_n = \overline{[\cup_{i=n}^{\infty} V_i]}$, $n = 1, 2, 3, \ldots$. Let $W_1 \subset Q$ algebraically and topologically. Suppose we are given a sesquilinear form $((v,u))$, W_1-elliptic, and sequences such that $\lim_{n\to\infty} f_n = f$ in Q', $\lim_{n\to\infty} g_n = g$ in $\overline{W}'$, $\lim_{n\to\infty} u_{0,n} = u_0$ in $W^{k,2}(\Omega)$. Then if u_n, u are the solutions of the corresponding problems, we have: $\lim_{n\to\infty} u_n = u$ in $W^{k,2}(\Omega)$.*

Proof. Let P_n be the projectors $P_n : W_1$ on W_n; we have: $P_1 \geq P_2 \geq \ldots$. and if $v \in W_1$, then $\lim_{n\to\infty} P_n v = Pv$, where P is the projector $P : W_1 \to V$. Denote by Z_n^*, Z^* the mapping of W_n onto W_n (resp. V onto V) defined by: $v, u \in W_n \Longrightarrow (Z_n^* v, u)_k = (v,u)_k$, (resp. $v, u \in V \Longrightarrow (Z^* v, u)_k = ((v,u))$; cf. Lemma 1.3.1.

We shall prove that $v \in V$ implies $\lim_{n\to\infty} Z_n^* v = Z^* v$.
Indeed: let $\omega \in W_1$, then $(Z_n^* v, \omega)_k = (Z_n^* v, P_n \omega)_k$, but $\lim_{n\to\infty} (Z_n^* v, (P - P_n)\omega)_k = 0$, we obtain $\lim_{n\to\infty} (Z_n^* v, P_n \omega)_k = ((v, P\omega)) = (Z^* v, P\omega)_k = (Z^* v, \omega)_k$. From this we can deduce:

$$\lim_{n\to\infty} (Z_n^* v, Z_n^* v)_k = \lim_{n\to\infty} ((v, Z_n^* v)) = \lim_{n\to\infty} (Z^* v, Z_n^* v)_k$$
$$= (Z^* v, Z^* v)_k = ((v, Z^* v)) = (Z^* v, Z^* v)_k.$$

This gives the result.

Now we prove that $\lim_{n\to\infty} u_n = u$ in the weak topology. Let $w \in W_1$, we have:

$$(w, u_n - u_{0,n} - u + u_0)_k = (w, u_0 - u_{0,n})_k + (w, u_n - u)_k.$$

But

$$\lim_{n\to\infty} (w, u_0 - u_{0,n})_k = 0,$$

hence it is sufficient to consider $(w, u_n - u)_k$. We have:

$$(w, u_n - u)_k = (P_n w, u_n - u - u_{0,n} + u_0)_k + (w, u_{0,n} - u_0)_k$$
$$= (Pw, u_n - u - u_{0,n} + u_0)_k + ((P_n - P)w, u_n - u - u_{0,n} + u_0)_k + (w, u_{0,n} - u_0)_k.$$

Using the hypotheses on P_n, P we have:

$$\lim_{n\to\infty} ((P_n - P)w, u_n - u - u_{0,n} + u_0)_k = 0.$$

Using the fact that $|u_n|_k$ are bounded, there exists exactly one element $v \in V$ such that $Pw = Z^* v$. Now we consider $(Z^* v, u_n - u - u_{0,n} + u_0)_k$. We have:

$$\lim_{n\to\infty} (Z^* v, u_n - u - u_{0,n} + u_0)_k = \lim_{n\to\infty} (Z_n^* v, u_n - u_{0,n})_k - (Z^* v, u - u_0)_k$$
$$= \lim_{n\to\infty} ((v, u_n - u_{0,n})) - ((v, u - u_0)) = \lim_{n\to\infty} (\langle v_n, \overline{f}_n\rangle + \overline{g}_n v_n - \langle v, \overline{f}\rangle - \overline{g} v) = 0,$$

where $v_n \in V_n$, $\lim\limits_{n\to\infty} v_n = v$ strongly in $W^{k,2}(\Omega)$, $\lim_{n\to\infty} u_n = u$ weakly. Now we have:

$$\lim_{n\to\infty} ((u_n - u - u_{0,n} + u_0, u_n - u - u_{0,n} + u_0)) = \lim_{n\to\infty} ((u_n - u_{0,n} - u + u_0, u_n - u))$$
$$= \lim_{n\to\infty} (\langle u - u_0, \overline{f}\rangle + \overline{g}(u - u_0) + \langle u_n - u_{0,n}, \overline{f}_n\rangle + \overline{g}_n(u_n - u_{0,n}))$$
$$- \lim_{n\to\infty} ((u - u_0, u_n)) - \lim_{n\to\infty} ((u_n - u_{0,n}, u)) = 0.$$

□

Example 6.4. Given $\Omega \in \mathfrak{N}^{k,1}$, if $k \geq 2$, $\Omega \in \mathfrak{N}^{0,1}$, if $k = 1$, we define on $\partial\Omega$ boundary operators B_{ns} (resp. B_s), $s = 1, 2, \ldots, \mu$, as in (1.2.6a). We suppose that F_{ns} are written in local coordinates (σ, t) as

$$F_{ns} = \sum_{|\alpha| \leq k-1} c_{ns\alpha} \frac{\partial^{|\alpha|}}{\partial \sigma_1^{\alpha_1} \partial \sigma_2^{\alpha_2} \ldots \partial \sigma_{N-1}^{\alpha_{N-1}} \partial t^{\alpha_N}}, \tag{3.153}$$

$\alpha_N = i_t$ if t is well chosen, with $\lim_{n\to\infty} c_{ns\alpha} = c_{s\alpha}$ in $C^{k-|\alpha|-1,1}(\partial\Omega)$. The spaces V_n (resp. V) are generated by the conditions $B_{ns} v = 0$ (resp. $B_s v = 0$) on $\partial\Omega$, $s = 1, 2, \ldots, \mu$. Moreover condition (3.145) is satisfied: we look for h such that $w = v - h$, $\partial^{i_t} h / \partial t^{i_t} = 0$ on $\partial\Omega$, $t = 1, 2, \ldots, k - \mu$, and $\partial^{j_s} h / \partial t^{j_s} = (F_{ns} - F_n) v$, $s = 1, 2, \ldots, \mu$. Such a function exists due to Theorems 2.5.7, 2.5.8; we have $|h|_k \leq \varepsilon_n |v|_k$, $\lim_{n\to\infty} \varepsilon_n = 0$. The same process is used for condition (3.146).

Remark 6.6. The conditions $B_{ns} v = 0$, resp. $B_s v = 0$, can be restricted to Γ, an open subset of $\partial\Omega$, without restrictions on the other part of $\partial\Omega$. In the previous Example 6.4, it is sufficient to have an extension operator:

$$[\prod_{i=0}^{k-1} W^{k-i-\frac{1}{2},2}(\Lambda) \to \prod_{i=0}^{k-1} W^{k-i-\frac{1}{2},2}(\partial\Omega)]$$

(cf. Corollary 2.5.3), or the operator from Theorem 2.5.8:

$$[\prod_{i=0}^{k-1} W^{k-i-\frac{1}{2},2}(\Lambda) \to W^{k,2}(\Omega)];$$

it is possible to use the construction from Chap. 2, 2.2.5.

Example 6.5. Let $\Omega \in \mathfrak{N}^{0,1}$, $\Lambda_n \subset \partial\Omega$, $\lim_{n\to\infty}\Lambda_n = \Lambda$, $\Lambda_n \subset \Lambda$, Λ and Λ_n open sets. Then we have:

$$\cup_{n=1}^{\infty} \cap_{i=n}^{\infty} \Lambda_i = \Lambda = \cap_{n=1}^{\infty} \cup_{i=n}^{\infty} \Lambda_i,$$

the spaces V_n, resp.V are given by the boundary conditions $\partial^i v/\partial n^i = 0$, $i = 1,2,\dots,k-1$ on Λ_n resp. Λ. We have in the sense of (3.151), (3.152) $\lim_{n\to\infty} V_n = V$; indeed, $V \subset V_n$, (3.151) is clear; concerning (3.152) we observe that: $V \subset \cap_{n=1}^{\infty}\overline{\cup_{i=n}^{\infty} V_i}$.

Let $v \in \cap_{n=1}^{\infty}\overline{\cup_{i=n}^{\infty} V_i}$; then there exist $v_n \in \cup_{i=n} V_i$, $\lim_{n\to\infty} v_n = v$ in $W^{k,2}(\Omega)$; but if $v_n \in \cup_i^{\infty} V_i$, we have $\partial^i v_n/\partial n^i = 0$, $i = 1,2,\dots,k-1$ in $\cap_i^{\infty}\Lambda_i$. Now using the Lebesgue theorem we have $\lim_{n\to\infty}\operatorname{meas}(\Lambda - \cap_{i=n}^{\infty}\Lambda_i) = 0$, hence $v \in V$.

3.6.6 Dependence on the Domain, the Dirichlet Problem

Now we investigate the dependence of solutions of our boundary value problems on the domain. Let Ω_n, Ω be such that $\lim_{n\to\infty}\Omega_n = \Omega$; we always assume $\Omega_n \subset \Omega$. I. Babuška [2, 3] has considered a more general case connected with the notion of stability of domains for the Dirichlet problem; for other results of the same type cf. J. Kautsky [1], A.Kufner [1].

The first steps in this direction can be found in N. Wiener [1], J. Keldysh [1]. The variational method was used by R. Courant, D. Hilbert [1], J. Deny, J.L. Lions [1].

We begin with the simplest:

Theorem 6.7. *Let Ω be a bounded (or unbounded) domain in $\mathbb{R}^N$, Ω_n, $n = 1,2,\dots$ a sequence of subdomains in Ω; $\cup_{n=1}^{\infty}\cap_{i=n}^{\infty}\Omega_i = \Omega$. Let $V(\Omega) = W_0^{k,2}(\Omega)$, $V(\Omega_n) = W_0^{k,2}(\Omega_n)$; we assume the functions in $V(\Omega_n)$ extended by zero to Ω and denote V_n the spaces of extended functions; we have $V_n \subset V$ and assume:*

$$v \in V \Longrightarrow \lim_{n\to\infty}\big(\inf_{v_n\in V_n} |v - v_n|_k\big) = 0. \tag{3.154}$$

Let $((v,u))$ be a V-elliptic sesquilinear form, $f_n, f \in W^{-k,2}(\Omega)$, $\lim_{n\to\infty} f_n = f$ in $W^{-k,2}(\Omega)$, $\lim_{n\to\infty} u_{0n} = u_0$ in $W^{k,2}(\Omega)$; u_n, u are the solutions of the corresponding boundary value problems. We extend u_n by u_{0n} outside of Ω_n. Then $\lim_{n\to\infty} u_n = u$ *in $W^{k,2}(\Omega)$.*

Proof. We prove first that $\lim_{n\to\infty} u_n = u$ weakly. Indeed, if $v \in V$, we have:

$$\lim_{n\to\infty}((v, u_n - u_{0,n} - u + u_0)) = \lim_{n\to\infty}((v, u_n - u)) = \lim_{n\to\infty}((v_n, u_n)) - ((v,u)),$$

where $v_n \in V_n, \lim_{n\to\infty} v_n = v$, by (3.154). But since

$$((v_n, u_n)) = \langle v_n \overline{f}_n\rangle, \quad ((v,u)) = \langle v, \overline{f}\rangle,$$

we get:

$$\lim_{n\to\infty}((v,u_n-u_{0,n}-u+u_0))=0.$$

Let us denote by Z^* the one-to-one, linear, continuous mapping $Z^*:V\to V$ defined by:

$$v,w\in V\Longrightarrow (Z^*v,w)_k=((v,w)),$$

cf. Lemma 1.3.1. We have:

$$\lim_{n\to\infty}(Z^*v,u_n-u_{0n}-u-u_0)_k=0;$$

but, by definition of $Z^*,Z^*(V)=V$, hence the result follows. (It is clear that the sequence $|u|_k$ is bounded). We can write:

$$\begin{aligned}
&\lim_{n\to\infty}((u_n-u_{0,n}-u+u_0,u_n-u_{0,n}-u+u_0))\\
&\qquad=\lim_{n\to\infty}((u_n-u_{0,n}-u+u_0,u_n-u))\\
&=\lim_{n\to\infty}((u_n-u_{0,n},u))-\lim_{n\to\infty}((u-u_0,u_n))-\lim_{n\to\infty}((u-u_0,u_n))+((u-u_0,u)).
\end{aligned}\tag{3.155}$$

We have:

$$\lim_{n\to\infty}((u_n-u_{0,n},u_n))=\lim_{n\to\infty}\langle u_n-u_{0,n},\overline{f}_n\rangle=\langle u-u_0,\overline{f}\rangle,$$

and finally using (3.155) and the V-ellipticity of the sesquilinear form the conclusion occurs.

□

We have obviously

Proposition 6.3. *Let Ω be a bounded domain, Ω_n, $n=1,2,\ldots$ a sequence of subdomains $\Omega_n\subset\Omega$ such that for every compact set $K\subset\Omega$, there exists n_0 such that $n\geq n_0\Longrightarrow K\subset\Omega_n$. Using the notations of the previous theorem we have (3.154).*

3.6.7 The General Case

Now we formulate a more general theorem; an adaptation is possible for the Dirichlet problem and other problems.

Theorem 6.8. *Let Ω be a bounded (or unbounded) domain in $\mathbb{R}^N$, Ω_n, $n=1,2,\ldots$ a sequence of subdomains in Ω, $\cup_{n=1}^{\infty}\cap_{i=n}^{\infty}\Omega_i=\Omega$. Let $V=V(\Omega)\subset W^{k,2}(\Omega)$, $V(\Omega_n)\subset W^{k,2}(\Omega_n)$. Let us assume the existence of a sequence of extension operators $P_n,P_n\in[V(\Omega_n)\to V(\Omega)]$, $|P_n|\leq const$, $n=1,2,\ldots$; we denote $V_n=P_n(V(\Omega_n))$.*

Moreover we also assume:

$$v \in V(\Omega) \Longrightarrow \lim_{n\to\infty}\big(\inf_{v_n\in V_n} |v-v_n|_k\big) = 0. \tag{3.156}$$

Let $V(\Omega_n) \subset Q(\Omega_n)$, $V(\Omega) \subset Q(\Omega)$ *algebraically and topologically,* $\overline{C_0^\infty(\Omega_n)} = Q(\Omega_n), \overline{C_0^\infty(\Omega)} = Q(\Omega)$; *let* $((v,u))_{\Omega_n}$, *resp.* $((v,u))_\Omega$ *be a sequence of sesquilinear forms on* $W^{k,2}(\Omega_n) \times W^{k,2}(\Omega_n)$ *resp. a sesquilinear form,* V*-elliptic on* $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$*, such that*

$$|((v,u))_{\Omega_n}| \le c_1 |v|_{W^{k,2}(\Omega_n)} |u|_{W^{k,2}(\Omega_n)}, \tag{3.157}$$

$$v \in V(\Omega_n) \Rightarrow |((v,v))_{\Omega_n}| \ge c_2 |v|^2_{V(\Omega_n)}, \tag{3.158}$$

$$v,u \in W^{k,2}(\Omega) \Rightarrow \lim_{n\to\infty} ((v,u))_{\Omega_n} = ((v,u))_\Omega, \tag{3.159}$$

the convergence being uniform with respect to u for fixed v and conversely.

Let $f \in Q'(\Omega), f_n \in Q'(\Omega_n)$. *Let us assume that for* $v \in V(\Omega_n)$

$$|f_n v - f P_n v| \le \varepsilon_n |v|_{W^{k,2}(\Omega_n)}, \quad \lim_{n\to\infty} \varepsilon_n = 0. \tag{3.160a}$$

Let $g_n \in \overline{V}(\Omega_n)', g \in \overline{V}(\Omega)'$ *the nonstable boundary conditions, assume that for* $v \in V(\Omega_n)$

$$|g_n v - g P_n v| \le \varepsilon'_n |v|_{W^{k,2}(\Omega_n)}, \quad \lim_{n\to\infty} \varepsilon'_n = 0. \tag{3.160b}$$

Let $\lim_{n\to\infty} u_{0,n} = u_0$ *in* $W^{k,2}(\Omega)$ *and denote by* u_n, u *the solutions of the corresponding boundary value problems.*

Then $\lim_{n\to\infty} P_n(u_n - u_{0,n}) = u - u_0$ *in* $W^{k,2}(\Omega)$.

Proof. We begin by proving that $\lim_{n\to\infty} P_n(u_n - u_{0,n}) = u - u_0$ weakly. Let $v \in V(\Omega)$ and let us consider $((v, P_n(u_n - u_{0,n})))_\Omega$. Condition (3.159) implies

$$\lim_{n\to\infty} ((v, P_n(u_n - u_{0,n})))_\Omega = \lim_{n\to\infty} ((v, u_n - u_{0,n}))_{\Omega_n};$$

now if $v_n \in V(\Omega_n)$, then by (3.156) $\lim_{n\to\infty} P_n v_n = v$. Using (3.157), we obtain:

$$\lim_{n\to\infty} ((v, u_n - u_{0,n}))_{\Omega_n} = \lim_{n\to\infty} ((v_n, u_n - u_{0,n}))_{\Omega_n}$$
$$= \lim_{n\to\infty} \langle v_n, \overline{f}_n \rangle + \lim_{n\to\infty} \overline{g}_n v_n - ((v,u_0))_\Omega = \langle v, \overline{f} \rangle + \overline{g} v - ((v,u_0))_\Omega;$$

this completes the first step of the proof.

Now we use the transformation $Z^* \in [V(\Omega) \to V(\Omega)]$, as in the proof of the previous theorem. Let $v_n \in V(\Omega_n)$ be a sequence such that

$$\lim_{n\to\infty} P_n v_n = u - u_0.$$

Using (3.157), (3.159), and the weak convergence proved above, we obtain:

$$\lim_{n\to\infty}((u_n-u_{0,n}-v_n,u_n-u_{0,n}-v_n))_{\Omega_n}=\lim_{n\to\infty}((u_n-u_{0,n},u_n-u_{0,n}))_{\Omega_n}$$
$$-\lim_{n\to\infty}((u_n-u_{0,n},v_n))_{\Omega_n}-\lim_{n\to\infty}((v_n,u_n-u_{0,n}))_{\Omega_n}+\lim_{n\to\infty}((v_n,v_n))_{\Omega_n}$$
$$=-\lim_{n\to\infty}((u_n-u_{0,n},u_0))_{\Omega_n}+\lim_{n\to\infty}\langle u_n-u_{0,n},\overline{f}_n\rangle+\lim_{n\to\infty}\overline{g}_n(u_n-u_0)$$
$$-\lim_{n\to\infty}((P_n(u_n-u_0),u-u_0))_{\Omega}-\lim_{n\to\infty}((u-u_0,P_n(u_n-u_{0,n})))_{\Omega}$$
$$+((u-u_0,u-u_0))_{\Omega}$$
$$=-((u-u_0,u_0))_{\Omega}+\langle u-u_0,\overline{f}\rangle+\overline{g}(u-u_0)-3((u-u_0,u-u_0))_{\Omega}=0;$$

the last result is a consequence of (3.158). Then finally:

$$\lim_{n\to\infty}|u_n-u_{0,n}-v_n|_{W^{k,2}(\Omega_n)}=0,$$

and hence

$$\lim_{n\to\infty}|P_n(u_n-u_{0,n}-v_n)|)=0.$$

□

Remark 6.7. The sesquilinear forms $((v,u))_{\Omega_n}$ depend on Ω_n. This can have two reasons:

1. The boundary forms $a_n(v,u)$ change with respect to Ω_n.

2. The coefficients of the operators change, and at the same time the sesquilinear forms $((v,u))_{\Omega_n}$ on $W^{k,2}(\Omega_n)\times W^{k,2}(\Omega_n)$ change with respect to Ω_n.

From this point of view, Theorem 6.8 can be considered as a generalization of Theorem 6.2 and is an example of simultaneous dependences.

Remark 6.8. In comparison with the Dirichlet problem, where the extension is the simple extension by zero, for other problems the situation is more complicated.

For the Neumann problem, consider Ω_n, $\Omega\in\mathfrak{N}^{k-1,1}$, and assume $\lim_{n\to\infty}\Omega_n=\Omega$ in the sense of 2.4.2. In the construction of P_n we can use Theorem 2.3.9, and we obtain $|P_n|\leq$const. If $\lim_{n\to\infty}\Omega_n=\Omega$ in $\mathfrak{N}^{0,1}$ in the sense of 2.4.2, we can use Theorem 2.3.10 and we obtain again $|P_n|\leq$const.

For other problems such as the oblique derivative, etc., in each case, where $V(\Omega_n)=W^{k,2}(\Omega_n)$, $V(\Omega)=W^{k,2}(\Omega)$, the situation is the same.

For all other problems, where $W_0^{k,2}(\Omega_n)\neq V(\Omega_n)\neq W^{k,2}(\Omega_n)$, we can combine different types of extensions.

Example 6.6. Let $\Omega=\{x\in\mathbb{R}^2,x_1^2+x_2^2<1,x_2>0\}$, $\Omega_n=\{x\in\mathbb{R}^2,x_1^2+x_2^2<1-\varepsilon_n,x_2>\varepsilon_n\},\varepsilon_n>0,\lim_{n\to\infty}\varepsilon_n=0$; $V(\Omega)=\{v\in W^{k,2}(\Omega);\partial^i v/\partial n^i=0$, $i=0,1,2,\dots,k-1$, for $x_2=0,|x_1|<1\}$, $V(\Omega_n)=\{v\in W^{k,2}(\Omega_n)$, $\partial^i v/\partial n^i=0$, $i=0,1,2,\dots,k-1$, for $x_2=\varepsilon_n,|x_1|<(1-\varepsilon_n-\varepsilon_n^2)^{1/2}\}$.

We define P_n by $P_n v=0$ for $x_2<\varepsilon_n$, $x_1^2+x_2^2<1-\varepsilon_n$, and we use Theorem 2.3.9.

Example 6.7. Let Ω be the unit disc, Ω_n the n-sided regular polygons with vertices on the circle $\partial\Omega$; let $V(\Omega) = \{v \in W^{2,2}(\Omega), v = 0 \text{ on } \partial\Omega)\}$, $V(\Omega_n) = \{v \in W^{2,2}(\Omega_n), v = 0 \text{ on } \partial\Omega_n\}$. The sequence P_n, as in Theorem 6.8, here does not exist. This is a result from I. Babuška [3]: let $u_n \in V(\Omega_n)$ be such that

$$\int_{\Omega_n} \sum_{|i|=2} \frac{2}{i!} D^i v D^i u_n \,\mathrm{d}x = \int_{\Omega_n} v f \,\mathrm{d}x, \quad v \in V(\Omega_n), \quad f \in L^2(\Omega),$$

and $u \in V(\Omega)$ such that

$$\int_{\Omega} \sum_{|i|=2} \frac{2}{i!} D^i v D^i u \,\mathrm{d}x = \int_{\Omega} v f \,\mathrm{d}x, \quad v \in V(\Omega).$$

Then there exists $u^* \in W^{1,2}(\Omega)$ such that

$$\lim_{n\to\infty} |u_n - u^*|_{W^{1,2}(\Omega_n)} = 0, \quad |u^* - u|_{W^{1,2}(\Omega)} \neq 0.$$

Remark 6.9. If $V(\Omega) = W^{k,2}(\Omega)$, $V(\Omega_n) = W^{k,2}(\Omega_n)$, usually the construction of the sequence P_n satisfies the hypothesis:

$$v \in V(\Omega) \Longrightarrow \lim_{n\to\infty} |P_n R v - v|_{W^{k,2}(\Omega)} = 0,$$

which implies (3.156).

Remark 6.10. Under certain hypotheses, it is not necessary to have $P(V(\Omega_n)) \subset V(\Omega)$; it is sufficient to require $P(V(\Omega_n)) \subset W^{k,2}(\Omega)$ and the condition (3.156) for $V_n = P(V(\Omega_n))$.

Remark 6.11. If $\partial\Omega, \partial\Omega_n$ are sufficiently smooth, sometimes it is more convenient to transform the problems on Ω_n to problems on Ω. Then we have the hypotheses from 6.1 to 6.5.

Example 6.8. Let $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^{0,1}$ in the sense of 2.4.2. Given:

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \big(a_{ij} \frac{\partial}{\partial x_j}\big) + b,$$

$$((v,u))_{\Omega_n} = \sum_{i,j=1}^{N} \int_{\Omega_n} \left(\overline{a}_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_j} + \overline{b} v \overline{u} \right) \mathrm{d}x + \int_{\partial\Omega_n} \overline{\sigma}_n v \overline{u} \,\mathrm{d}S,$$

$$((v,u))_{\Omega} = \sum_{i,j=1}^{N} \int_{\Omega} \left(\overline{a}_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_j} + \overline{b} v \overline{u} \right) \mathrm{d}x + \int_{\partial\Omega} \overline{\sigma} v \overline{u} \,\mathrm{d}S,$$

let us assume:

$$|\sigma_n(x'_r, a_{r,n}(x'_r))| \leq c, \quad \lim_{n\to\infty} \sigma_n(x'_r, a_{r,n}(x'_r)) = \sigma(x'_r, a_r(x'_r)) \text{ in sense of measure,}$$

and for complex numbers ζ_i, $i = 1, 2, \ldots, N$:

$$\sum_{i,j=1}^{N} \frac{\overline{a_{ij}} + a_{ji}}{2} \zeta_i \overline{\zeta}_j \geq \sum_{i=1}^{N} |\zeta_i|^2, \quad \text{Re}\, b \geq 0, \quad \text{Re}\, b \neq 0.$$

We take:

$$Q(\Omega_n) = L^2(\Omega_n),\ Q(\Omega) = L^2(\Omega);$$

$$f_n \in L^2(\Omega_n),\ f \in L^2(\Omega), \quad \lim_{n\to\infty} |f_n - f|_{L^2(\Omega_n)} = 0.$$

Let be $g_n \in L^2(\partial\Omega_n)$, $g \in L^2(\partial\Omega)$, such that

$$\lim_{n\to\infty} g_n(x'_r, a_{r,n}(x'_r)) = g(x'_r, a_r(x'_r)) \text{ in } L^2(\Delta_r), \quad \lim_{n\to\infty} u_{0,n} = u_0 \text{ in } W^{1,2}(\Omega).$$

We are in the setting of Theorem 6.8; indeed: $V(\Omega_n) = W^{1,2}(\Omega_n)$, $V(\Omega) = W^{1,2}(\Omega)$, using Theorem 2.3.9 we construct the sequence P_n. We have also Remark 6.9. It is clear that (3.157) is satisfied, because the following inequality holds:

$$\int_{\partial\Omega_n} |v|^2 dS \leq c_2 |v|^2_{W^{1,2}(\Omega_n)}. \tag{3.161}$$

If n is sufficiently large, then by Theorem 2.7.4 we have:

$$\text{Re}\ ((v,v))_{\Omega_n} \geq c_3 |v|^2_{W^{1,2}(\Omega_n)}, \quad \text{Re}\ ((v,v))_{\Omega} \geq c_3 |v|^2_{W^{1,2}(\Omega)}.$$

If $v, u \in W^{1,2}(\Omega), r = 1, 2, \ldots .m$, let us consider:

$$\int_{\Delta_r} \overline{\sigma}(x'_r, a_r(x'_r)) v(x'_r, a_r(x'_r)) \overline{u}(x'_r, a_r(x'_r)) \big(1 + \sum_{i=1}^{N-1} \big(\frac{\partial a_r}{\partial x_{ri}}(x'_r)\big)^2\big)^{1/2} \mathrm{d}x'_r$$
$$- \int_{\Delta_r} \overline{\sigma}_n(x'_r, a_{r,n}(x'_r)) v(x'_r, a_{r,n}(x'_r)) \overline{u}(x'_r, a_{r,n}(x'_r)) \big(1 + \sum_{i=1}^{N-1} \big(\frac{\partial a_{r,n}}{\partial x_{ri}}(x'_r)\big)^2\big)^{1/2} \mathrm{d}x'_r$$
$$\equiv \int_{\Delta} \overline{\sigma} v \overline{u} p \,\mathrm{d}x' - \int_{\Delta} \overline{\sigma}_n v_n \overline{u}_n p_n \,\mathrm{d}x'. \tag{3.161 bis}$$

We take v fixed, and $|u|_{W^{1,2}(\Omega)} \leq 1$;we obtain:

$$\lim_{n\to\infty} \sup_{|u|_1 \leq 1} \left| \int_{\Delta} (\overline{\sigma} - \overline{\sigma}_n) v \overline{u} p \,\mathrm{d}x' \right| = 0. \tag{3.162}$$

Indeed: let $\varepsilon > 0$ and $M_n \subset \Delta$ the set such that $|\sigma - \sigma_n| \geq \varepsilon$;
we have $\lim_{n\to\infty} \operatorname{meas}(M_n) = 0$, and hence

$$\int_\Delta |\overline{\sigma} - \overline{\sigma}_n||v||u||p|\,\mathrm{d}x' \leq c_4\varepsilon \left(\int_{C-M_n} |v|^2\,\mathrm{d}x'\right)^{1/2} \left(\int_{C-M_n} |u|^2\,\mathrm{d}x'\right)^{1/2}$$
$$+ c_5 \left(\int_{M_n} |v|^2\,\mathrm{d}x'\right)^{1/2} \left(\int_C |u|^2 dx'\right)^{1/2}.$$

Then (3.161) implies (3.162). Now using Theorem 2.4.5, we obtain:

$$\lim_{n\to\infty} \sup_{|u|_1 \leq 1} \left| \int_\Delta \overline{\sigma}_n (v - v_n)\overline{u} p\, dx' \right| = 0. \tag{3.163}$$

Then it follows:

$$\lim_{n\to\infty} \sup_{|u|_1 \leq 1} \left| \int_\Delta \overline{\sigma}_n v_n (\overline{u} - \overline{u}_n) p dx' \right| = 0, \tag{3.164}$$

and

$$\lim_{n\to\infty} \sup_{|u|_1 \leq 1} \left| \int_\Delta \overline{\sigma}_n v_n \overline{u}_n (p - p_n)\,\mathrm{d}x' \right| = 0. \tag{3.165}$$

Indeed: we have:

$$\left| \int_\Delta (\overline{\sigma}_n v_n \overline{u}_n (p - p_n)\,\mathrm{d}x' \right| \leq c_6 \big(\int_\Delta |v_n|^q\,\mathrm{d}x' \big)^{1/q} \big(\int_\Delta |u_n|^q\,\mathrm{d}x' \big)^{1/q} |a - a_n|_{W^{1,2N-2}(\Delta)}$$

for $N \geq 3$ with $1/q = \frac{1}{2} - \frac{1}{2}[1/(N-1)]$; for $N = 2$, the modification is clear; we use Theorems 2.4.2 and 2.4.6. We must also use the property:

$$\lim_{n\to\infty} |a - a_n|_{W^{1,p}(\Delta)} = 0$$

for all $p \geq 2$, but this is a consequence of the fact that

$$\lim_{n\to\infty} |a - a_n|_{W^{1,2}(\Delta)} = 0$$

and of $|a_n|_{C^{0,1}(\overline{\Delta})} \leq$ const; we can use inequality (2.4.6 bis), applied to $\partial a_n/\partial x_i$. Using (3.162)–(3.165) we deduce that the last quantity in (3.161 bis) tends to zero if $n \to \infty$, uniformly with respect to $|u|_k \leq 1$ which implies (3.159). (3.160a) is clear; using the inequality:

$$\left| \int_\Delta v_n g_n p_n\,\mathrm{d}x' - \int_\Delta v g p\,\mathrm{d}x' \right| \leq c_7 (\max_{x\in\overline{\Delta}} |a_{r,n} - a_r|^{1/2} |v|_1 |g|_{L^2(\partial\Omega)})$$
$$+ c_7 (|v|_1 |g_n - g|_{L^2(\Delta)} + c_7 |v|_1 |g_n|_{L^2(\Delta)} |a_r - a_{r,n}|_{W^{1,p}(\Delta)},$$

we obtain (3.161) with $p > 4$, if $N = 2$, and with $p = 4N - 4$, if $N \geq 3$. We must use (2.4.6).

Exercise 6.3. Let $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^{0,1}$, $V(\Omega_n) = W^{k,2}(\Omega_n)$, $V(\Omega) = W^{k,2}(\Omega)$; we assume the existence of a sequence $P_n, |P_n| \leq c$, and (3.156); let us denote $Q(\Omega_n) = L^2(\Omega_n)$, $Q(\Omega) = L^2(\Omega)$. Let A be the operator

$$A = \sum_{|i|,|j|=k} D^i(a_{ij}D^j);$$

let us set:

$$((v,u))_{\Omega_n} = \int_{\Omega_n} \sum_{|i|,|j|=k} \overline{a}_{ij} D^i v D^j \overline{u} \, \mathrm{d}x, \quad ((v,u))_{\Omega} = \int_{\Omega} \sum_{|i|,|j|=k} \overline{a}_{ij} D^i v D^j \overline{u} \, \mathrm{d}x$$

and assume:

$$|((\tilde{v},\tilde{v}))_{\Omega_n}| \geq c_1 |\tilde{v}|^2_{W^{k,2}(\Omega_n)/P_{(k-1)}}, \quad |((\tilde{v},\tilde{v}))_{\Omega}| \geq c_1 |\tilde{v}|^2_{W^{k,2}(\Omega)/P_{(k-1)}}.$$

Let

$$f_n \in L^2(\Omega_n),\ f \in L^2(\Omega), \quad \lim_{n\to\infty} |f_n - f|_{L^2(\Omega_n)} = 0.$$

Finally we assume:

$$p \in P_{(k-1)} \Longrightarrow \int_{\Omega} p\overline{f} dx = \int_{\Omega_n} p\overline{f}_n dx = 0$$

Let u, resp.u_n, be the solution of the homogenous Neumann problem. Prove that in $W^{k,2}(\Omega)/P_{(k-1)}$, $\lim_{n\to\infty} \widetilde{P_n u_n} = \tilde{u}$.

3.6.8 Dependence on the Domain, Another Method

Let us come back to the case $\lim_{n\to\infty} \Omega_n = \Omega$, $V(\Omega_n) = W^{k,2}(\Omega_n)$, $V(\Omega) = W^{k,2}(\Omega)$. For simplicity, we consider only the homogeneous case; let us assume that the following sesquilinear form is given:

$$A(v,u)_{\Omega} = \int_{\Omega} \sum_{|i|,|j|\leq k} \overline{a}_{ij} D^i v D^j \overline{u} \, \mathrm{d}x,$$

and let us define:

$$A(v,u)_{\Omega_n} = \int_{\Omega_n} \sum_{|i|,|j|\leq k} \overline{a}_{ij} D^i v D^j \overline{u} \, \mathrm{d}x;$$

the reader can make other generalizations. We have $Q(\Omega_n)$, $Q(\Omega)$, $f_n \in Q'(\Omega_n), f \in Q'(\Omega)$. Let $u_n \in W^{k,2}(\Omega_n)$, $u \in W^{k,2}(\Omega)$. We shall say that $\lim_{n\to\infty} u_n = u$ *almost in* $W^{k,2}(\Omega)$, if

$$\lim_{n\to\infty} |u_n - u|_{W^{k,2}(\Omega_n)} = 0,$$

this will be written as $\lim_{n\to\infty} u_n = u$-almost in $W^{k,2}(\Omega)$, if $|u_n|_{W^{k,2}(\Omega_n)} \leq$const and if for Ω_n, $n = 1,2,\dots,u_n$ converges to u weakly.

We shall assume that f_n converges to f in the following sense: if $\lim_{n\to\infty} u_n = u$- almost in $W^{k,2}(\Omega)$, then

$$\lim_{n\to\infty} \langle u_n, \overline{f}_n \rangle = \langle u, \overline{f} \rangle, \ |f_n|_{Q'(\Omega_n)} \leq \text{const.} \tag{3.166}$$

For instance, this is the case if:

$$Q(\Omega_n) = L^2(\Omega_n), \quad Q(\Omega) = L^2(\Omega), \quad f_n \in L^2(\Omega_n), \quad f \in L^2(\Omega),$$
$$\lim_{n\to\infty} |f_n - f|_{L^2(\Omega_n)} = 0.$$

We have:

Theorem 6.9. *Let* Ω *be a bounded domain,* $\lim_{n\to\infty} \Omega_n = \Omega$ *in the following sense: if* $K = \overline{K} \subset \Omega$*, then there exists* n_0*, such that* $n \geq n_0 \Longrightarrow K \subset \Omega_n$*. Let us assume, that for* $v \in W^{k,2}(\Omega)$*,*

$$|A(v,v)_\Omega| \geq c_1 |v|^2_{W^{k,2}(\Omega)}, \tag{3.167}$$

and for $v \in W^{k,2}(\Omega_n)$

$$|A(v,v)_{\Omega_n}| \geq c_1 |v|^2_{W^{k,2}(\Omega_n)}. \tag{3.168}$$

Let $f_n \in Q'(\Omega_n)$*,* $f \in Q'(\Omega)$*,* $\lim_{n\to\infty} f_n = f$ *in the sense given by (3.166),* u_n *the solution of the problem*

$$A(v,u_n)_{\Omega_n} = \langle v, \overline{f}_n \rangle, \quad v \in W^{k,2}(\Omega_n),$$

and u *the solution of the problem*

$$A(v,u)_\Omega = \langle v, \overline{f} \rangle, \quad v \in W^{k,2}(\Omega).$$

Then $\lim_{n\to\infty} u_n = u$*−almost in* $W^{k,2}(\Omega)$*.*

Proof. First of all we have $|u_n|_{W^{k,2}(\Omega_n)} \leq c_1$; using the diagonal process we can extract from the sequence u_n a subsequence u_{n_i} which converges weakly almost in $W^{k,2}(\Omega)$. Let u^* be this limit; now we claim $u = u^*$ almost everywhere in Ω. Indeed: let $v \in W^{k,2}(\Omega)$, we do not loss any generality if we assume $\lim_{n\to\infty} D^\alpha u_{n_i} = D^\alpha u^*-$ almost in $L^2(\Omega)$, $|\alpha| \leq k$; this is a direct consequence of $\lim_{n\to\infty} u_{n_i} = u^*-$almost in $W^{k,2}(\Omega)$. We have:

$$A(v,u^*)_\Omega = \lim_{i\to\infty} A(v,u^*)_{\Omega_{n_i}} = \lim_{i\to\infty} A(v,u_{n_i})_{\Omega_{n_i}} = \lim_{i\to\infty}\langle v,\overline{f}_{n_i}\rangle = \langle v,\overline{f}\rangle.$$

Now we prove $\lim_{n\to\infty} u_{n_i} = u$ strongly. Indeed: we have

$$\lim_{i\to\infty} A(u_{n_i}-u,u_{n_i}-u)_{\Omega_{n_i}} = \lim_{i\to\infty}\langle u_{n_i},\overline{f}_{n_i}\rangle - \lim_{i\to\infty}\langle u,\overline{f}_{n_i}\rangle$$
$$+\lim_{i\to\infty} A(u,u)_{\Omega_{n_i}} - \lim_{i\to\infty} A(u_{n_i},u)_{\Omega_{n_i}} = 0,$$

then the result follows from (3.168); this implies the strong convergence. Indeed: if not, there would exist a subsequence u_{n_j} such that $|u_{n_j}-u|_{W^{k,2}(\Omega_{n_j})} \geq \varepsilon > 0$, weakly convergent almost in $W^{k,2}(\Omega)$. If $\lim_{n\to\infty} u_{n_j} = u^*$ weakly, then as above $u^* = u$ and $\lim_{n\to\infty} u_{n_j} = u$ which is a contradiction. □

We can see that for the Neumann problem, for the mixed problem, etc, Ω_n, Ω can be very general and we still obtain the continuous dependence on the domain.

Remark 6.12. We can generalize Theorem 6.9 in other cases, for instance if we assume the existence of a sequence of operators $R_n \in [V(\Omega) \to V(\Omega_n)]$, $|R_n| \leq$ const., such that

$$\text{for } v \in V(\Omega), \quad \lim_{n\to\infty} |R_n v - v|_{W^{k,2}(\Omega_n)} = 0;$$

or it is sufficient to assume that for $v \in V(\Omega')$ there exists a sequence $v_n \in V(\Omega_n)$ such that

$$\lim_{n\to\infty} |v - v_n|_{W^{k,2}(\Omega_n)} = 0.$$

Clearly we can define the continuous dependence on the domains if $\Omega_n \supset \Omega$, $\lim_{n\to\infty} \Omega_n = \Omega$ in some sense. For the Dirichlet problem, we have a positive answer if $W_0^{k,2}(\Omega) = \cap_{i=1}^{\infty} W_0^{k,2}(\Omega_i)$; the details can be found in I. Babuška [2, 3].

3.7 Elliptic Systems

We add this section to complete and to generalize the previous considerations for systems of equations, but the results will be not given in complete generality. We will not return to systems in the remaining part of this book; the reasons for this choice is due to the fact that the investigation of systems is not so developed in comparison with the equations and the state of the art for systems is presently not sufficient.

We follow the ideas introduced in the works of L. Nirenberg [1], J.L. Lions [4], M.I. Vishik [1], in particular we shall give new results concerning the coercivity which can be easily applied.

For particular questions we recommend various references given later in this section.

3.7.1 Elliptic Systems and Sesquilinear Forms

Let us define $m \times m$ differential operators in the form

$$A_{rs} = \sum_{|i| \le \kappa_r, |j| \le \kappa_s} (-1)^{|i|} D^i (a_{ij}^{rs} D^j), \tag{3.169}$$

where κ_r, κ_s, $r,s = 1,2,\dots,m$, are positive integers, a_{ij}^{rs} are measurable bounded functions on Ω. The system (3.169) is called *elliptic at the point x*, if for $\xi \in \mathbb{R}^N$, $\xi \neq 0$, the determinant:

$$\det \sum_{|i|=\kappa_r, |j|=\kappa_s} a_{ij}^{rs}(x) \xi^{i+j} \neq 0,$$

and *elliptic in Ω* if the ellipticity property holds almost everywhere in Ω.

Let $\mathbf{u} = (u_1, u_2, \dots, u_m)$ be a vector from the product $\prod_{i=1}^m W^{\kappa_i,2}(\Omega)$, and $\mathbf{v}$ another vector from this product; we associate the system (3.169) with the *sesquilinear form*

$$A(\mathbf{v},\mathbf{u}) = \int_\Omega \Big(\sum_{r,s=1}^m \sum_{|i| \le \kappa_r, |j| \le \kappa_s} \overline{a}_{ij}^{rs} D^i v_r D^j \overline{u}_s \Big) \,\mathrm{d}x. \tag{3.170}$$

Together with (3.170) we define a *boundary sesquilinear form*, say $a(\mathbf{v},\mathbf{u})$, such that $a(\mathbf{v},\mathbf{u}) = 0$ if at least one of $\mathbf{v},\mathbf{u} \in \prod_{i=1}^m W_0^{\kappa_i,2}(\Omega)$.

The following proposition is clear:

Proposition 7.1. *Let $\Omega \in \mathfrak{N}^{0,1}$, and*

$$a(\mathbf{v},\mathbf{u}) = \int_{\partial\Omega} \sum_{r,s=1}^m \sum_{i=0}^{\kappa_r - 1} \sum_{|\alpha| \le \kappa_s - 1} \overline{b}_{i\alpha}^{rs} \frac{\partial^i v_r}{\partial n^i} D^\alpha \overline{u}_s \,\mathrm{d}S, \tag{3.171}$$

with $b_{i\alpha}^{rs} \in L^\infty(\partial\Omega)$. Then $a(\mathbf{v},\mathbf{u})$ is a boundary form.

According to Lemma 2.5, we obtain:

Theorem 7.1. *Let*

$$\kappa = \max_{r=1,2,\dots,m} \kappa_r, \quad \Omega \in \mathfrak{N}^{2\kappa,1}, \quad b_{i\alpha}^{rs} \in L^\infty(\partial\Omega) \qquad \textit{for} \quad |\alpha| < \kappa_s,$$

$$b_{i\alpha}^{rs} \in C^{|\alpha| - \kappa_s, 1}(\partial\Omega) \qquad \textit{for} \quad |\alpha| - \kappa_s \ge 0.$$

If the operators

$$\sum_{|\alpha| \le \kappa_r + \kappa_s - i - 1} b_{i\alpha}^{rs} D^\alpha$$

are at most $\kappa_s - 1$ transversal (cf. 1.2), then

$$a(\mathbf{v},\mathbf{u}) = \int_{\partial\Omega} \sum_{r,s=1}^{m} \sum_{i=0}^{\kappa_r-1} \sum_{|\alpha|\le\kappa_r+\kappa_s-i-1} \overline{b}_{i\alpha}^{rs} \frac{\partial^i v_r}{\partial n^i} D^{\alpha} \overline{u}_s \, \mathrm{d}S \tag{3.172}$$

is a boundary form.

3.7.2 Boundary Value Problems

Now we define the *boundary value problem* for an elliptic system; consider

$$\Omega \in \mathfrak{N}^{0,1}, \quad \partial\Omega = \cup_{l=1}^{\lambda} \Gamma_l \tag{3.173a}$$

(except a set of zero superficial measure on the boundary), Γ_l open disjointed sets in $\partial\Omega$;

the system (3.169) and the associated sesquilinear form (3.170); (3.173b)

a boundary sesquilinear form of the form (3.171) or (3.172), depending on the regularity of $\partial\Omega$; (3.173c)

indices j_{lra}, $l = 1,2,\dots,\lambda$, $r = 1,2,\dots,m$, such that $0 \le j_{lra} \le \kappa_r - 1$, $a = 1,2,\dots,\mu_{lr}$, $j_{lr1} < j_{lr2} <,\dots,< j_{lr\mu_{lr}}$, complementary indices to i_{lrb} from the set $0,1,2,\dots,\ \kappa_{r-1}$ such that $0 \le i_{lrb} \le \kappa_r - 1$, $b = 1,2,\dots,$ $\kappa_r - \mu_{lr}$, operators for $l = 1,2,\dots,\lambda$, $r = 1,2,\dots,m$, $a = 1,2,\dots,\mu_{lr}$: (3.173d)

$$\frac{\partial^{j_{lr\alpha}} v_r}{\partial n^{j_{lr\alpha}}} = \sum_{s=1}^{m} \sum_{b=1}^{\kappa_s-\mu_{ls}} c_{lrsab} \frac{\partial^{i_{lsb}} v_s}{\partial n^{i_{lsb}}}, \tag{3.174}$$

with coefficients c_{lrsab} which are complex functions in general; if the sets Γ_l are locally $(\kappa+1)$-times continuously differentiable, the operators can be written as

$$\frac{\partial^{j_{lra}} v_r}{\partial n^{j_{lra}}} = \sum_{s=1}^{m} \sum_{|\alpha|\le\kappa_s-1} c_{lrsab} \frac{\partial^{|\alpha|} v_s}{\partial\sigma_1^{\alpha_1} \partial\sigma_2^{\alpha_2} \partial\sigma_{N-1}^{\alpha_{N-1}} \partial n^{\alpha_N}}, \tag{3.175}$$

α_N is one of the indices i_{lsb}; we denote V the subset of $\mathbf{v}$ satisfying (3.174), (3.175);

a Banach space Q, $V \subset Q$ algebraically and topologically such that $[C_0^{\infty}(\Omega)]^m$ is dense is Q; (3.176)

a vector $\mathbf{u_0} \in \prod_{i=1}^{m} W^{\kappa_i,2}(\Omega)$, a distribution $\mathbf{f} \in Q'$, functionals $g_{lsb} \in \overline{V}'$, $l = 1,2,\dots,\lambda$; $s = 1,2,\dots,m$; $b = 1,2,\dots,\kappa_s - \mu_{ls}$ such that $g_{lsb} v_s = 0$ if on Λ_l (3.177)

$$\frac{\partial^{i_{lsb}} v_s}{\partial n^{i_{lsb}}} = 0.$$

We set

$$g\mathbf{v} = \sum_{l=1}^{\lambda} \sum_{s=1}^{m} \sum_{b=1}^{\kappa_s-\mu_{ls}} g_{lsb} v_s.$$

The vector $\mathbf{u} \in \prod_{i=1}^{m} W^{\kappa_i,2}(\Omega)$ is *the solution of the boundary value problem*, if:

$$\mathbf{u} - \mathbf{u_0} \in V, \tag{7.8a}$$

and for all $\mathbf{v} \in V$

$$A(\mathbf{v},\mathbf{u}) + a(\mathbf{v},\mathbf{u}) = \overline{f}\mathbf{v} + \overline{g}\mathbf{v}. \tag{7.8b}$$

Now can proceed as in Chap. 1; using Green's formula we can give a formal interpretation of the conditions (3.178b).

Obviously we obtain, in the sense of distributions:

$$\sum_{s=1}^{m} A_{rs} u_s = f_r,$$

with

$$\mathbf{f}\mathbf{v} = \sum_{s=1}^{m} f_s v_s.$$

3.7.3 Strongly Elliptic Systems

The sesquilinear form $((\mathbf{v},\mathbf{u})) = A(\mathbf{v},\mathbf{u}) + a(\mathbf{v},\mathbf{u})$ is called *V-elliptic,* if for $\mathbf{v} \in V$

$$|((\mathbf{v},\mathbf{v}))| \geq c|\mathbf{v}|_m^2. \tag{3.179}$$

where $|\mathbf{v}|_m = (\sum_{r=1}^{m} |v_r|_{W^{\kappa_r,2}})^{1/2}$.

As in Chap. 1, we have

Theorem 7.2. *For a given boundary value problem with the sesquilinear form $((\mathbf{v},\mathbf{u}))$ V-elliptic, there exists a unique solution of the problem, and we have the estimate:*

$$|\mathbf{u}|_{\mathbf{W}^{\kappa,2}} \leq c[|f|_{Q'} + |\mathbf{u}_0|_{\mathbf{W}^{\kappa,2}} + |g|_{V'}].\text{[9]} \tag{3.180}$$

Remark 7.1. The definition of the Green operator and Fredholm alternative can be obtained by the same approach as for equations, cf. Sect. 3; if $((\mathbf{v},\mathbf{u}))$ is hermitian, we generalize immediately the results obtained in Chap. 1, Sect. 6.

The system (3.169) is called *uniformly elliptic in Ω,* if almost everywhere in Ω:

$$|\det \sum_{|i|=\kappa_r, |j|=\kappa_s} a_{ij}^{rs}(x)\xi^{i+j}| \geq c|\xi|^{2\sum_{r=1}^{m}\kappa_r}.$$

The system is called *strongly elliptic at x,* if for all $\xi \in \mathbb{R}^N$, $\xi \neq 0$, and for all complex numbers $\eta_1, \eta_2, \ldots, \eta_m$,

[9]Here we denote $\mathbf{W}^{\kappa,\mathbf{2}} = \prod_{i=1}^{m} W^{\kappa_i,2}(\Omega)$.

$$\operatorname{Re} \sum_{r,s=1}^{m} \sum_{|i|=\kappa_r,|j|=\kappa_s} \overline{a}_{ij}^{rs}(x)\xi^{i+j}\eta_r\overline{\eta}_s \geq c(x)\sum_{r=1}^{m}|\eta_r|^2|\xi|^{2\kappa_r}, \quad c(x)>0, \qquad (3.181a)$$

and *uniformly strongly elliptic* if $c(x) \geq c > 0$.

Theorem 7.3. *Assume that the operators A_{rs} in (3.169) are of order $\kappa_r + \kappa_s$, with constant coefficients, and the system is strongly elliptic in the bounded domain Ω. Then the sesquilinear form $A(\mathbf{v},\mathbf{u})$ is $\prod_{i=1}^{m} W_0^{\kappa_i,2}(\Omega)$-elliptic.*

Proof. Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m) \in [C_0^\infty(\Omega)]^m$; we have:

$$\begin{aligned}
\operatorname{Re} A(\varphi,\varphi) &= \operatorname{Re} \int_\Omega \Big(\sum_{r,s=1}^{m} \sum_{|i|=\kappa_r,|j|=\kappa_s} \overline{a}_{ij}^{rs} D^i\varphi_r D^j\overline{\varphi_s}\Big)\,\mathrm{d}x \\
&= \operatorname{Re} \frac{1}{2\pi\mathrm{i}^N}\int_{\mathbb{R}^N} \sum_{r,s=1}^{m} \sum_{|i|=\kappa_r,|j|=\kappa_s} \overline{a}_{ij}^{rs}\xi^{i+j}(i^{\kappa_r}\hat{\varphi}_r)\overline{(i^{\kappa_s}\hat{\varphi}_s)}\,\mathrm{d}\xi \\
&\geq c_1 \int_{\mathbb{R}^N} \sum_{r=1}^{m} |\hat{\varphi}_r|^2|\xi|^{2\kappa_r}\,\mathrm{d}\xi = c_1(2\pi)^N \int_{\mathbb{R}^N} \sum_{r=1}^{m}\sum_{|i|=\kappa_r} \frac{\kappa_r!}{i!}|D^i\varphi_r|^2\,\mathrm{d}x.
\end{aligned}$$

□

Let us observe that if $\partial\Omega$ is sufficiently smooth, the necessary and sufficient conditions of V-coercivity (cf. Sect. 4) have been obtained by D.G. Figueiredo [1] using a method developed by S. Agmon (cf. [1]).

By the same approach as used in the proof of Theorem 3.5, we prove:

Theorem 7.4. *Let us assume: Ω bounded, the coefficients a_{ij}^{rs} in (3.169) for $|i| = \kappa_r, |j| = \kappa_s$ in $C^0(\overline{\Omega})$. If the system is uniformly elliptic for λ sufficiently large, then the form*

$$A(\mathbf{v},\mathbf{u}) + \lambda\sum_{r=1}^{m}(v_r,u_r) \qquad (3.181b)$$

is $\prod_{i=1}^{m} W_0^{\kappa_i,2}(\Omega)$-elliptic (the form $A(\mathbf{v},\mathbf{u})$ is $\prod_{i=1}^{m} W_0^{\kappa_i,2}(\Omega)$-coercive).

3.7.4 Algebraically Complete, Formally Positive Forms

As far as concerns the $\prod_{i=1}^{m} W^{\kappa_i,2}(\Omega)$-ellipticity of the forms $((\mathbf{v},\mathbf{u}))$, we have a simple theorem:

Theorem 7.5. *Let ζ_i^r be complex numbers, $r = 1,2,\ldots,m$, $|i| = \kappa_r$. We assume that almost everywhere in $\Omega \in \mathfrak{N}^0$*

$$\operatorname{Re} \sum_{r,s=1}^{m} \sum_{|i|=\kappa_r,|j|=\kappa_s} \overline{a}_{ij}^{rs} \zeta_i^r \overline{\zeta_j^s} \geq c \sum_{r=1}^{m} \sum_{|i|=\kappa_r} |\zeta_i^r|^2, \quad c > 0. \tag{3.181c}$$

Then for λ sufficiently large, the form (3.181b) is $\prod_{i=1}^{m} W^{\kappa_i,2}(\Omega)$-elliptic.

Proof. Indeed: we have

$$\operatorname{Re} \int_\Omega \sum_{r,s=1}^{m} \sum_{|i|=\kappa_r,|j|=\kappa_s} \overline{a}_{ij}^{rs} D^i v_r D^j \overline{v}_s \, dx \geq c \int_\Omega \sum_{r=1}^{m} \sum_{|i|=\kappa_r} |D^i v_r|^2 \, dx, \quad c > 0;$$

then we proceed as in Theorem 4.1.

□

Let us now consider the operators

$$N_l \mathbf{v} = \sum_{s=1}^{m} \sum_{|\alpha| \leq \kappa_s} a_{ls\alpha} D^\alpha v_s, \quad a_{ls\alpha} \in L^\infty(\Omega), \quad l = 1, 2, \ldots, h, \tag{3.182}$$

and the forms

$$A(\mathbf{v}, \mathbf{u}) = \int_\Omega \Big(\sum_{l=1}^{h} N_l \mathbf{v} \overline{N_l \mathbf{u}}\Big) \, dx. \tag{3.183}$$

Such a form is called *V-coercive* if there exists $\lambda \geq 0$ such that for all $\mathbf{v} \in V$

$$A(\mathbf{v}, \mathbf{v}) + \lambda \sum_{s=1}^{m} (v_s, v_s) \geq c \sum_{s=1}^{m} |v_s|^2_{W^{\kappa_s,2}(\Omega)}. \tag{3.184}$$

If Ω is sufficiently smooth, $m = 1$, $V = W^{k,2}(\Omega)$, a necessary and sufficient condition for V-coercivity was given in N. Aronszajn [2]; for $m = 1$ and V more general see M. Schechter [1, 3]; for $m = 1$, $h = 1$ cf. also Sect. 5. For $m \geq 1$ cf. M.S. Agranovich, A.S. Dynin [1], S. Agmon, A. Douglis, L. Nirenberg [2]: $V = \prod_{i=1}^{m} V_i$ with V_i of the same type as in Sect. 5.

We consider the problem of $\prod_{i=1}^{m} W^{\kappa_i,2}(\Omega)$-coercivity for (3.183) in the case $\Omega \in \mathfrak{N}^{0,1}$.

We call the system $N_i \mathbf{v}$ a *complete* system, if the form (3.183) is $\prod_{i=1}^{m} W^{\kappa_i,2}(\Omega)$-coercive.

Now we prove the following

Lemma 7.1. *Let $\Omega \in \mathfrak{N}^{0,1}$, l an integer, $f \in W^{-l,2}(\Omega)$. Then*

$$|f|_{W^{-l,2}(\Omega)} \leq c \left(\sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i} \right|_{W^{-l-1,2}(\Omega)} + |f|_{W^{-l-1,2}(\Omega)} \right). \tag{3.185}$$

Proof. If $l < 0$, the lemma is trivial. Let $l \geq 0$; we use the same notations as in 1.2.4. $C_0^\infty(\Omega)$ is dense in $W^{-l,2}(\Omega)$, and $C_0^\infty(\Omega) \subset W^{-l,2}(\Omega)$ in the sense of distributions, hence it is sufficient to consider $f \in C_0^\infty(\Omega)$. Let $f_r = f\varphi_r$. For $r = 1, 2, \dots, m+1$, we have obviously:

$$\sum_{i=1}^{N} \left| \frac{\partial f_r}{\partial x_i} \right|_{W^{-l-1,2}(\Omega)} + |f_r|_{W^{-l-1,2}(\Omega)} \leq c_1 \left(\sum_{i=1}^{N} \left| \frac{\partial f}{\partial x_i} \right|_{W^{-l-1,2}(\Omega)} + |f|_{W^{-l-1,2}(\Omega)} \right). \tag{3.186}$$

Let us consider first f_{m+1}; we have $\operatorname{supp} f_{m+1} \subset U_{m+1}$, then

$$\begin{aligned} &\sum_{i=1}^{N} \left| \frac{\partial f_{m+1}}{\partial x_i} \right|_{W^{-l-1,2}(\mathbb{R}^N)} + |f_{m+1}|_{W^{-l-1}(\mathbb{R}^N)} \\ &\quad \leq c_2(U_{m+1}) \left(\sum_{i=1}^{N} \left| \frac{\partial f_{m+1}}{\partial x_i} \right|_{W^{-l-1,2}(\Omega)} + |f_{m+1}|_{W^{-l-1,2}(\Omega)} \right). \end{aligned} \tag{3.187}$$

On the other hand:

$$\begin{aligned} |f_{m+1}|^2_{W^{-l,2}(\Omega)} \leq |f_{m+1}|^2_{W^{-l,2}(\mathbb{R}^N)} &\leq c_3 \int_{\mathbb{R}^N} |\hat{f}_{m+1}(\xi)|^2 (1+|\xi|^2)^{-l} \,\mathrm{d}\xi \\ &= c_3 \int_{\mathbb{R}^N} |\hat{f}_{m+1}(\xi)|^2 (1+|\xi|^2)(1+|\xi|^2)^{-l-1} \,\mathrm{d}\xi \\ &\leq c_4 \left(\sum_{i=1}^{N} \left| \frac{\partial f_{m+1}}{\partial x_i} \right|^2_{W^{-l-1,2}(\mathbb{R}^N)} + |f_{m+1}|^2_{W^{-l-1,2}(\mathbb{R}^N)} \right), \end{aligned} \tag{3.188}$$

hence, using (3.187), we obtain (3.185) for $r = m+1$.

Now let us put

$$a_r(h, x_r') = \frac{1}{\kappa h^{N-1}} \int_{\Delta_r(\alpha)} \exp \frac{|x_r' - \xi_r'|^2}{|x_r' - \xi_r'|^2 - h^2} a_r(\xi_r') \,\mathrm{d}\xi_r'.$$

We choose $0 < \alpha' < \alpha$, $0 < \beta' < \beta$, $\alpha - \alpha'$, $\beta - \beta'$ sufficiently small such that U_i', $i = 1, 2, \dots, m$, and U_{m+1} cover Ω and that $\varphi \in C_0^\infty(U_i')$, $i = 1, 2, \dots, m$. We choose $0 < h < \alpha - \alpha'$. By simple computations, using the hypothesis $a_r \in C^{0,1}(\overline{\Delta_r(\alpha)})$, we obtain $a_r(h, x_r') \in C^0(\overline{(0, \alpha-\alpha') \times \Delta_r(\alpha')})$, $a_r(h, x_r')$ infinitely differentiable in $(0, \alpha - \alpha') \times \Delta_r(\alpha')$, and

$$\left| \frac{\partial^{|i|} a_r}{\partial x_{r_1}^{i_1} \partial x_{r_2}^{i_2} \dots \partial x_{r_{N-1}}^{i_{N-1}} \partial h^{i_N}} \right| \leq \frac{c_5}{h^{|i|} - 1}, \quad |i| \geq 1. \tag{3.189}$$

Let $\varepsilon > 0$ be sufficiently small such that for $0 < \eta < \beta'$

$$\left| \frac{\partial}{\partial \eta} a_r(\varepsilon\eta, x_r') \right| \leq \frac{1}{2} \left(\frac{\beta}{\beta'} - 1 \right) = c_6.$$

We denote $K_r = (0,\beta') \times \Delta_r(\alpha')$ and define $T : K_r \to V_r^* = \{x_r' \in \Delta_r(\alpha'),\ a_r(x_r') < x_{rN} < a_r(x_r') + (1+2c_6)\beta'\}$ setting:

$$y_r' = x_r', \quad a_r(\varepsilon y_{rN}, y_r') + (1+c_6)y_{rN} = x_{rN}. \tag{3.190}$$

We have $T(K_r) \supset V_r^*, T$ is a one-to-one mapping such that the Jacobian,

$$\frac{dT}{dy} = \varepsilon \frac{\partial a_r}{\partial h} + 1 + c_{6,}$$

is infinitely differentiable in K_r.

If $m = 0,1,\dots, \psi \in C_0^\infty(K_r)$, and if we denote $\chi(x) = \psi(T^{-1}(x))$, we have

$$|\chi|_{W^{m,2}(V_r^*)} \le c_7 |\psi|_{W^{m,2}(K_r)}. \tag{3.191}$$

This result is a consequence of the following inequality which we must prove:

$$\int_{V_r^*} |D^\alpha \psi(T^{-1}(x))|^2 (x_{rN} - a_r(x_r'))^{2(|\alpha|-m)}\,\mathrm{d}x \le c_8 |\psi|^2_{W^{m,2}(K_r)}, \quad |\alpha| \le m. \tag{3.192}$$

According to Lemma 2.3.2:

$$\int_{V_r^*} |D^\alpha \psi(T^{-1}(x))|^2 (x_{rN} - a_r(x_r'))^{2(|\alpha|-m)}\,\mathrm{d}x \le c_9 \int_{K_r} |D^\alpha \psi(y)|^2 y_{rN}^{2(|\alpha|-m)}\,\mathrm{d}y,$$

and (3.192) follows by the Hardy inequality (cf. Lemma 2.5.1). Let us denote $f_r(T(y)) = g_r(y)$ and let us prove:

$$\sum_{i=1}^N \left|\frac{\partial g_r}{\partial y_{ri}}\right|_{W^{-l-1,2}(K_r)} + |g_r|_{W^{-l-1,2}(K_r)} \le c_{10}\left(\sum_{i=1}^N \left|\frac{\partial fr}{\partial x_i}\right|_{W^{-l-1,2}(V_r)} + |fr|_{W^{-l-1,2}(V_r)}\right). \tag{3.193}$$

Indeed, if $\psi \in W_0^{l+1,2}(K_r)$, we obtain:

$$\int_{K_r} \frac{\partial g_r}{\partial y_{ri}} \psi(y)\,\mathrm{d}y = \int_{V_r'} \sum_{j=1}^N \frac{\partial f_r}{\partial x_{rj}} \frac{\partial x_{rj}}{\partial y_{ri}} \chi(x) \left(\frac{dT}{dy}\right)^{-1} \mathrm{d}x.$$

But $\chi \in W_0^{l+1,2}(V_r^*)$ by (3.191) and we have as in (3.192),

$$\left|\frac{\partial x_{rj}}{\partial y_{ri}} \left(\frac{dT}{dy}\right)^{-1} \chi\right|_{W^{l+1,2}(V_r^*)} \le c_{11} |\chi|_{W^{l-1,2}(V_r^*)}.$$

Then using (3.191), with the same argument for $\int_{K_r} g_r \psi dy$, we obtain (3.193).

Let us denote $M_r = \{y, y'_r \in \Delta_r(\alpha'), |y_{rN}| < \beta'\}$ and let λ_i, $i = 1, 2, \dots, l+2$, be solutions of the linear system:

$$\sum_{i=1}^{l+2} (-i)^h \lambda_i = 1, \quad h = 0, 1, \dots, l+1. \tag{3.194}$$

If $-\beta' < y_{rN} < 0$, let us put

$$g_r(y'_r, y_{rN}) = \sum_{i=1}^{l+2} \lambda_i g\left(y'_r, -\frac{y_{rN}}{i}\right).$$

We have:

$$\sum_{i=1}^{N} \left|\frac{\partial g_r}{\partial y_{ri}}\right|_{W^{-l-1,2}(M_r)} + |g_r|_{W^{-l-1,2}(M_r)} \le c_{12}\left(\sum_{i=1}^{N}\left|\frac{\partial g_r}{\partial x_i}\right|_{W^{-l-1,2}(K_r)} + |g_r|_{W^{-l-1,2}(K_r)}\right). \tag{3.195}$$

Indeed, if $v \in W_0^{l+1,2}(M_r)$, $i = 1, 2, \dots, N-1$, then

$$\begin{aligned}\int_{M_r} \frac{\partial g_r}{\partial y_{ri}} v\,dy &= \int_{K_r} \frac{\partial g_r}{\partial y_{ri}}(y'_r, y_{rN}) v(y)\,\mathrm{d}y + \int_{K_{r_-}} \left(\sum_{i=1}^{l+2} \lambda_i \frac{\partial g_r}{\partial y_{ri}}\left(y'_r, -\frac{y_{rN}}{i}\right)\right) v(y)\,\mathrm{d}y \\ &= \int_{K_r} \frac{\partial g_r}{\partial y_{ri}}(y'_r, y_{rN})\big(v(y'_r, y_{rN}) + \sum_{i=1}^{l+2} \lambda_i i v(y'_r, -i y_{rN})\big)\,\mathrm{d}y \\ &= \int_{K_r} \frac{\partial g_r}{\partial y_{ri}}(y'_r, y_{rN}) w(y'_r, y_{rN})\,\mathrm{d}y,\end{aligned}$$

where $w(y) = v(y'_r,\ y_{rN}) + \sum_{i=1}^{l+2} \lambda_i i v(y'_r, -i y_{rN})$; by (3.194), $w \in W_0^{l+1,2}(K_r)$ and satisfies

$$|w|_{W^{l+1,2}(K_r)} \le c_{13} |v|_{W^{l+1,2}(M_r)};$$

the same argument works for $\int_{M_r} g_r v dy$. Now we consider:

$$\begin{aligned}\int_{M_r} \frac{\partial g_r}{\partial y_{rN}} v\,\mathrm{d}y &= \int_{K_r} \frac{\partial g_r}{\partial y_{rN}}(y'_r, y_{rN}) v(y)\,\mathrm{d}y + \int_{K_{r_-}} \left(\sum_{i=1}^{l+2} \frac{-\lambda_i}{i} \frac{\partial g_r}{\partial y_{rN}}\left(y'_r, -\frac{y_{rN}}{i}\right)\right) v(y)\,\mathrm{d}y \\ &= \int_{K_r} \frac{\partial g_r}{\partial y_{rN}}(y'_r, y_{rN})\big(v(y'_r, y_{rN}) - \sum_{i=1}^{l+2} \lambda_i i v(y'_r, -i y_{rN})\big)\,\mathrm{d}y \\ &= \int_{K_r} \frac{\partial g_r}{\partial y_{rN}}(y'_r, y_{rN}) \omega(y'_r, y_{rN})\,\mathrm{d}y.\end{aligned}$$

It follows again from (3.194) that $\omega \in W_0^{l+1,2}(K_r)$ and that

$$|\omega|_{W^{l+1,2}(K_r)} \le c_{14}|v|_{W^{l+1,2}(M_r)};$$

these inequalities imply (3.195). The support of g_r is compact in M_r, hence it follows as for f_{m+1},:

$$|g_r|_{W^{-l,2}(M_r)} \le c_{15}\left(\sum_{i=1}^{N}\left|\frac{\partial g_r}{\partial y_{ri}}\right|_{W^{-l-1,2}(M_r)} + |g_r|_{W^{-l-1,2}(M_r)}\right). \tag{3.196}$$

Obviously, we also have:

$$|g_r|_{W^{-l,2}(K_r)} \le |g_r|_{W^{-l,2}(M_r)},$$

then with the same approach as used in the proof of (3.193) we get the inequality:

$$|f_r|_{W^{-l,2}(V_r')} \le |g_r|_{W^{-l,2}(K_r)}$$

and (3.185) is a consequence of (3.186), (3.193), and (3.196). □

Arguing by recurrence, we obtain from Lemma 7.1

Lemma 7.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, $f \in W^{-l,2}(\Omega), l$ and ν integers, $\nu \ge 0$. Then*

$$|f|_{W^{-l,2}(\Omega)} \le c\left(\sum_{|\alpha|=\nu}^{N} |D^\alpha f|_{W^{-l-\nu,2}(\Omega)} + |f|_{W^{-l-\nu,2}(\Omega)}\right). \tag{3.197}$$

Now we come back to the properties of the operators (3.182). Let us put for $\xi \in \mathbb{R}^N$

$$N_{is}\xi = \sum_{|\alpha|=\kappa_s} a_{is\alpha}(x)\xi^\alpha.$$

We say that the operators (3.182) constitute at x a system *algebraically complete*, it there exist positive integers ν_r, $r = 1,2,\dots,m$, such that for all $|\beta| = \nu_r + \kappa_r$ we can find homogenous polynomials of degree $\nu_r, B_l\xi$, $l = 1,2,\dots,h$, such that

$$\sum_{l=1}^{h} \overline{B_l\xi} N_{lr}\xi = \xi^\beta, \quad \sum_{l=1}^{h} \overline{B_l\xi} N_{ls}\xi = 0, \quad s \ne r. \tag{3.198}$$

Theorem 7.6. *Let $\Omega \in \mathfrak{N}^{0,1}$, $a_{is\alpha}$ constants and*

$$N_l\mathbf{v} = \sum_{s=1}^{m} \sum_{|\alpha|=\kappa_s} a_{is\alpha} D^\alpha v_s,$$

an algebraically complete system. Then it is complete, i.e the form given in (3.183) is $\prod_{i=1}^m W^{\kappa_i,2}(\Omega)$-coercive.

Proof. Let $\mathbf{v} \in \left[C_0^\infty(\mathbb{R}^N)\right]^m$, and consider for $\varphi \in C_0^\infty(\Omega)$, $\quad |\gamma| = \kappa_r \quad |\delta| = \nu_r$ the integral

$$\int_\Omega D^\gamma v_r D^\delta \overline{\varphi}\,\mathrm{d}x.$$

We have:

$$\begin{aligned}\int_\Omega D^\gamma v_r D^\delta \overline{\varphi}\,\mathrm{d}x &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} (\mathrm{i})^{\kappa_r-\nu_r}\xi^{\gamma+\delta}\hat{v}_r(\xi)\overline{\hat{\varphi}(\xi)}\,\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\sum_{s=1}^m (\mathrm{i})^{\kappa_r-\nu_r}(\sum_{l=1}^h \overline{B_l\xi}N_{ls}\xi)\hat{v}_s(\xi)\overline{\hat{\varphi}(\xi)}\,\mathrm{d}\xi \qquad (3.199) \\ &= \int_\Omega (\sum_{l=s}^m\sum_{l=1}^h \overline{(B_l\varphi)}N_{ls}v_s)\,\mathrm{d}x = \int_\Omega(\sum_{l=1}^h \overline{B_l\varphi}N_l\mathbf{v})\,\mathrm{d}x,\end{aligned}$$

where $B_l\xi$ are as in (3.198). Now using Lemma 7.2 with $l = 0$, $\nu = \nu_r$, $r = 1,2,\ldots,m$, for any $\gamma, |\delta| = \nu_r$, we obtain the inequalities:

$$\begin{aligned}|D^\gamma v_r|_{L^2(\Omega)} &\le c_1(|D^\gamma v_r|_{W^{-\nu_r,2}(\Omega)} + \sum_{l=1}^h |N_i\mathbf{v}|_{L^2(\Omega)}) \\ &\le c_2(|v_r|_{W^{\kappa_r-\nu_r,2}(\Omega)} + \sum_{l=1}^h |N_l\mathbf{v}|_{L^2(\Omega)}).\end{aligned}$$

For γ arbitrary with length $|\gamma| = \kappa_r$, we obtain:

$$\sum_{s=1}^m |v_s|^2_{W^{\kappa_s,2}(\Omega)} \le c_3(\sum_{s=1}^m |v_s|^2_{W^{\kappa_s-\nu_s,2}(\Omega)} + \sum_{s=1}^m |v_s|^2_{L^2(\Omega)} + \sum_{l=1}^h |N_l\mathbf{v}|^2_{L^2(\Omega)}),$$

and the result follows from Lemma 2.6.1. □

Using Lemma 2.6.1 and the partition of unity given in Theorem 4.5, we prove:

Theorem 7.7. *Let be $\Omega \in \mathfrak{N}^{0,1}$, $a_{ls\alpha} \in C^0(\overline{\Omega})$ for $|\alpha| = \kappa_s$, and*

$$N_l\mathbf{v} = \sum_{s=1}^m \sum_{|\alpha|\le\kappa_s} a_{ls\alpha}D^\alpha v_s.$$

Let $N_l(x_0)\mathbf{v}$ be algebraically complete for every $x_0 \in \overline{\Omega}$. Then the system is complete, i.e. the form in (3.183) is $\prod_{i=1}^m W^{\kappa_i,2}(\Omega)$-coercive.

By definition, the operators in (3.182) constitute at x an *algebraically complete system*, if for all $\xi \neq 0, \xi_i$ complex numbers, we have:

$$\operatorname{rank}(N_{ls}\xi) = m. \qquad (3.200)$$

We have (cf. J. Nečas [15]):

Theorem 7.8. *Let $\Omega \in \mathfrak{N}^{0,1}$, $N_l\mathbf{v}$ as in Theorem 7.6. Then the system is complete if and only if condition (3.200) is satisfied. The definitions (3.198) and (3.200) are equivalent.*

Proof. (a) The condition is necessary: if for $\xi \neq 0$, we have rank $(N_{ls}\xi) < m$, then we can find numbers $b_1, b_2, \dots, b_m$ such that $\sum_{s=1}^m N_{ls}\xi \cdot b_s = 0$, $l = 1,2,\dots,h$, $\sum_{s=1}^m |b_s| \neq 0$.
For each complex $\lambda \neq 0$ we define $\mathbf{u}_\lambda$ by $u_{r\lambda}(x) = \lambda^{-\kappa_r} b_r e^{\lambda(\xi,x)}$. By (3.184), any linear combination of $\mathbf{u}_\lambda$, say $\mathbf{u}$, satisfies $\sum_{r=1}^m |u_r|^2_{W^{\kappa_r,2}(\Omega)} \le c_1 \sum_{r=1}^m |u_r|^2_{L^2(\Omega)}$, which is not possible because the set of such $\mathbf{u}$ is a finite dimensional space and the identity imbedding $W^{\kappa_r,2}(\Omega) \to L^2(\Omega)$ is compact (cf. Theorem 1.14).

(b)The condition is sufficient: Let us denote by $\Delta_i\xi$, $i = 1,2,\dots,\tau$, the $m \times m$ determinants of the matrix $N_{ls}\xi$ and by $\Delta_{i,rl}$ the $(m-1)\times(m-1)$ associated algebraic minors; if $\Delta_{i,rl}\xi$ is not defined, we set $\Delta_{i,rl}\xi = 0$. The only common root of $\Delta_i\xi$ is 0. By the Hilbert theorem – cf. Van der Waerden [1] – we know that there exist polynomials $P_j\xi$ such that $\sum_{j=1}^\tau P_j\xi\Delta_j\xi = \xi_1^\rho$, ρ a positive integer. If we consider $B_l\xi = \sum_{j=1}^\tau \Delta_{j,rl}\xi P_j\xi$ we obtain (3.198) with ξ_1^ρ, hence (3.198) in the general case. $\square$

Remark 7.2. A proof of Theorem 7.8 in L^p- spaces, $1 < p < \infty$, is given in J. Nečas [15].

If $\Omega = \mathbb{R}^N$, then Lemmas 7.2, 7.3 are obvious; as in Theorem 7.6 we can prove with $\varphi \in C_0^\infty(\mathbb{R}^N)$

Proposition 7.2. *Let $\Omega \in \mathbb{R}^N$ be bounded, $a_{ls\alpha}$ constants,*

$$N_l\mathbf{v} = \sum_{s=1}^m \sum_{|\alpha|=\kappa_s} a_{ls\alpha} D^\alpha v_s$$

an algebraically complete system. Then the form (3.183) is $\prod_{i=1}^m W_0^{\kappa_i,2}(\Omega)$-coercive.

3.7.5 Examples

Example 7.1. Let $\Omega \in \mathfrak{N}^{0,1}$, $N = 2$, $m = 1$, $N_1 v = \partial^k v/\partial x_1^k$, $N_2 v = \partial^k v/\partial x_2^k$, $k \ge 1$. Then N_1, N_2 form an algebraically complete system. For $v \in W^{k,2}(\Omega)$ we have:

$$\int_\Omega \sum_{|\alpha|\le k} |D^\alpha v|^2 \, \mathrm{d}x \le c_1 \int_\Omega \left(|v|^2 + \left|\frac{\partial^k v}{\partial x_1^k}\right|^2 + \left|\frac{\partial^k v}{\partial x_2^k}\right|^2 \right) \mathrm{d}x.$$

Now, let us consider a particular case, called *the system of elasticity*:

$$\sum_{s=1}^{3} A_{rs}\mathbf{u} = -\mu\triangle u_r - (\lambda+\mu)\frac{\partial}{\partial x_r}\operatorname{div}\mathbf{u}, \quad \mu > 0,\ \lambda \geq 0. \tag{3.201}$$

We immediately obtain:

Proposition 7.3. *The system (3.201) is strongly elliptic. If we consider the Laplacian written in the form* $-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}[\mathbf{1}\frac{\partial}{\partial x_i}]$*, we obtain immediately the condition (3.181a) for the sesquilinear form* $A(\mathbf{v},\mathbf{u})$ *generated by (3.201).*

In elasticity problems we use another decomposition:

$$-\lambda\frac{\partial}{\partial x_r}\operatorname{div}\mathbf{u} - 2\mu\operatorname{div}\varepsilon_r, \quad \varepsilon_r = (\varepsilon_{r1},\varepsilon_{r2},\varepsilon_{r3}),$$

where

$$\varepsilon_{rs} = \frac{1}{2}\left(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r}\right). \tag{3.202}$$

The V-ellipticity of the Dirichlet problem does not depend on the decomposition of (3.201). For the investigation of coercivity of $A(\mathbf{v},\mathbf{u})$ corresponding to (3.202) cf. K.O. Friedrichs [1], D.M. Ejdus [1], S.G. Mikhlin [3], S. Campanato [1–4]; the inequality

$$A(\mathbf{v},\mathbf{v}) \geq c_1\sum_{s=1}^{3}\sum_{i=1}^{3}\left|\frac{\partial v_s}{\partial x_i}\right|^2_{L^2(\Omega)}$$

is known as *Korn's inequality*.

We have for $\mathbf{v}\in\prod_{i=1}^{3}W^{1,2}(\Omega)$ obviously

$$A(\mathbf{v},\mathbf{v}) = \int_\Omega(\lambda|\operatorname{div}\mathbf{v}|^2 + 2\mu\sum_{i,j=1}^{3}|\varepsilon_{ij}|^2)\,\mathrm{d}x. \tag{3.203}$$

We prove without difficulty that the system ε_{ij} is algebraically complete. If we neglect the term λ div $\mathbf{v}$ ($\lambda = 0$ is possible), we have $h = 6$ and

$$N_1\mathbf{v} = \frac{\partial v_1}{\partial x_1}, \quad N_2\mathbf{v} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right), \quad N_3\mathbf{v} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right),$$
$$N_4\mathbf{v} = \frac{\partial v_2}{\partial x_2}, \quad N_5\mathbf{v} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right), \quad N_6\mathbf{v} = \frac{\partial v_3}{\partial x_3}.$$

Let us consider the case $r = 1$ and let us write for ξ^α the polynomials $(B_1\xi, B_2\xi, \dots, B_6\xi)$ from (3.198):

$$\begin{aligned}
\xi_1^2 &\sim (\xi_1, 0, 0, 0, 0, 0),\\
\xi_1\xi_2 &\sim (\xi_2, 0, 0, 0, 0, 0),\\
\xi_1\xi_3 &\sim (\xi_3, 0, 0, 0, 0, 0),\\
\xi_2^2 &\sim (0, 2\xi_2, 0, -\xi_1, 0, 0),\\
\xi_2\xi_3 &\sim (0, \xi_3, \xi_2, 0, -\xi_1, 0),\\
\xi_3^2 &\sim (0, 0, 2\xi_3, 0, 0, -\xi_1).
\end{aligned}$$

Then we get:

Theorem 7.9. *If $\Omega \in \mathfrak{N}^{0,1}$, then the sesquilinear form (3.203) is $\prod_{i=1}^{3} W^{1,2}(\Omega)-$ coercive.*

If we solve one of the elasticity problems, for instance: given V, a closed subspace in $[W^{1,2}(\Omega)]^3$, we look for $\mathbf{u} \in V$ such that for all $\mathbf{v} \in V$,

$$A(\mathbf{v}, \mathbf{u}) = \sum_{i=1}^{3} (v_i, f_i), \quad f_i \in L^2(\Omega),$$

where $A(\mathbf{v}, \mathbf{u})$ is given by (3.203); then it is sufficient to apply the Fredholm alternative. We find a finite dimensional subspace $H \subset V$ and the solution exists if

$$0 = \sum_{r=1}^{3} (h_r, f_r), \quad h = (h_1, h_2, h_3), \quad h \in H.$$

If this last condition is satisfied, the solution is unique if we require:

$$\sum_{r=1}^{3} (h_r, u_r) = 0 \text{ for } h \in H.$$

We recover well-known results, cf. S.G. Mikhlin [3].

Let us observe, that for $\Omega \in \mathfrak{N}^{0,1}$ the results obtained by S. Campanato [1, 2] are a consequence of Theorem 7.8.

We close this chapter with a result of L. Nirenberg [1].

Example 7.2. Let $N = 2$, $m = 2$, $\kappa_1 = 1$, $\kappa_2 = 3$, L_1, M_1, L_3, M_3, N_2 operators with constant coefficients of orders indicated by the indices. Let us define:

$$A_1\mathbf{u} = -\triangle u_1 + L_1 u_1 + \frac{\partial^4 u_2}{\partial x_2^4} + L_3 u_2 + \lambda u_1,$$

$$A_2\mathbf{u} = -\frac{\partial^4 u_1}{\partial x_2^4} - \triangle^3 u_2 + M_3 u_1 + N_2 M_1 u_1 + \lambda u_2.$$

Using these operators we define the following corresponding sesquilinear form:

$$A(\mathbf{v},\mathbf{u}) = \int_\Omega \Big(\frac{\partial v_1}{\partial x_1}\frac{\partial \overline{u}_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\frac{\partial \overline{u}_1}{\partial x_2} + v_1\overline{L_1 u_1} - \frac{\partial v_1}{\partial x_2}\frac{\partial^3 \overline{u}_2}{\partial x_3^3} + v_1\overline{L_3 u_2} + \frac{\partial^3 v_2}{\partial x_2^3}\frac{\partial \overline{u}_1}{\partial x_2}$$
$$+ \frac{\partial^3 v_2}{\partial x_1^3}\frac{\partial^3 \overline{u}_2}{\partial x_1^3} + \frac{\partial^3 v_2}{\partial x_2^3}\frac{\partial^3 \overline{u}_2}{\partial x_2^3} + 3\frac{\partial^3 v_2}{\partial x_1^2 \partial x_2}\frac{\partial^3 \overline{u}_2}{\partial^2 x_1 \partial x_2} + 3\frac{\partial^3 v_2}{\partial x_1 \partial x_2^2}\frac{\partial^3 \overline{u}_2}{\partial x_1 \partial^2 x_2}$$
$$+ M_3^* v_2 \overline{u}_1 + N_2^* v_2 \overline{M_1 u_1} + \overline{\lambda} v_1 \overline{u}_1 + \overline{\lambda} v_2 \overline{u}_2 \Big)\, \mathrm{d}x;$$

the system is strongly elliptic, moreover it is easy to see that $A(\mathbf{v},\mathbf{u})$ is $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$-coercive, for instance if $\Omega \in \mathfrak{N}^0$. A delicate problem is the computation of κ_1, κ_2. Let us observe that the term $-\partial^4 u_1/\partial x_2^4$ in A_2 "corresponds" to $\kappa_1 = 2$.

Chapter 4
Regularity of the Solution

In Chap. 3, we have found the solution for a large class of problems. Here we adapt the difference method, used by L. Nirenberg [1], E. Magenes, G. Stampacchia [1], J.L. Lions [4], N. Aronszajn [1], M. Schechter [2, 4] and others to prove the smoothness of the weak solution inside of the domain if the coefficients and the right hand side of the equation are smooth; moreover we introduce the concept of a very weak solution; cf. S.L. Sobolev [1], M.I. Vishik, G. Fichera [3–5], J.L. Lions [4], J.L. Lions, E. Magenes [1–3, 5–8], E. Magenes [3],... and we prove for the very weak solution some regularity theorems. The Green kernels are a particular case of very weak solutions, cf. L. Schwartz [3], J.L. Lions [4]. For questions concerning the regularity of solutions cf. also K.O. Friedrichs [2].

For other references cf. also: S.Agmon [2], S. Agmon, A. Douglis, L. Nirenberg [1, 2], M.S. Agranovich [1], M.S. Agranovich, A.S. Dynin [1], M.S. Agranovich, L.R. Volevich, A.S. Dynin [1], Ju.M. Berezanskii, S.G. Krein, Ja.A. Rojtberg [1], S.N. Bernstein [1], M.S. Birman, T.E. Skvorcov [1], F.E. Browder [3–5, 7, 8], R. Cacciopoli [1], A. Douglis, L. Nirenberg [1], A.S. Dynin [1], V.V. Fufaev [2], I.M. Gelfand [1], G. Geymonat, P. Grisvard [1], D. Greco [1], L. Hörmander [2, 3], F. John [1], J. Kadlec [2, 3], A.J. Koshelev [1], O.A. Ladyzhenskaya [1], Ja.B. Lopatinskii [1], E. Magenes [2, 3, 5], B. Malgrange [3], N.G. Meyers [1], C. Miranda [3], Ch.B. Morrey jr, L. Nirenberg [1], A.L. Mullikin, K.T. Smith [1], J. Nečas [9], L. Nirenberg [2, 3], J. Peetre [1, 3], Ja.A. Roitberg, Z.T. Sheftel [1], P.C. Rosenblum [1], E. Shamir [1], M. Schechter [3, 5, 7–10], L.N. Slobodetskii [1–4], V.A. Solonnikov [1], [2], S.L. Sobolev, M.I. Vishik [1], B.Ju. Sternin [1], Z.T. Sheftel [1], M.I. Vishik, B.E. Shilov [1], A.I. Volpert [1], L.R. Volevich [1].

In this chapter, we also consider the behaviour of the solution in a neighborhood of the boundary. We use the compensation method of N. Aronszajn and we prove that this method can be used in all problems considered in Chap. 3.

We obtain also results concerning boundary value problems for properly elliptic operators and boundary operators satisfying a covering condition, cf. Chap. 3, Sect. 3.5. The reader will find the references for this topic in Sect. 4.3.

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4.1 Interior Regularity

4.1.1 Regularity of the Weak Solution

Let us consider $u \in W^{k,p}(\Omega)$, $p \geq 1$, $\Omega' \subset \Omega$, $\overline{\Omega}' \subset \Omega$. Let $|h| < \text{dist}(\partial\Omega, \overline{\Omega}')$, denote $h^{(i)} = (0,...,0,h,0,...,0) \in \mathbb{R}^N$: i.e. the coordinate on the i-th place equals h, the other coordinates are zero. Let us denote $\Delta_h^i u = (u(x+h^{(i)}) - u(x))/h$.

Lemma 1.1. *Let $\overline{\Omega}' \subset \Omega$, Ω' bounded, dist $(\partial\Omega, \overline{\Omega}') > 0$ and $|h| <$ dist $(\partial\Omega, \overline{\Omega}')$, $u \in W^{k,p}(\Omega)$. Then*

$$|\Delta_h^i|_{W^{k-1,p}(\Omega')} \leq c|u|_{W^{k,p}(\Omega)}, \tag{4.1}$$

$$\lim_{h\to 0} \left| \Delta_h^i u - \frac{\partial u}{\partial x_i} \right|_{W^{k-1,p}(\Omega')} = 0. \tag{4.2}$$

Proof. Without loss of generality, we can assume $\overline{C^\infty(\overline{\Omega})} = W^{k,p}(\Omega)$; if not we replace Ω by a subdomain $\Omega^* \subset \Omega$, with $\partial\Omega^*$ smooth, such that $\overline{C^\infty(\overline{\Omega^*})} = W^{k,p}(\Omega^*)$. We use Theorem 2.3.1. Let $u \in C^\infty(\overline{\Omega})$. We have:

$$\Delta_h^i u(x) = \int_0^1 \frac{\partial u}{\partial x_i}(x + h^{(i)}t)\, \mathrm{d}t.$$

Let α be a multi-index such that $|\alpha| \leq k-1$. Then

$$\begin{aligned} &\int_{\Omega'} \left| \int_0^1 \left[\frac{\partial D^\alpha u}{\partial x_i}(x + h^{(i)}t) - \frac{\partial D^\alpha u}{\partial x_i}(x) \right] \mathrm{d}t \right|^p \mathrm{d}x \\ &\leq \int_0^1 \mathrm{d}t \int_{\Omega'} \left| \frac{\partial D^\alpha u}{\partial x_i}(x + h^{(i)}t) - \frac{\partial D^\alpha u}{\partial x_i}(x) \right|^p \mathrm{d}x, \end{aligned}$$

hence

$$\int_{\Omega'} \left| \Delta_h^i D^\alpha u - \frac{\partial}{\partial x_i} D^\alpha u \right|^p \mathrm{d}x \leq \int_0^1 \mathrm{d}t \int_{\Omega'} \left| \frac{\partial D^\alpha u}{\partial x_i}(x + h^{(i)}t) - \frac{\partial D^\alpha u}{\partial x_i}(x) \right|^p \mathrm{d}x. \tag{4.3}$$

By a limiting procedure, we obtain (4.3) for $u \in W^{k,p}(\Omega)$, and Theorem 2.1.1 implies (4.2). The proof of (4.1) can be done in the same way. □

Theorem 1.1. *Let Ω be a bounded domain, A an operator from 1.2.1 with the associated sesquilinear form $A(v,u)$ $W_0^{k,2}(\Omega)-elliptic$. Assume that the coefficients a_{ij} of A belong to $C^{0,1}(\overline{\Omega})$. Let us consider $f \in W^{-k+1,2}(\Omega)$, and let $u \in W^{k,2}(\Omega)$ be a weak solution of $Au = f$, i.e. in the sense of distributions (cf. Remark 3.2.1). Then for each subdomain $\Omega' \subset \overline{\Omega}' \subset \Omega$, $u \in W^{k+1,2}(\Omega')$ and the following inequality holds:*

$$|u|_{W^{k+1,2}(\Omega')} \leq c(\Omega')\big(|u|_{W^{k,2}(\Omega)} + |f|_{W^{-k+1,2}(\Omega)}\big). \tag{4.4}$$

Proof. Let $\overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega$, $\psi \in C_0^\infty(\Omega'')$ be such that $\psi(x) = 1$ for $x \in \overline{\Omega}'$. Let us denote $w = \psi u$, $\delta > 0$ sufficiently small such that if $|h| < \delta$ then $x + h^{(\tau)} \in \Omega$, $1 \le \tau \le N$, for $x \in \overline{\Omega}''$. Let $\varphi \in W_0^{k,2}(\Omega'')$. Then

$$\begin{aligned} A(\varphi, \Delta_h^\tau w) &= \int_\Omega \sum_{|i|,|j| \le k} \overline{a}_{ij} D^i \varphi D^j (\Delta_h^\tau \overline{w}) \, \mathrm{d}x \\ &= -\int_\Omega \sum_{|i|,|j| \le k} \overline{a}_{ij}(x) \Delta_{-h}^\tau (D^i \varphi(x)) D^j \overline{w}(x) \, \mathrm{d}x \\ &\quad - \int_\Omega \sum_{|i|,|j| \le k} \Delta_{-h}^{(\tau)} \overline{a}_{ij}(x) D^i \varphi(x - h^{(\tau)}) D^j \overline{w}(x) \, \mathrm{d}x. \end{aligned} \tag{4.5}$$

Moreover,

$$\begin{aligned} &\int_\Omega \sum_{|i|,|j| \le k} \overline{a}_{ij}(x) \Delta_{-h}^\tau (D^i \varphi(x)) D^j \overline{(u(x)\psi(x))} \, \mathrm{d}x \\ =&\int_\Omega \sum_{|i|,|j| \le k} \overline{a}_{ij}(x) \Delta_{-h}^\tau (D^i (\varphi(x) \overline{\psi(x)})) D^j \overline{u(x)} \, \mathrm{d}x + I_1(\varphi, u) \\ =&\overline{f}(\Delta_{-h}^\tau (\varphi \overline{\psi})) + I_1(\varphi, u), \end{aligned} \tag{4.6}$$

where $I_1(\varphi, u)$ is a sum of terms of the following types:

$$\int_\Omega a \Delta_{-h}^\tau D^i \varphi D^j \overline{u} \, \mathrm{d}x, \quad |i| \le k, \quad |j| \le k-1, \quad a \in C^{0,1}(\overline{\Omega}), \tag{4.7}$$

or

$$\int_\Omega a D^i \varphi D^j \overline{u} \, \mathrm{d}x, \quad |i| \le k, \quad |j| \le k, \quad a \in C^{0,1}(\overline{\Omega}), \tag{4.8}$$

or

$$\int_\Omega a \Delta_{-h}^\tau D^i \varphi D^j \overline{u} \, \mathrm{d}x, \quad |i| \le k-1, \quad |j| \le k, \quad a \in C^{0,1}(\overline{\Omega}). \tag{4.8 bis}$$

We integrate by parts and transpose (4.7) into (4.8 bis). Then Lemma 1.1 implies:

$$|I_1(\varphi, u)| \le c_1 |\varphi|_{W^{k,2}(\Omega)} |u|_{W^{k,2}(\Omega)}; \tag{4.9}$$

moreover (4.5), (4.6), (4.9) lead to

$$|A(\varphi, \Delta_h^\tau w)| \le c_2 (|\varphi|_{W^{k,2}(\Omega)} |u|_{W^{k,2}(\Omega)} + |f|_{W^{-k+1,2}(\Omega)} |\varphi|_{W^{k,2}(\Omega)}). \tag{4.10}$$

We can choose δ sufficiently small such that $\Delta_h^\tau w \in W_0^{k,2}(\Omega'')$, thus by (4.10) and by the $W_0^{k,2}(\Omega)$-ellipticity of $A(v, u)$, taking $\varphi = \Delta_h^\tau w$, we obtain:

$$|\Delta_h^\tau w|_{W^{k,2}(\Omega)} \le c_3 (|u|_{W^{k,2}(\Omega)} + |f|_{W^{-k+1,2}(\Omega)}). \tag{4.11}$$

Due to Lemma 1.1 $\lim_{h\to 0}\Delta_h^\tau w = \partial w/\partial x_\tau$ in $W^{k-1,2}(\Omega)$; the theorem follows by (4.11) and by Proposition 2.2.4; we take $\tau = 1,2,\dots,N$. □

In L. Nirenberg [1] we have with more restrictive hypotheses:

Theorem 1.2. *Let Ω be bounded, A the operator from 1.2.1 such that $A(v,u)$ is $W_0^{k,2}(\Omega)$-elliptic. Let us assume that the coefficients $a_{ij} \in C^{\alpha_i,1}(\overline{\Omega})$, $\alpha_i = \max(0,|i|+l-k-1)$, $l \geq 0$ an integer, and $f \in W^{-k,2}(\Omega)$ such that $D^\alpha f \in W^{-k+1,2}(\Omega)$, $|\alpha| \leq l-1$. Let $u \in W^{k,2}(\Omega)$ be a weak solution of the equation $Au = f$ in Ω. Then for all subdomains $\Omega' \subset \overline{\Omega}' \subset \Omega$, u belongs to $W^{k+l,2}(\Omega')$ and the inequality*

$$|u|_{W^{k+1,2}(\Omega')} \leq c(\Omega')\big(|u|_{W^{k,2}(\Omega)} + \sum_{|\alpha|\leq l-1} |D^\alpha f|_{W^{-k+1,2}(\Omega)}\big) \tag{4.12}$$

holds.

Proof. We proceed by induction on l: if $l=1$, we have the hypotheses of Theorem 1.1; let us assume that the theorem is proved for $l-1$, $l \geq 2$, hence we have the inequality

$$|u|_{W^{k+l-1,2}(\Omega'')} \leq c(\Omega'')\big(|u|_{W^{k,2}(\Omega)} + \sum_{|\alpha|\leq l-2} |D^\alpha f|_{W^{-k+l,2}(\Omega)}\big) \tag{4.13}$$

with

$$\overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega.$$

If $\varphi \in C_0^\infty(\Omega)$, $1 \leq \tau \leq N$, then

$$\begin{aligned} A\Big(\varphi, \frac{\partial u}{\partial x_\tau}\Big) &= -\int_\Omega \sum_{|i|,|j|\leq k} \frac{\partial \overline{a}_{ij}}{\partial x_\tau} D^i\varphi D^j\overline{u}\,\mathrm{d}x - \int_\Omega \sum_{|i|,|j|\leq k} \overline{a}_{ij} D^i \frac{\partial \varphi}{\partial x_\tau} D^j(\overline{u})\,\mathrm{d}x \\ &= -\overline{f}\Big(\frac{\partial\varphi}{\partial x_\tau}\Big) - I(\varphi). \end{aligned}$$

If $|\alpha| \leq l-2$, we have:

$$I(D^\alpha\varphi) = \int_\Omega \sum_{|i|,|j|\leq k} \frac{\partial \overline{a}_{ij}}{\partial x_\tau} D^{\alpha+i}\varphi D^j\overline{u}\,\mathrm{d}x.$$

Let us consider $\int_\Omega \frac{\partial \overline{a}_{ij}}{\partial x_\tau} D^{\alpha+i}\varphi D^j\overline{u}\,\mathrm{d}x$. In case $|\alpha+i| \leq k-1$, we do not change this form. If $|\alpha+i| \geq k$, we can write it in the following way

$$\int_\Omega \Big(\frac{\partial \overline{a}_{ij}}{\partial x_\tau}\Big) D^{\alpha+i}\varphi D^j\overline{u}\,\mathrm{d}x = \int_\Omega D^\beta\varphi D^\gamma\Big(\frac{\partial \overline{a}_{ij}}{\partial x_\tau} D^j\overline{u}\Big)\,\mathrm{d}x,$$

where $|\beta| = k-1$, $|\gamma| = |\alpha+i| - k + 1 \le l - 1 + |i| - k$; then by (4.13) $I(\varphi)$ satisfies the hypotheses of Theorem 1.2, playing the role of the right hand side for the case $l-1$ and Ω''. Then $\partial u/\partial x_\tau \in W^{k+l-1,2}(\Omega')$. □

Remark 1.1. Theorems 1.1, 1.2 give local results. For Ω we can take a subdomain of the given domain where the weak solution should exist.

Exercise 1.1. The results of Theorems 1.1, 1.2 hold if the sesquilinear form $A(v,u) + \lambda(v,u)$ is $W_0^{k,2}(\Omega)$-elliptic for λ sufficiently large.

Obviously we have:

Proposition 1.1. *Suppose $f \in W^{-k+l,2}(\Omega), l \ge 1$ an integer. Then f satisfies the hypotheses of Theorem 1.2.*

Remark 1.2. If $l = k$, Theorem 1.2 implies that $u \in W^{2k,2}(\Omega')$. The equation $Au = f$ holds almost everywhere. If $l > k + \frac{1}{2}N$, Theorem 2.3.8 implies that $u \in C^{2k}(\overline{\Omega})$ and hence u is a classical solution of $Au = f$.

4.1.2 Regularity of the Very Weak Solution

Let us consider an operator A as in 1.2.1 with coefficients in $C^{\kappa,1}(\overline{\Omega})$, $\kappa \ge 0$ integer. Let us consider $f \in W^{-k-\kappa-1,2}(\Omega)$. The distribution $u \in W^{k-1-\kappa,2}(\Omega)$ is called a *very weak solution* of the equation $Au = f$ if $Au = f$ holds in the sense of distributions, i.e., for all $\varphi \in C_0^\infty(\Omega)$, $\langle A^*\varphi, u\rangle = \langle \varphi, \overline{f}\rangle$.

Theorem 1.3. *Let Ω be a bounded domain, A an operator as in 1.2.1, with $a_{ij} \in C^{\kappa,1}(\overline{\Omega})$, $\kappa \ge 0$ integer. The form $A(v,u)$ is $W_0^{k,2}(\Omega)$-coercive (cf. 3.4.1), moreover we assume $a_{ij} \in C^{\alpha_i,1}(\overline{\Omega})$, $\alpha_i = \max(0, |i| + l - 1 - k)$, $l > 0$ an integer. Let us consider $f \in W^{-k-\kappa-1+l,2}(\Omega)$ and let $u \in W^{k-\kappa-1,2}(\Omega)$ be a very weak solution of $Au = f$. Then for each subdomain $\Omega' \subset \overline{\Omega}' \subset \Omega$, we have $u \in W^{k-\kappa-1+l,2}(\Omega')$ and the following inequality holds:*

$$|u|_{W^{k-\kappa-1+l,2}(\Omega')} \le c(\Omega')\big(|u|_{W^{k-\kappa-1,2}(\Omega)} + |f|_{W^{-k-\kappa-1+l,2}(\Omega)}\big).$$

Proof. Let us consider $s = \kappa + 1$ and let $w \in W_0^{k,2}(\Omega)$ be the solution of the equation $(-1)^s \triangle^s w = u$; by Theorem 1.2 we have $w \in W^{k+s,2}(\Omega'')$, $\overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega$. If $v \in {W_0}^{k,2}(\Omega'')$, then we obtain:

$$\begin{aligned}\langle v, \overline{Au}\rangle &= \langle v, (-1)^s \overline{A\triangle^s w}\rangle \\ &= (-1)^s \int_{\Omega''} [\sum_{|i|,|j|\le k} \overline{a}_{ij} D^i v D^j (\sum_{|\gamma|=|\delta|=s} D^\gamma(\Delta_{\gamma\delta}(D^\delta \overline{w}))] \,\mathrm{d}x,\end{aligned}$$

with $\Delta_{\gamma\delta} \ne 0$ only for $\gamma = \delta$, $\triangle_{\gamma\gamma} = s!/\gamma!$. We have

$$B(v,w) = (-1)^s \int_{\Omega''} [\sum_{|i|,|j| \le k} \overline{a}_{ij} D^i v D^j (\sum_{|\gamma|=s} D^\gamma (\Delta_{\gamma\gamma}(D^\gamma \overline{w}))] \, \mathrm{d}x$$
$$= \int_{\Omega''} (\sum_{|i|,|j| \le k} \sum_{|\gamma|=s} \overline{a}_{ij} \Delta_{\gamma\gamma} D^{i+\gamma} v D^{j+\gamma} \overline{w}) \, \mathrm{d}x$$
$$+ \int_{\Omega''} (\sum_{\substack{|\mu| \le k+s-1 \\ |\nu| \le k+s}} \overline{c}_{\mu\nu} D^\mu v D^\nu \overline{w}) \, \mathrm{d}x,$$

where $c_{\mu\nu} \in C^{\alpha_\mu,1}(\overline{\Omega}'')$, $\alpha_\mu = \max(0, |\mu| + l - 1 - k - s)$. But for $v \in W_0^{k+s,2}(\Omega'')$ and λ_0 sufficiently large we have:

$$\mathrm{Re} \int_{\Omega''} (\sum_{|i|,|j| \le k} \sum_{|\gamma|=s} \overline{a}_{ij} \Delta_{\gamma\gamma} D^{i+\gamma} v D^{j+\gamma} \overline{v}) \, \mathrm{d}x$$
$$+ \lambda_0 \int_{\Omega''} (\sum_{|\gamma|=s} |D^\gamma v|^2) \, \mathrm{d}x \ge c_1 |v|^2_{W^{k+s,2}(\Omega'')},$$

and according to Lemma 2.6.1, the form $B(v,w)$ is $W_0^{k+s,2}(\Omega'')$-coercive. We apply Theorem 1.2, and obtain:

$$\begin{aligned} |w|_{W^{k+s+l,2}(\Omega')} &\le c_1(\Omega') \big(|w|_{W^{k+s,2}(\Omega'')} + |f|_{W^{-k-\kappa-1+l,2}(\Omega'')} \big) \\ &\le c_2(\Omega') \big(|f|_{W^{-k-\kappa-1+l,2}(\Omega)} + |u|_{W^{k-\kappa-1,2}(\Omega)} \big), \end{aligned} \tag{4.14}$$

where $(-1)^s \triangle^s w = u$ which gives us the assertion of Theorem 1.3. □

Let A be an operator with infinitely differentiable coefficients in Ω. Let $v \in \mathscr{D}'(\Omega)$ be such that $Av = f$ in Ω. The operator A is called *hypoelliptic* in Ω if $f \in C^\infty(\Omega)$ implies $v \in C^\infty(\Omega)$. Each distribution is locally in $W^{-m,2}(\Omega)$, cf. L. Schwartz [1, 2], hence Theorem 1.3 implies that every $W_0^{k,2}(\Omega)$-elliptic operator with coefficients in $C^\infty(\Omega)$ is hypoelliptic. The converse is not necessarily true, the heat operator is also hypoelliptic.

If A has constant coefficients, we have a necessary and sufficient condition for the hypoellipticity, cf. L. Hörmander [1–3], where also the case of variable coefficients is considered. For elliptic operators with coefficients in $C^\infty(\Omega)$ it is well known that they are hypoelliptic, cf. L. Schwartz [3]; for these questions cf. also B. Malgrange [4], A. Friedman [1], F. Trèves [1].

Following Hilbert we can ask whether the solution, being a distribution, is analytic in the case where the right hand side f and the coefficients of A are analytic; the answer is positive if A is elliptic, cf. for instance F. John [1]. In the papers of C.B. Morrey, L. Nirenberg [1], E. Magenes, G. Stampacchia [1], the analyticity of the solution is considered in a neighborhood of an analytic boundary.

We will end this section with the following theorem (cf. J.L. Lions [4]):

Theorem 1.4. *Let Ω be bounded, A elliptic in $\overline{\Omega}$, with coefficients in $C^\infty(\Omega)$, $f \in C^\infty(\Omega)$, $u \in W^{-m,2}(\Omega)$ the weak solution of $Au = f$. Then $u \in C^\infty(\Omega)$.*

Proof. Since A is elliptic it can be written as in (3.51); A^*A is uniformly strongly elliptic in $\overline{\Omega}$, hence also $W_0^{k,2}(\Omega)$-coercive. We can apply Theorem 1.3 and 2.3.8. □

4.2 Regularity of the Solution in the Neighborhood of the Boundary

4.2.1 The Second Order Operator

To give the reader the fundamental idea, we consider the case of a second order equation and of the Dirichlet problem; then we shall prove the general case.

Theorem 2.1. *Let $\Omega \in \mathfrak{N}^{1,1}$,*

$$A = \sum_{|i|,|j|\le 1} (-1)^{|i|} D^i(a_{ij}D^j), \quad a_{ij} \in C^{0,1}(\overline{\Omega}), \quad f \in L^2(\Omega), \quad u_0 \in W^{2,2}(\Omega),$$

where the form $A(v,u)$ is $W_0^{1,2}(\Omega)$-elliptic. Let $u \in W^{1,2}(\Omega)$ be the solution of the Dirichlet problem $Au = f$ in Ω, $u - u_0 \in W_0^{1,2}(\Omega)$. Then $u \in W^{2,2}(\Omega)$ and:

$$|u|_{W^{2,2}(\Omega)} \le c(|f|_{L^2(\Omega)} + |u_0|_{W^{2,2}(\Omega)}).$$

Proof. We use the notation 1.2.4. Let us set $w = u - u_0$; w solves the Dirichlet problem $Aw = f - Au_0$ in Ω, $w \in W_0^{1,2}(\Omega)$. Let us denote $w_r = w\varphi_r$. If $r = M+1$, Theorem 1.1 implies:

$$|w_{M+1}|_{W^{2,2}(\Omega)} \le c_1(|f|_{L^2(\Omega)} + |u_0|_{W^{2,2}(\Omega)}). \tag{4.15}$$

Now let us consider $1 \le r \le M$. The function w_r solves the Dirichlet problem $Aw_r = f\varphi_r + g_r$, $w_r \in W_0^{1,2}(\Omega)$, with

$$|g_r|_{L^2(\Omega)} \le c_2(|f|_{L^2(\Omega)} + |u_0|_{W^{2,2}(\Omega)}). \tag{4.16}$$

Using a linear mapping we transform the coordinate x into (x'_r, x_{rN}); the $W_0^{1,2}$-ellipticity of $A(v,u)$ does not change if the coefficients are in $C^{0,1}(\overline{\Omega})$. Without loss of generality, we can assume that the system (x'_r, x_{rN}) coincides with the original system. Let K be the prism defined by: $|y_i| < \alpha, i = 1,2,\dots,N-1$, $0 < y_N < \beta$ and $\Lambda = \{y \in \mathbb{R}^N,\ y_N = 0,\ |y_i| < \alpha,\ i = 1,2,\dots,N-1\}$. We define the mapping $T : K \to V_r$ by:

$$x'_r = y'_r, \quad x_{rN} = y_{rN} + a_r(y'_r). \tag{4.17}$$

T is one-to-one, regular, with Jacobian equal 1, the first derivatives of T are lipschitzian in $\overline{K}$. It follows from Lemma 2.3.4 that $w_r(T(y)) \in W_0^{1,2}(K)$. The operator A is changed to $\widetilde{A}$, with $\widetilde{A}(v,u)$ $W_0^{1,2}(K)$-elliptic, with coefficients in $C^{0,1}(\overline{K})$. In K we have:

$$\widetilde{A}w_r = g_r + f\varphi_r, \tag{4.18}$$

where

$$\widetilde{A} = \sum_{|i|,|j|\le 1} (-1)^{|i|} D^i(\widetilde{a}_{ij}D^j).$$

After the mapping T we denote the functions w_r, g_r, $f\varphi_r$ by the same symbols, for simplicity. Let us remark that $\operatorname{supp} w_r \subset K \cup \Lambda$. We use the same approach as in Theorem 1.1 with $h^{(\tau)}$, $\tau = 1,2,\ldots,N-1$. Let us denote $f_r = f\varphi_r$, then:

$$|D^\alpha w_r|_{L^2(K)} \le c_3(|f_r|_{L^2(K)} + |g_r|_{L^2(K)}) \tag{4.19}$$

for

$$|\alpha| = 2, \quad \alpha \ne (0,0,\ldots,0,2).$$

From Theorem 3.4.5 it follows that $c_{(0,0,\ldots,0,2)} \ne 0$ in $\overline{K}$. Then (4.18) and (4.19) imply inequality (4.19) also for $\alpha = (0,0,\ldots,0,2)$. □

4.2.2 The Regularizable Problem

Now we consider the general case for an operator of order $2k$. We define the *regularizable problem* in $\bar{\Omega}$ as follows:

The boundary $\partial\Omega$ is sufficiently smooth (the smoothness will be specified later). (4.20a)

There is given an operator A of order $2k$ with the associated sesquilinear form $A(v,u)$ and the boundary form $a(v,u)$, with sufficiently smooth coefficients. (4.20b)

There are given boundary operators B_s, $s = 1,2,\ldots,\mu$, $\mu \ge k$ written as in (1.34a), (1.34b), i.e.: let μ be an integer, $0 \le \mu \le k$, a set of indices j_s, $s = 1,2,\ldots,\mu$, $0 \le j_1 < j_2 < \cdots < j_\mu \le k-1$; this set can be empty, in this case we write $\mu = 0$. We denote the complementary indices by i_t, $t = 1,2,\ldots,k-\mu$, $i_1 < i_2 < \cdots < i_{k-\mu}$; $0 \le i_\tau \le k-1$. Then

$$B_s = \frac{\partial^{j_s}}{\partial n^{j_s}} - \sum_{|\alpha|\le j_s} h_{s\alpha} D^\alpha,$$

where $h_{s\alpha}$ are sufficiently smooth,

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial \sigma_1^{\alpha_1} \partial \sigma_2^{\alpha_2} \dots \partial \sigma_{N-1}^{\alpha_{N-1}} \partial n^{\alpha_N}},$$

where (σ, n) are the local coordinates as in 1.2.4 with $n = -t$, α_N is one of the indices i_t. The space V is defined as usual by: $V = \{v \in W^{k,2}(\Omega), B_s v = 0 \text{ on } \partial\Omega, s = 1, 2, \dots, \mu\}$. (4.20$c$)

The sesquilinear form $((v,u)) = A(v,u) + a(v,u)$ is assumed to be V-elliptic. (4.20d)

The problem is given by conditions (3.15f)–(3.15i), with the boundary conditions sufficiently smooth. (4.20e)

We prove similarly as in Lemma 1.1:

Lemma 2.1. *Let* $u \in W^{k,p}(\mathbb{R}_+^N)$; ($\mathbb{R}_+^N = \{x \in \mathbb{R}^N, x_N > 0\}$). *Then*

$$|\Delta_h^{\tau} u|_{W^{k-1,p}(\mathbb{R}_+^N)} \le c|u|_{W^{k,p}(\Omega)(\mathbb{R}_+^N)},$$

$$\lim_{h \to 0} \left| \Delta_h^{\tau} u - \frac{\partial u}{\partial x_{\tau}} \right|_{W^{k-1,p}(\mathbb{R}_+^N)} = 0, \quad \tau = 1, 2, \dots, N-1.$$

We recall that for $u, \varphi \in C_0^{\infty}(\mathbb{R}^N)$:

$$\Delta_h^{\tau}(u(x)\varphi(x)) = \Delta_h^{\tau} u(x) \varphi(x + h^{(\tau)}) + u(x) \Delta_h^{\tau} \varphi(x), \tag{4.21a}$$

$$\int_{\mathbb{R}^N} (\Delta_h^{\tau} u(x)) \varphi(x) \, \mathrm{d}x = -\int_{\mathbb{R}^N} u(x) \Delta_{-h}^{\tau} \varphi(x) \, \mathrm{d}x. \tag{4.21b}$$

Let $l \ge 1$ be an integer. We say that the boundary value problem (4.20a)–(4.20e) satisfies $k + l$ conditions of regularity or that it is *(k+l)-regularizable,* if:

$$\Omega \in \mathfrak{N}^{2k+l,1}, \tag{4.22a}$$

$$a_{ij} \in C^{\alpha_i,1}(\overline{\Omega}), \quad \alpha_i = \max(0, |i| + l - k - 1),$$
$$a_{(0,0,\dots,0,k)(0,0,\dots,0,k)} \in W^{l+k-1,\infty}(\Omega), \tag{4.22b}$$

the coefficients $b_{i\alpha}$ of the boundary sesquilinear form belong to $C^{|\alpha|+l-k,1}(\partial\Omega)$ for $|\alpha| - k \ge 0$, and to $C^{l-1,1}(\partial\Omega)$ for $|\alpha| - k < 0$, (4.22c)

the coefficients $h_{s\alpha}$ of the boundary operators B_s belong to $C^{k-j_s+l-1,1}(\partial\Omega)$, (4.22$d$)

the right hand side f of the equation belongs to Q' and if $l < k$ then

$$\sup_{v \in V, |v|_{W^{k-l,2}(\Omega)} \leq 1} |fv| \equiv |f| < \infty; \tag{4.22e}$$

for $k \leq l$ let $Q \subset L^2(\Omega)$, $f \in W^{k-l,2}(\Omega)$, $|f|_{W^{l-k,2}(\Omega)} = |f|$.

$$u_0 \in W^{k+l,2}(\Omega), \tag{4.22f}$$

$$g_{i_\tau} \in W^{i_\tau+1/2+l-k,2}(\partial\Omega). \tag{4.22g}$$

Remark 2.1. The hypotheses (4.22a)–(4.22g) are almost the same as the necessary conditions implying that the solution belongs to $W^{k+l,2}(\Omega)$. We can make other refinements. Briefly, the conditions (4.16a)–(4.16g) are satisfied if Ω, the coefficients and the data are sufficiently smooth. This is the case if $\Omega \in \mathfrak{N}^\infty$, $a_{ij} \in C^\infty(\overline{\Omega})$, $b_{i\alpha} \in C^\infty(\partial\Omega)$, $h_{s\alpha} \in C^\infty(\partial\Omega)$, the form $((v,u))$ is V-elliptic, $f \in C^\infty(\overline{\Omega})$, $u_0 \in C^\infty(\overline{\Omega})$, $g_{i_\tau} \in C^\infty(\partial\Omega)$.

Lemma 2.2. *Suppose $\Omega \in \mathfrak{N}^{l+k,1}$, $g_{j_s} \in W^{l+k-j_s-1/2,2}(\partial\Omega)$, $h_{s\alpha} \in L^\infty(\partial\Omega)$. Then there exists a mapping:*

$$T \in \Big[\prod_{s=1}^{\mu} W^{l+k-j_s-1/2,2}(\partial\Omega) \to W^{l+k,2}(\Omega)\Big],$$

such that $T(g_{j_1}, g_{j_2}, \dots, g_{j_\mu}) = u_0$, $B_s u_0 = g_{j_s}$, $s = 1, 2, \dots, \mu$.

Proof. We follow the same method as in the proof of Theorem 2.5.8: We are looking for $u_0 \in W^{k+l,2}(\Omega)$ such that $\partial^{j_s} u_0/\partial n^{j_s} = g_{j_s}$ on $\partial\Omega$, $s = 1, 2, \dots, \mu$, $\partial^{i_t} u_0/\partial n^{i_t} = 0$ on $\partial\Omega$, $t = 1, 2, \dots, k-\mu$, $\partial^i u_0/\partial n^i = 0$ on $\partial\Omega$, $i = k, k+1, \dots k+l-1$. Clearly we have $B_s u_0 = g_{j_s}$, which gives us the assertion of Lemma 2.2. □

4.2.3 Lemmas

Let us consider open sets G_i as in 1.2.4, with the same notations as in this section, and denote $V_i = \{v \in V, \operatorname{supp} v \subset G_i \cap \overline{\Omega}\}$. We use the mapping T defined in (1.35), $T : K_+ \to G_{i_+} = G_i \cap \Omega$. We set: $z(T(y)) = z^\bullet(y)$, $V^\bullet = \{v \in W^{k,2}(K_+), v(y) = \omega(T(y)), \omega \in V_i\}$. If $v \in V^\bullet$, let $v_\bullet \in V_i$ be such that $(v_\bullet)^\bullet = v$; we set $f^\bullet v = f v_\bullet$, $g v_\bullet = g^\bullet v_\bullet$. If $u \in W^{k,2}(\Omega)$, we define for $v \in V^\bullet$: $((v,u^\bullet))^\bullet = ((v_\bullet, u))$. We have the following lemma: □

Lemma 2.3. *Suppose the boundary value problem is $(l+k)$-regularizable, let u be the solution; for all $v \in V^\bullet$: $((v,u^\bullet))^\bullet = \overline{f}^\bullet v + \overline{g}^\bullet v - ((v,u_0^\bullet))^\bullet$, where for $v,u \in W^{k,2}(K_+)$, we have:*

$$((v,u))^\bullet = \int_{K_+} \Big(\sum_{|i|,|j| \leq k} \overline{a}^\bullet_{ij} D^i v D^j \overline{u}\Big)\,\mathrm{d}y + \int_\Delta \left(\sum_{i=1}^{N} \sum_{|\alpha| \leq 2k-i-1} \overline{b}^\bullet_{i\alpha} \frac{\partial^i v}{\partial y_N^i} D^\alpha \overline{u}\right) \mathrm{d}y'; \tag{4.23}$$

$a^{\bullet}_{ij} \in C^{\alpha_i,1}(\overline{K_+})$, *$\alpha_i$ defined in (4.22b),*

$$b^{\bullet}_{i\alpha} \in C^{|\alpha|+i+l-k,1}(\overline{\Delta}) \quad \textit{for} \quad |\alpha|-k \geq 0,$$
$$b^{\bullet}_{i\alpha} \in C^{l-1,1}(\overline{\Delta}) \quad \textit{for} \quad |\alpha|-k < 0.$$

We have $v \in V^{\bullet} \Leftrightarrow v \in W^{k,2}(K_+)$, supp $v \subset K_+ \cup \Delta$,

$$\frac{\partial^{j_s} v}{\partial y_N^{j_s}} - \sum_{|\alpha| \leq j_s} h^{\bullet}_{s\alpha} \frac{\partial^{|\alpha|} v}{\partial y_1^{\alpha_1} \partial y_2^{\alpha_2} \dots \partial y_N^{\alpha_N}} = 0,$$

where $\alpha_N = i_t$ *for well chosen* $t, h^{\bullet}_{s\alpha} \in C^{k-j_s+l-1,1}(\overline{\Delta})$. *If* $v \in V^{\bullet}$, *then* $|((v,v))^{\bullet}| \geq c|v|^2_{W^{k,2}(K_+)}$.

Proof: We must take into account the fact that up to order $\leq 2k+l-1$ the derivatives of T (resp. T^{-1}) are lipschitzian in $\overline{K}$ (resp. in $\overline{G}^+$). □

Now we give two lemmas concerning the "compensation". We denote, for $0 < \varepsilon < \min(\alpha, \gamma)$:

$$V^{\bullet}_{\varepsilon} = \{v \in V^{\bullet}, \text{supp}\, v \subset K_{\varepsilon} \cup \Delta_{\varepsilon}\},$$

where

$$\Delta_{\varepsilon} = \{y \in \mathbb{R}^N, y_N = 0, |y_i| < \alpha - \varepsilon, i = 1,2,\dots,N-1\},$$
$$K_{\varepsilon} = \{y \in \mathbb{R}^N, |y_i| < \alpha - \varepsilon, i = 1,2,\dots,N-1, 0 < y_N < \gamma - \varepsilon\}.$$

Lemma 2.4. *Let us consider* $|h| < (1/2)\varepsilon$, $1 \leq \tau \leq N-1$. *Then there exists* $Z^{\tau}_h \in [V^{\bullet}_{\varepsilon} \to W^{k,2}(\Omega)(K_+)]$ *such that for all* $v \in V^{\bullet}_s, (\Delta^{\tau}_h v - Z^{\tau}_h v) \in V^{\bullet}_{\varepsilon'}$, $\varepsilon' > 0$, $|Z^{\tau}_h| \leq c$.

Proof. Setting

$$B^{\bullet}_s = \frac{\partial^{j_s}}{\partial y_N^{j_s}} - \sum_{|\alpha| \leq j_s} h^{\bullet}_{s\alpha} D^{\alpha}$$

we have

$$B^{\bullet}_s(\Delta^{\tau}_h v) = \sum_{|\alpha| \leq j_s} \Delta^{\tau}_h h^{\bullet}_{s\alpha}(x) D^{\alpha} v(x + h^{(\tau)}).$$

We extend $v(x)$ by zero to $\mathbb{R}^N_+$ and use the operator Z defined in Theorem 2.5.4:

$$Z \in \Big[\prod_{i=0}^{k-1} W^{k-l-1/2,2}(\Delta) \to W^{k,2}(K_+)\Big].$$

Let us set $\varphi_{i_\tau} = 0$ on Δ, $\varphi_{j_s} = B^{\bullet}_s \Delta^{\tau}_h v, \widetilde{Z}^{\tau}_h v = Z(\varphi_0, \varphi_2, \dots, \varphi_{k-1})$. Let $\psi \in C^{\infty}_0(\mathbb{R}^N)$ such that supp $\psi \subset K, x \in \Delta_{\varepsilon} \Rightarrow \psi(x) = 1$. and let us set:

$$Z^{\tau}_h v = \psi \widetilde{Z}^{\tau}_h v. \qquad (4.23\,\text{bis})$$

Now Lemma 2.5.5 implies the assertion. □

Use (4.23 bis) to define Z_h^τ for $v \in W^{k,2}(K_+)$ with supp $v \subset K_\varepsilon^+ \cup \Lambda_\varepsilon$, and denote $W_{k,\varepsilon} = \{v \in W^{k,2}(K_+),\ \operatorname{supp} v \subset K_\varepsilon^+ \cup \Delta_s\}$. We have:

Lemma 2.5. *Let us assume that the boundary value problem is* $(l+k)$*-regularizable. Then the operator* Z_h^τ *defined in (4.23 bis) has the following properties:*

$$Z_h^\tau \in [W_{k+\mu,\varepsilon} \to W_{k+\mu,0}], \quad \mu = 0,1,\dots,l-1,$$

and $\lim_{h\to 0} Z_h^\tau = Z_0^\tau$ *in* $[W_{k+\mu,\varepsilon} \to W_{k+\mu,0}]$. *If* $v \in W_{k+\mu,\varepsilon}$, *then for* $|\alpha| \le \mu$, $\alpha_N = 0$:

$$Z_h^\tau D^\alpha v = \sum_{|\beta|\le|\alpha|} D^\beta Z_{h,\beta}^\tau v, \tag{4.24}$$

where

$$Z_{h,\beta}^\tau \in [W_{k+\mu,\varepsilon} \to W^{k+|\beta|,2}(K_+)]$$

and $\lim_{h\to 0} Z_{h,\beta}^\tau = Z_{0,\beta}^\tau$ *in this space. Let us consider* $|\alpha| \le \mu - 1$, τ_1 *an integer,* $1 \le \tau_1 \le N-1$. *If* $|h|$ *is sufficiently small,* $v \in W_{k+\mu,2\varepsilon}$ *then*

$$Z_0^\tau D^\alpha \Delta_h^{\tau_1} v = \sum_{|\beta|\le|\alpha|} \Delta_h^{\tau_1} D^\beta Y_\beta v + \sum_{|\beta|\le|\alpha|} D^\beta \Gamma_{\beta,h} v, \tag{4.25}$$

with

$$Y_\beta \in [W_{k+\mu,2\varepsilon} \to W_{k+|\beta|+1,\varepsilon'}], \quad \varepsilon' > 0,$$
$$\Gamma_{\beta h} \in [W_{k+\mu,2\varepsilon} \to W_{k+|\beta|,\varepsilon'}],$$

and in this space $\lim_{h\to 0} \Gamma_{\beta h} = \Gamma_\beta$.

Proof. First, let us prove (4.24) for $|\alpha| = 1$; the general case can be proved by induction: let be $i \le N-1$, and

$$Z_h^\tau \frac{\partial v}{\partial x_i} = \psi \widetilde{Z}_h^\tau \frac{\partial v}{\partial x_i} = \psi Z(\psi_0,\psi_1,\dots,\psi_{k-1}) \quad \text{with} \quad \psi_{i_\tau} = 0,$$
$$\psi_{j_s} = \sum_{|\alpha|\le j_s} \Delta_h^\tau h_{s\alpha}(x) D^\alpha \frac{\partial}{\partial x_i} v(x+h^{(\tau)}),$$
$$\frac{\partial}{\partial x_i}\varphi_{j_s} \equiv \psi_{j_s} + \sum_{|\alpha|\le j_s} \Delta_h^\tau \frac{\partial}{\partial x_i} h_{s\alpha}(x) D^\alpha v(x+h^{(\tau)}).$$

We have:

$$Z\left(\frac{\partial \varphi_0}{\partial x_i},\dots,\frac{\partial \varphi_{k-1}}{\partial x_i}\right) = \frac{\partial}{\partial x_i} Z(\varphi_0,\dots,\varphi_{k-1}),$$

thus,

$$Z_h^\tau \frac{\partial v}{\partial x_i} = \psi \frac{\partial}{\partial x_i} \widetilde{Z}_h^\tau v - \psi Z(\mu_0, \mu_1, \ldots, \mu_{k-1}) \quad \text{with} \quad \mu_{i_\tau} = 0,$$

$$\mu_{j_s} = \sum_{|\alpha| \le j_s} \Delta_h^\tau \frac{\partial}{\partial x_i} h_{s\alpha}(x) D^\alpha v(x + h^{(\tau)}).$$

This implies:

$$Z_h^\tau \frac{\partial v}{\partial x_i} = \frac{\partial}{\partial x_i} Z_h^\tau v - \frac{\partial \psi}{\partial x_i} \widetilde{Z}_h^\tau v - \psi Z(\mu_0, \mu_1, \ldots, \mu_{k_1}),$$

and (4.24) follows. □

Lemma 2.6. *Given a $(k+l)$-regularizable problem, let $\chi \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp} \chi \subset K_+ \cup \Delta$. Then there exists a mapping $R \in [W_{k,\varepsilon} \to W_{k,0}]$ such that for $0 \le \mu \le l-1$, $R \in [W_{k+\mu,\varepsilon} \to W_{k+\mu,0}]$, for $v \in V_\varepsilon^\bullet$, $R \in [V_\varepsilon^\bullet \cap W^{k+\mu,2}(K_+) \to W^{k+\mu+1,2}(K_+)]$, and such that $v \in V_\varepsilon^\bullet \Rightarrow \chi v - Rv \in V^\bullet$, and for $v \in W_{k,\varepsilon}$, $\operatorname{supp} Rv \subset M \subset \overline{M} \subset K_+ \cup \Delta$.*

Proof. If $v \in W_{k,\varepsilon,}$ let us set $\varphi_{i_t} = 0$, $\varphi_{j_s} = B_s(v\chi)$ on Δ, and let $\nu \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} \nu \cap (\mathbb{R}^N - K_+ - \Delta) = \emptyset$, with $x \in \operatorname{supp} \chi \Rightarrow \nu(x) = 1$. Let us set $Rv = \nu Z(\varphi_0, \varphi_1, \ldots, \varphi_{k-1})$ where Z is as in Lemma 2.4; clearly R satisfies the hypotheses of Lemma 2.6. □

As in Lemma 2.5, we prove:

Lemma 2.7. *Under the assumptions of the previous lemma and if $v \in W_{k+\mu,\varepsilon}$, $0 \le \mu \le l-1$, $|\alpha| \le \mu, \alpha_N = 0$, we obtain:*

$$RD^\alpha v = \sum_{|\beta| \le |\alpha|} D^\beta R_\beta v \tag{4.26}$$

with

$$R_\beta \in [W_{k+|\beta|,\varepsilon} \to W^{k+|\beta|,2}(K_+)],$$

and

$$\operatorname{supp} R_\beta v \subset N \subset \overline{N} \subset K_+ \cup \Delta.$$

Now we prove a lemma about density:

Lemma 2.8. *The space $V^\bullet \cap W^{k+l,2}(K_+)$ is dense in $V^\bullet$.*

Proof. Let $v \in V^\bullet$; we extend v to K using (2.48); $v \in W^{k,2}(K)$, $\operatorname{supp} v \subset K$. Now using the regularizing operator we construct a sequence $v_n \in C^\infty(K)$, $\lim_{n\to\infty} v_n = v$ in $W^{k,2}(K)$. Let M be an open set such that $\operatorname{supp} v \subset M \subset \overline{M} \subset K$. If n is sufficiently large we have $\operatorname{supp} v_n \subset \overline{M}$; consequently $B_s^\bullet v_n \in W^{(k+l-j_s-1/2,2}(\Delta)$ and $\lim_{n\to\infty} B_s^\bullet v_n = 0$ in $W^{k-j_s-1/2,2}(\Delta)$ hold. Applying Theorem 2.5.6 and assuming $w_n \in W^{k+l,2}(K_+)$ such that

$$\frac{\partial^{j_s} w_n}{\partial y_N^{j_s}} = B_s^\bullet v_n, \quad s = 1, 2, \ldots, \mu, \quad \frac{\partial^{i_t} w_n}{\partial y_N^{i_t}} = 0, \quad t = 1, 2, \ldots, \mu,$$

we have $\lim_{n\to\infty} w_n = 0$ in $W^{k,2}(K_+)$. Moreover, let $\psi \in C_0^\infty(K)$ be such that if $x \in M$, then $\psi(x) = 1$. Denoting $\omega_n = \psi w_n$ we have

$$\frac{\partial^{j_s}\omega_n}{\partial y_N^{j_s}} = B_s^\bullet v_n, \quad \frac{\partial^{i_t}\omega_n}{\partial y_N^{i_t}} = 0$$

on Δ, $\lim_{n\to\infty}\omega_n = 0$ in $W^{k,2}(K_+)$. Finally, if we set $u_n = v_n - \omega_n$, we obtain $u_n \in V^\bullet \cap W^{k+l,2}(K_+)$ and $\lim_{n\to\infty} u_n = v$ in $W^{k,2}(K_+)$. □

Also we get

Lemma 2.9. *Let $g \in W^{1/2,2}(\Delta)$, $\Delta = (-1,1)^{N-1}$, $\Delta_{1/2} = (-1/2,1/2)^{N-1}$. Then*

$$\sup_{|v|_{W^{1/2,2}(\Delta)}\le 1,\, \operatorname{supp} v\subset \Delta_{1/2}} |\langle \Delta_h^\tau v, g\rangle| \le c|g|_{W^{1/2,2}(\Delta)}, \quad |h| < 1/2.$$

Proof. Let be $K = \Delta\times(0,1)$, $g \in W^{1,2}(K)$ such that $g(x',0) = g(x')$. By Theorem 2.5.6 we can find g such that $|g|_{W^{1,2}(K)} \le c_1|g|_{W^{1/2,2}(\Delta)}$. Let T be the mapping as in Theorem 2.5.6, i.e. T is an extension operator, $T \in [W^{1/2,2}(\Delta) \to W^{1,2}(K)]$, let $\varphi \in C_0^\infty(\mathbb{R}^N)$ be such that $x \in \Delta_{1/2} \Rightarrow \varphi(x) = 1$, supp $\varphi \subset K\cup\Delta$; for $v \in W^{1/2,2}(\Delta)$, let us set $Zv = \varphi T v$. Denote $Zv = v$. Now we have:

$$\int_\Delta \Delta_h^\tau v g\,\mathrm{d}x' = -\int_K \frac{\partial}{\partial x_N}(\Delta_h^\tau v g)\,\mathrm{d}x = -\int_K \Delta_h^\tau v(x)\frac{\partial g}{\partial x_N}(x)\,\mathrm{d}x + \int_K \frac{\partial v}{\partial x_N}\Delta_{-h}^\tau g\,\mathrm{d}x$$

and the result follows from Lemma 2.1. □

4.2.4 Lemmas (Continuation)

In K_+ we define functionals F_l and G_l of the following type: $F_l(v)$ is a sum of integrals of the form $\int_{K_+} D^\alpha v\bar{h}\,\mathrm{d}y$, $|\alpha| \le k-1$, $v \in W^{k,2}(K_+)$, $h \in W^{l-k+|\alpha|,2}(K_+)$ for $l-k+|\alpha| \ge 0$, $h \in L^2(K_+)$ for $l-k+|\alpha| < 0$, and $G_l(v)$ is a sum of integrals of the form $\int_\Delta D^\alpha v\bar{h}\,\mathrm{d}y'$, $|\alpha| \le k-1$, $v \in W^{k,2}(K_+)$, $h \in W^{(l-k+1/2+|\alpha|,2}(\Delta)$, supp $v \subset K_+\cup\Delta$.

We prove now:

Lemma 2.10. *Given a $(k+l)$-regularizable problem, and with the same notation as in Lemma 2.3, let $\psi \in C_0^\infty(\mathbb{R}^N)$, supp $\psi \subset K$. Then for $v \in V^\bullet$:*

$$((v, w\psi))^\bullet = \overline{f}^\bullet(v\overline{\psi} - Rv) + \overline{g}^\bullet(v\overline{\psi} - Rv) + F_l(v) + G_l(v) + ((Rv, u^\bullet))^\bullet, \quad (4.27)$$

if we assume $w = u^\bullet - u_0^\bullet \in W^{k+l-1,2}(K_\varepsilon)$, $\varepsilon > 0$. Here F_l, G_l, are the functionals defined above and R is the operator defined in Lemma 2.6.

Proof. Let $v \in V$, then it is easy to see that $((v, w\psi))^\bullet - ((v\overline{\psi}, w))^\bullet$ is a linear combination of the following terms:

$$\int_{K_+} \overline{a}^\bullet_{ij} D^i v D^p \overline{w} D^q \overline{\psi}\, \mathrm{d}x, \quad p+q = j, \quad |q| \geq 1, \tag{4.28}$$

$$\int_{K_+} \overline{a}^\bullet_{ij} D^p v D^q \overline{\psi} D^i \overline{w}\, \mathrm{d}x, \quad p+q = i, \quad |q| \geq 1, \tag{4.29}$$

$$\int_{\Delta} \overline{b}^\bullet_{i\alpha} \frac{\partial^i v}{\partial n^i} D^p \overline{w} D^q \overline{\psi}\, \mathrm{d}S, \quad p+q = \alpha, \quad |q| \geq 1, \tag{4.30}$$

$$\int_{\Delta} \overline{b}^\bullet_{i\alpha} \frac{\partial^p v}{\partial n^p} \frac{\partial^q \overline{\psi}}{\partial n^q} D^\alpha \overline{w}\, \mathrm{d}S, \quad p+q = i, \quad |q| \geq 1. \tag{4.31}$$

Let us consider (4.28); if $|i| \leq k-1$, as $|p| \leq k-1$, (4.28) is of type F_l. If $|i| = k$, by integration by parts, we obtain:

$$\int_{K_+} \overline{a}^\bullet_{ij} D^i v D^p \overline{w} D^q \overline{\psi}\, \mathrm{d}x = \int_{\Delta} \overline{a}^\bullet_{ij} D^{\tilde{i}} v D^p \overline{w} D^q \overline{\psi} n_\tau\, \mathrm{d}x' - \int_{K_+} D^{\tilde{i}} v \frac{\partial}{\partial x_\tau} (\overline{a}^*_{ij} D^p \overline{w} D^q \overline{\psi})\, \mathrm{d}x.$$

Here $n_\tau = 0$ for $\tau \leq N-1$, $n_N = -1$. The first integral on the right hand side is obviously of G_l-type, the second is of F_l-type.

The integral in (4.29) is obviously of F_l-type.

The integral (4.30) is clearly of G_l-type: if $|p| \leq k-1$, we see it immediately, if $|p| \geq k$, we integrate $(|p| - k + 1)$-times by parts, which is possible because D^α is at most $(k-1)-$transversal. After this integration we have our assertion.

Concerning (4.31), everything is clear if $|\alpha| \leq k-1$: the integral is of G_l-type; if $|\alpha| > k-1$, we integrate $(|\alpha| - k + 1)$-times by parts and obtain the result.

Let us observe that everywhere we have used Lemma 2.5.5.

We have now:

$$\begin{aligned} ((v\overline{\psi}, w^\bullet))^\bullet &= ((v\overline{\psi}, u^\bullet - u_0^\bullet))^\bullet = ((v\overline{\psi}, u^\bullet))^\bullet - ((v\overline{\psi}, u_0^\bullet))^\bullet \\ &= (v\overline{\psi} - Rv, u^\bullet))^\bullet + ((Rv, u^\bullet))^\bullet - ((v\overline{\psi}, u_0^\bullet))^\bullet \\ &= \overline{f}^\bullet(v\overline{\psi} - Rv) + \overline{g}(v\overline{\psi} - Rv) + ((Rv, u^\bullet))^\bullet - ((v\overline{\psi}, u_0^\bullet))^\bullet. \end{aligned}$$

We see, without difficulty, that $((v\overline{\psi},\, u_0^\bullet))^\bullet$ is a sum of integrals of F_l and G_l-type. □

4.2.5 A Modification of Lions' Lemma

We give here a lemma playing the role of the "Lions' lemma", so called in E. Magenes, G. Stampacchia [1]:

Lemma 2.11. *For $u, v \in W^{k,2}(K_+)$, suppose we are given the sesquilinear form:*

$$[v,u] = \int_{K_+} \sum_{|i|,|j| \le k} \overline{a}^{\bullet}_{ij} D^i v D^j \overline{u} \mathrm{d}y,$$

with $a^{\bullet}_{ij}$ smooth as in (4.29), $a^{\bullet}_{(0,0,\ldots,0,k),(0,0,\ldots,0,k)} \in W^{l+k-1,\infty}(K_+)$; we assume also that $[v,u]$ is $W_0^{k,2}(K_+)$-elliptic. Let $F \in W^{-k+l,2}(K_+)$, $u \in W^{k,2}(K_+)$ be such that for all $v \in C_0^\infty(K)$ with $\operatorname{supp} v \subset K_+ \cup \Delta$,

$$[v,u] = \langle v, \overline{F} \rangle, \tag{4.32}$$

and that $D^\alpha u \in L^2(K_+)$ for $|\alpha| \le l+k$, $\alpha \ne (0,0,\ldots,0,l+k) \equiv \alpha^{l+k}$. Then $u \in W^{k+l,2}(K_+)$ and we have the estimate

$$\begin{aligned} |u|_{W^{k+l,2}(K_+)} &\le c(\operatorname{supp} u)\Big(|F|_{W^{-k+l,2}(K_+)} + \sum_{|\alpha| \le l+k, \alpha \ne \alpha^{l+k}} |D^\alpha u|_{L^2(K_+)}\Big) \\ &\equiv c(\operatorname{supp} u) M. \end{aligned}$$

Proof. Let us prove, first, that for $|\beta| \le l-1$, $|i|,|j| \le k$, $\varphi \in C_0^\infty(K_+)$,

$$\left| \int_{K_+} D^{i+\beta} \varphi D^j \overline{u} \,\mathrm{d}y \right| \le c_1 M |\varphi|_{W^{k-1,2}(K_+)}. \tag{4.33}$$

Indeed, for $j + \beta + i - \widetilde{i} \ne \beta^{(l)}$, $|i - \widetilde{i}| \le 1$,

$$\int_{K_+} D^{i+\beta} \varphi D^j \overline{u} \,\mathrm{d}y = (-1)^{|\beta|+1} \int_{K_+} D^{\widetilde{i}} \varphi D^{j+\beta+i-\widetilde{i}} \overline{u} \,\mathrm{d}y,$$

hence we have the result in this case. We must now consider the term

$$\int_{K_+} D^{\alpha^{(l+k-1)}} \varphi D^{\alpha^{(k)}} \overline{w} \,\mathrm{d}y.$$

From (4.32), we get:

$$\left| \int_{K_+} \overline{a}^{\bullet}_{\alpha^{(k)}\alpha^{(k)}} D^{\alpha^{(l+k-1)}} \varphi D^{\alpha^{(k)}} \overline{w} \,\mathrm{d}y \right| \le c_2 M |\varphi|_{W^{k-1,2}(K_+)}. \tag{4.34}$$

But $\overline{a}^{\bullet}_{\alpha^{(k)}\alpha^{(k)}} \in W^{l+k-1,\infty}(K_+)$, thus (4.34) implies:

$$\left| \int_{K_+} D^{\alpha^{(l+k-1)}} (\overline{a}^{\bullet}_{\alpha^{(k)}\alpha^{(k)}} \varphi) D^{\alpha^{(k)}} \overline{w} \,\mathrm{d}y \right| \le c_3 M |\varphi|_{W^{k-1,2}}(K_+).$$

According to Theorem 3.4.5, we have $a^{\bullet}_{\alpha^{(k)}\alpha^{(k)}}(y) \neq 0$ in $\overline{K}_+$, hence $a^{\bullet}_{\alpha^{(k)}\alpha^{(k)}}\varphi$ is an isomorphism of and onto $W_0^{k-1,2}(K_+)$; this implies (4.33). Now we can apply Lemma 3.7.2; for pedagogical reasons, we use again the same argument but from another point of view. Let us set for $y_N < 0$:

$$u(y', y_N) = \sum_{r=1}^{2k+l-1} \lambda_r u\left(y', -\frac{y_N}{r}\right),$$

where the numbers λ_r satisfy:

$$\sum_{r=1}^{2k+l-1} \lambda_r \left(-\frac{1}{r}\right)^h = 1, \quad h = 0, 1, \ldots, 2k+l-2.$$

Now we consider the sesquilinear form

$$(v, u)_k = \int_K \sum_{|i|=k} D^i v D^i \overline{u}\, \mathrm{d}y, \quad v \in W_0^{k,2}(K).$$

We have for $|\beta| \leq l-1, v \in C_0^\infty(K)$:

$$\left|(D^\beta v, u)_k\right| \leq c_4 M |v|_{W^{k-1,2}(K)}. \tag{4.35}$$

Indeed: the term

$$\int_K D^{i+\beta} v D^i \overline{u} \mathrm{d}y$$

can be estimated as (4.33), if $\beta + 2i - \widetilde{i} \neq \beta^{(l)}$. Now we consider the term

$$\int_K D^{\beta^{(l+k-1)}} v D^{\beta^{(k)}} \overline{u}\, \mathrm{d}y.$$

We get:

$$\begin{aligned}
\int_K D^{\beta^{(l+k-1)}} v(y', y_N) D^{\beta^{(k)}} \overline{u}(y', y_N)\, \mathrm{d}y &= \int_{K_+} D^{\beta^{(l+k-1)}} v(y', y_N) D^{\beta^{(k)}} \overline{u}(y', y_N)\, \mathrm{d}y \\
&+ \int_{K_-} \sum_{r=1}^{k+l-1} D^{\beta^{(l+k-1)}} v(y', y_N) \lambda_r \left(-\frac{1}{r}\right)^k D^{\beta^{(k)}} \overline{u}(y', -\frac{y_N}{r})\, \mathrm{d}y \\
&= \int_{K_+} \left(D^{\beta^{(l+k-1)}} v(y', y_N) - \sum_{r=1}^{2k+l-1} D^{\beta^{(2l+k-1)}} v(y', -r y_N) \lambda_r \left(-\frac{1}{r}\right)^{k-1} \right) \times \\
&\qquad \times D^{\beta^{(k)}} \overline{u}(y', y_N)\, \mathrm{d}y.
\end{aligned}$$

Now if we set:

$$\omega(y',y_N) = v(y',y_N) - \sum_{r=1}^{2k+l-1} \lambda_r \left(-\frac{1}{r}\right)^{l+2k-2} v(y',-ry_N),$$

we obtain $\omega \in W_0^{k+l-1,2}(K_+)$, and

$$D^{\beta^{(k+l-1)}}\omega(y',y_N) = D^{\beta^{(k+l-1)}}v(y',y_N) - \sum_{r=1}^{k+l-1} \lambda_r \left(-\frac{1}{r}\right)^{k-1} D^{\beta^{(l-1)}}v(y',-\frac{y_N}{r}),$$

hence (4.35) follows. With $u \in W^{k,2}(K)$, we are in the framework of Theorem 1.2. □

4.2.6 A Fundamental Lemma

Now we prove a lemma which will play a fundamental role:

Lemma 2.12. *Let us define a sesquilinear form on* $W^{k,2}(K_+) \times W^{k,2}(K_+)$*:*

$$[v,u] = \int_{K_+} \sum_{|i|,|j|\le k} a^{\bullet}_{ij} D^i v D^j \overline{u}\,\mathrm{d}y + \int_{\Delta} \sum_{i=0}^{k-1} \sum_{|\alpha|\le 2k-i-1} b^{\bullet}_{i\alpha} \frac{\partial^i v}{\partial y_N^i} D^{\alpha}\overline{u}\,\mathrm{d}y'$$

with coefficients satisfying

$$a^{\bullet}_{ij} \in C^{\alpha_i,1}(\overline{K}^+), \quad \alpha_i = \max(0,|i|+l-k-1), \quad a^{\bullet}_{(0,0,\dots,k)(0,0,\dots,k)} \in W^{l+k-1,\infty}(K_+),$$
$$b^{\bullet}_{i\alpha} \in C^{|\alpha|+l-k,1}(\overline{\Delta}) \quad \text{for } |\alpha|-k \ge 0, \qquad b^{\bullet}_{i\alpha} \in C^{l-1,1}(\overline{\Delta}) \quad \text{for } |\alpha|-k<0.$$

Moreover we assume that for $v \in V^{\bullet}$

$$|[v,v]| \ge c_1 |v|^2_{W^{k,2}(K_+)}. \tag{4.36}$$

Let F be a functional on $W^{k,2}(K_+)$, $|F| \le M$, *such that for* $|\alpha| \le l-1$, $v \in C_0^{\infty}(K_+)$

$$\sup_{|v|_{W^{k-1,2}(K_+)}\le 1} |FD^{\alpha}v| \le M, \tag{4.37}$$

and for $|\alpha| \le l-1,\ \alpha_N = 0,\ \tau \le N-1$, $v \in W_{k+|\alpha|,0}$,

$$\sup_{|v|_{W^{k,2}(K_+)}\le 1} |FD^{\alpha}\Delta_h^{\tau} v| \le M. \tag{4.38}$$

Let $u \in V^\bullet$ be such that $v \in V^\bullet \Longrightarrow [v,u] = Fv$. Then $u \in W^{k+l,2}(K_+)$ and

$$|u|_{W^{k+l,2}(K_+)} \leq c(\operatorname{supp} u)M. \tag{4.39}$$

Proof. In this proof we use induction with respect to l. Let $l = 1$. For $\tau = 1,2,\dots,N-1$, $v \in V^\bullet$, we have:

$$[v,\Delta_h^\tau u] = -[\Delta_{-h}^\tau v,u] - \int_{K_+} \sum_{|i|,|j|\leq k} \Delta_{-h}^\tau \overline{a}'_{ij}(y) D^i v(y-h^{(\tau)}) D^j \overline{u}\,\mathrm{d}y$$

$$-\int_\Delta \sum_{i=0}^{k-1} \sum_{|\alpha|\leq 2k-1-i} \Delta_{-h}^\tau \overline{b}^\bullet_{i\alpha}(y) \frac{\partial^i v}{\partial y_N^i}(y-h^{(\tau)}) D^\alpha \overline{u}\,\mathrm{d}y'. \tag{4.40}$$

Let $|h| < \varepsilon$ and Z_h^τ from Lemma 2.4 such that $Z_h^\tau \in [W_{k,5\varepsilon} \to W^{k,2}(K_+)]$ and for $v \in V_{5\varepsilon}^\bullet$, $Z_h^\tau v - \Delta_h^\tau v \in W_{k,3\varepsilon}$. Let X_h^τ be an another compensation operator such that

$$\text{for } |h| < \varepsilon \quad X_h^\tau \in [W_{k,3\varepsilon} \to W^{k,2}(K_+)],$$

for $v \in V_{3\varepsilon}^\bullet$, $X_h^\tau v - \Delta_h^\tau v \in V_\varepsilon^\bullet$.

We have for $|h| < \varepsilon, v \in V_{3\varepsilon}^\bullet$:

$$[\Delta_{-h}^\tau v,u] = [\Delta_{-h}^\tau v - X_{-h}^\tau v,u] + [X_{-h}^\tau v,u] = F(\Delta_{-h}^\tau v - X_{-h}^\tau v) + [X_{-h}^\tau v,u]. \tag{4.41}$$

Moreover

$$[v,\Delta_h^\tau u - Z_h^\tau u] = [v,\Delta_h^\tau u] - [v,Z_h^\tau u] \tag{4.42}$$

and if we put $v = \Delta_h^\tau u - Z_h^\tau u$, it follows from (4.36), (4.40), (4.41), that

$$c_1|\Delta_h^\tau u - Z_h^\tau u|^2_{W^{k,2}(K_+)} \leq M|\Delta_h^\tau u - Z_h^\tau u|_{W^{k,2}(K_+)} + c_2|\Delta_h^\tau u - Z_h^\tau u|_{W^{k,2}(K_+)}|u|_{W^{k,2}(K_+)},$$

and hence

$$|\Delta_h^\tau u|_{W^{k,2}(K_+)} \leq c_4(\operatorname{supp} u)(|u|_{W^{k,2}(K_+)} + M). \tag{4.43}$$

As a consequence of Lemma 2.1, Proposition 2.2.4 and (4.43), we get:

$$|D^\alpha u|_{L^2(K_+)} \leq c_5(\operatorname{supp} u)(|u|_{W^{k,2}(K_+)} + M), \quad |\alpha| = k+1, \alpha \neq \alpha^{(k+1)}. \tag{4.44}$$

On the other hand, using Lemma 2.11 and (4.36) we deduce:

$$|u|_{W^{k+1,2}(K_+)} \leq c_6(\operatorname{supp} u)(|u|_{W^{k,2}(K_+)} + M),$$

and our lemma is proved for $l = 1$.

Assuming that the lemma has been proved for $l \geq 1$, we consider the case $l+1$. We have again (4.40) and set:

$$\int_{K_+} \sum_{|i|,|j| \leq k} \Delta^{\tau}_{-h} \overline{a}^{\bullet}_{ij}(y) D^i v(y-h^{(\tau)}) D^j \overline{u}\, \mathrm{d}y +$$

$$\int_{\Delta} \sum_{i=0}^{k-1} \sum_{|\alpha| \leq 2k-1-i} \Delta^{\tau}_{-h} \overline{b^{\bullet}_{i\alpha}(y)} \frac{\partial^i v}{\partial y^i_N}(y-h^{(\tau)}) D^{\alpha} \overline{u}\, \mathrm{d}y' = I_h v.$$

If $\omega = \Delta^{\tau}_h u - Z^{\tau}_h u$, $|h| < \varepsilon$, we obtain $\omega \in V^{\bullet}_{3\varepsilon}$; for $v \in V^{\bullet}_{3\varepsilon}$ and by (4.40), (4.42), we get

$$[v,\omega] = [v,\Delta^{\tau}_h u] - [v,Z^{\tau}_h u] = -F(\Delta^{\tau}_{-h} v - X^{\tau}_{-h} v) - [X^{\tau}_{-h} v, u] - I_h v - [v, Z^{\tau}_h u]. \quad (4.45)$$

Let us denote: $F_h v = -F(\Delta^{\tau}_{-h} v - X^{\tau}_{-h} v)$. We have $\lim_{h\to 0} F_h v = F_1 v$, $v \in W_{k,3\varepsilon}$, $|F_1| \leq c_7 M$. Indeed: if we consider $F\Delta^{\tau}_{-h} v$, then by (4.38) $|F\Delta^{\tau}_{-h} v| \leq M|v|_{W^{k,2}(K_+)}$. On the other hand the set of $v \in C^{\infty}(\overline{K}^+)$ with $\operatorname{supp} v \subset K_{3\varepsilon} \cup \Delta_{3\varepsilon}$ is dense in $W_{k,3\varepsilon}$ (cf. Lemma 2.8), and we have:

$$\lim_{h\to 0} F\Delta^{\tau}_{-h} v = \frac{\partial v}{\partial x_{\tau}};$$

then

$$\lim_{h\to 0} F\Delta^{\tau}_{-h} v = F_2 v \quad \text{for } v \in W_{k,3\varepsilon} \text{ and } |F_2| \leq M.$$

Finally, using Lemma 2.5, we obtain;

$$\lim_{h\to 0} FX^{\tau}_{-h} v = FX^{\tau}_0 v.$$

The functional F_1 satisfies (4.37) with l and $M; F_1$ satisfies (4.38) with l: it is sufficient to consider again $v \in C^{\infty}(\overline{K}_+)$, $\operatorname{supp} v \subset K_{3\varepsilon} \cup \Delta_{3\varepsilon}$, and we obtain:

$$|F_1 D^{\alpha} \Delta^{\tau_1}_{h_1} v| \leq \left| FD^{\alpha} \frac{\partial}{\partial x_{\tau}} \Delta^{\tau_1}_{h_1} v \right| + |FX^{\tau}_0 D^{\alpha} \Delta^{\tau_1}_{h_1} v| \leq (M + c_8)|v|_{W^{k,2}(K_+)}; \quad (4.46)$$

this is a consequence of (4.38) applied to F and of Lemma 2.5.

If $v \in W_{k,3\varepsilon}$, then

$$\lim_{h\to 0} [X^{\tau}_{-h} v, u] = F_3 v; \quad (4.47)$$

due to Lemma 2.5, F_3 satisfies the hypotheses (4.37), (4.38) with

$$|\sup_{v \in W_{k+|\alpha|,\varepsilon}, |v|_{W^{k,2}(K_+)} \leq 1} F_3 D^{\alpha} \Delta^{\tau}_h v| \leq c_{10} |u|_{W^{k+l,2}(K_+)}. \quad (4.48)$$

Moreover if $v \in C_0^\infty(K_+)$ then $F_3 D^\alpha v = 0$. We obtain (4.48) considering integrals of following types:

$$\int_{K_+} \overline{a} D^i D^\alpha \Delta_h^\tau v D^j \overline{u}\, \mathrm{d}y, \quad \int_\Delta \overline{b} \frac{\partial}{\partial y_N^l} D^\alpha \Delta_h^\tau v D^\beta \overline{u}\, \mathrm{d}y',$$

with $|i| \le k$, $|j| \le k$, $l \le k-1$, $|\beta| \le 2k-l-1$, $\beta_N \le k-1$, a,b are smooth as $a_{ij}^\bullet$, $b_{i\alpha}^\bullet$ for $l+1$. By successive integrations by parts with respect to y' the first integral can be rewritten into the following form:

$$\int_{K_+} D^\gamma v \Delta_{-h}^\tau D^\lambda (\overline{a} D^j \overline{u}) \mathrm{d}y \quad \text{with} \quad |\lambda| + |j| + 1 \le l + k, \quad |\gamma| \le k,$$

which is possible. The second integral will be treated similarly.

We can use the same method to obtain the existence of

$$\lim_{h\to 0} I_h v = I_0 v, \quad \lim_{h\to 0} [v, Z_h^\tau u] = [v, Z_0^\tau u] = F_4 v, \quad v \in W_{k,3\varepsilon},$$

with the conditions

$$\sup_{v\in C_0^\infty(K_+), |v|_{W^{k-1,2}(K_+)} \le 1} |I_h D^\alpha v| \le c_{11} |u|_{W^{k+l,2}(K_+)}, \quad |\alpha| \le l-1, \tag{4.49}$$

$$\sup_{v\in W_{k+|\alpha|,3\varepsilon}, |v|_{W^{k,2}(K_+)} \le 1} |I_0 D^\alpha \Delta_h^\tau v| \le c_{12} |u|_{W^{k+l,2}(K_+)}, \quad |\alpha| \le l-1, \alpha_N = 0. \tag{4.50}$$

According to Lemma 2.5 we obtain the condition:

$$\sup_{v\in W_{k+|\alpha|,3\varepsilon}, |v|_{W^{k,2}(K_+)} \le 1} |[D^\alpha \Delta_0^\tau v, Z_0^\tau u]| \le c_{13} |u|_{W^{k+l,2}(K_+)}, \quad |\alpha| \le l-1. \tag{4.51}$$

If $h \to 0$ in (4.45) we obtain for $v \in V_{3\varepsilon}^\bullet$:

$$\left[v, \frac{\partial u}{\partial x_\tau} - Z_0^\tau u \right] = F_1 v - F_3 v - I_0 v - F_4 v. \tag{4.52}$$

But

$$\frac{\partial u}{\partial x_\tau} - Z_0^\tau u \in V_{3\varepsilon}^\bullet,$$

and $F_1 - F_3 - I_0 - F_4$ is a bounded functional on $W_{k,3\varepsilon}$ satisfying (4.37), (4.38), thus we get:

$$\frac{\partial u}{\partial x_\tau} - Z_0^\tau u \in W^{k+l,2}(K_+);$$

but $Z_0^\tau u \in W^{k+i,2}(K_+)$, and we have:

$$\frac{\partial u}{\partial x_\tau} \in W^{k+l,2}(K_+)$$

and the following inequality:

$$\left| D^\alpha \frac{\partial u}{\partial x_\tau} \right|_{L^2(K_+)} \leq c(\operatorname{supp} u)(|u|_{W^{k+l,2}(K_+)} + M), \quad |\alpha| \leq k+l.$$

Now we apply Lemma 2.11 and Lemma 2.12 is proved for $l+1$. □

4.2.7 Regularity of the Solution in a Neighborhood of the Boundary

Theorem 2.2. *Let us consider a boundary value problem satisfying the conditions of $(k+l)$-regularity. Then the solution u is in $W^{k+l,2}(\Omega)$ and we have:*

$$|u|_{W^{k+l,2}(\Omega)} \leq c(|f|_{Q'} + |f| + |u_0|_{W^{k+l,2}(\Omega)} + \sum_{t=1}^{k-\mu} |g_{i_t}|_{W^{i_t+1/2+l-k,2}(\partial\Omega)}),$$

where

$$|f| = \sup_{v \in V,\, |v|_{W^{k-l,2}(\Omega)} \leq 1} |fv| \quad \textit{for} \quad l < k,$$

and $|f| = |f|_{W^{l-k,2}(\Omega)}$ for $l \geq k$ (here we assume $Q \subset L^2(\Omega)$).

Proof. Using Theorem 1.2, we have $u \in W^{k+l,2}(\Omega')$ for $\Omega' \subset \overline{\Omega}' \subset \Omega$ and the estimate

$$|u|_{W^{k+l,2}(\Omega')} \leq c_1(\Omega')(|f|_{Q'} + |f|). \tag{4.53}$$

First let us apply Lemma 2.3 and then Lemma 2.10; we obtain using the notation introduced in Lemmas 2.3, 2.10 and 2.6, for $v \in V^\bullet$,

$$\begin{aligned} ((v, w^\bullet \psi - Rw^\bullet))^\bullet &= \overline{f}^\bullet(v\overline{\psi} - Rv) + \overline{g}^\bullet(v\overline{\psi} - Rv) \\ &\quad + F_l v + G_l v + ((Rv, u^\bullet))^\bullet - ((v, Rw^\bullet))^\bullet. \end{aligned} \tag{4.54}$$

We claim: if $u^\bullet \in W^{k+\mu-1,2}(K_+)$, $1 \leq \mu \leq l$, then the right hand side of the last equality (4.54) satisfies the hypotheses of Lemma 2.12 where μ is in the place of l. Let us consider $\overline{f}^\bullet(v\overline{\psi} - Rv)$; if $l < k$, $f^\bullet$ is defined on the closure of $V^\bullet$ in $W^{k-l,2}(K_+)$. By the Hahn-Banach theorem we extend $f^\bullet$ to $W^{k-l,2}(K_+)$ with the same norm. Let $v \in C_0^\infty(K_+)$, $|\alpha| \leq \mu - 1$. Then

$$f^\bullet((D^\alpha v)\overline{\psi}) = f^\bullet(\sum_{|\beta| \leq |\alpha|} D^\beta(\overline{\psi}_\beta v)), \quad \psi_\beta \in C_0^\infty(\mathbb{R}^N).$$

We have the estimate:

$$|f^\bullet(D^\beta(\overline{\psi}_\beta v))| \le c_1 |D^\beta(\overline{\psi}_\beta v|_{W^{k-l,2}(K_+)} \le c_2 |v|_{W^{k-1,2}(K_+)},$$

which implies (4.37) with c_2. If $l \ge k$, we have:

$$f^\bullet(D^\beta(\overline{\psi}_\beta v)) = \int_{K_+} D^\beta(\overline{\psi}_\beta v) f \,\mathrm{d}y, \quad f \in W^{l-k,2}(K_+);$$

integrating by parts $(|\beta| - k + 1)$ times, if $|\beta| - k + 1 > 0$, we obtain (4.37) with c_3.

Let us assume $v \in W_{k+|\alpha|,0}$, $|\alpha| \le \mu - 1$; first let us consider the case $k - l > 0$. We have:

$$f^\bullet(\overline{\psi} D^\alpha \Delta_h^\tau v) = \sum_{|\beta| \le |\alpha|} f^\bullet(D^\beta \Delta_h^\tau \overline{\psi}_{\beta,h} v), \quad \psi_{\beta,h} \in C_0^\infty(\mathbb{R}^N),$$

$\psi_{\beta,h}$ bounded in $C_0^\infty(\mathbb{R}^N)$ for $|h|$ sufficiently small. Then we have:

$$|f^\bullet(\overline{\psi} D^\alpha \Delta_h^\tau v)| \le c_4 |v|_{W^{k,2}(K_+)}. \tag{4.55}$$

If $k - l \le 0$ we obtain (4.55) with another constant.

We must consider the terms $f^\bullet(RD^\alpha \Delta_h^\tau v)$, $|\alpha| \le \mu - 1$; according to Lemma 2.5 we finally have:

$$|f^\bullet(\overline{\psi} D^\alpha \Delta_h^\tau v - RD^\alpha \Delta_h^\tau v)| \le c_5 |v|_{W^{k,2}(K_+)}. \tag{4.56}$$

Now we consider the term $\overline{g}^\bullet(v\overline{\psi} - Rv)$; we have:

$$\overline{g}^\bullet(v\overline{\psi} - Rv) = \sum_{t=1}^{k-\mu} \langle \frac{\partial^{i_t}}{\partial y_N^{i_t}}(v\overline{\psi} - Rv), g_t \rangle, \quad g_t \in W^{i_t+1/2+l-k,2}(\Delta).$$

Let us consider $v \in W_{k+|\alpha|,0}$, $|\beta| \le \mu - 1$, $1 \le \mu \le l$. By the same method as above we must estimate terms of the following type:

$$\langle \frac{\partial^{i_t}}{\partial y_N^{i_t}} D^\beta \Delta_h^\tau (v\chi_1), g_t \rangle, \quad \langle \frac{\partial^{i_t}}{\partial y_N^{i_t}} D^\beta (v\chi_2), g_t \rangle,$$

$\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^N)$ with supports in K. Let us consider for instance a term of the first type. If $i_t + l - k < 0$,

$$\frac{\partial^{i_t}}{\partial y_N^{i_t}} D^\beta \Delta_h^\tau (v\chi_1)$$

is bounded in $W_0^{k-l-i_t-1/2,2}(\Delta)$ and independent on h, hence (4.38) holds with

$$M \le c_6 |g_t|_{W^{i_t+l+1/2-k,2}(\Delta)}.$$

If $i_t + l - k \geq 0$, $|\beta| \geq i_t + l - k + 1$, we integrate by parts $l + i_t - k + 1$ times and arrive at the same situation as above. If $|\beta| < i_s + l - k + 1$, we integrate by parts $|\beta|$ times. If $k - 1 > i_t$ we are in the previous case; if $k - 1 = i_t$, we apply Lemma 2.9. With the same approach we estimate the term of second type: finally we obtain (4.38) for $\overline{g}^{\bullet}(v\overline{\psi} - Rv)$, with:

$$M \leq c_7 \sum_{t=1}^{k-\mu} |g_t|_{W^{l+i_t+1/2-k}(\Delta)}.$$

Concerning F_l, G_l, we obtain the conditions of Lemma 2.12 for $\mu \leq l$ after integration by parts, and using Lemma 2.9 if necessary. Finally we have:

$$M \leq c_7 |u|_{W^{k+\mu-1,2}(K_+)}.$$

In the same way we estimate the term $((Rv, u^{\bullet}))^{\bullet}$. We must take into account Lemma 2.7. For the term $((v, Rw^{\bullet}))^{\bullet}$ we use Lemma 2.6, i.e.

$$R \in [V^{\bullet} \cap W_{k+\mu-1,\varepsilon} \to W_{k+\mu,0}], \quad \mu \leq l.$$

Then we have the estimate of all terms.

Now let $\mu = 1$; then

$$|Rw^{\bullet}|_{W^{k+1,2}(K_+)} \leq c_8 \big(|u|_{W^{k,2}(K_+)} + |u_0|_{W^{k,2}(K_+)}\big).$$

Let us set $\omega = w^{\bullet}\psi - Rw^{\bullet}$. According to Lemma 2.12, we get $\omega \in W^{k+1,2}(K_+)$ and

$$|\omega|_{W^{k+l,2}(K_+)} \leq c_9 \Big(|f|_{Q'} + |f| + |u_0|_{W^{k+l,2}(\Omega)} + \sum_{t=1}^{k-\mu} |g_{i_t}|_{W^{i_t+1/2+l-k,2}(\partial\Omega)}\Big).$$

Then we deduce (4.48) for $l = 1$. We apply again Lemma 2.12 for (4.54), with $\mu = 2$; this finishes the proof of Theorem 2.2; it is sufficient to take $\psi(y) = 1$ in K_ε, with ε sufficiently small, for each G_i. □

4.2.8 Strong Solutions

Lemma 2.7 can be interpreted as a property of regularity for w, the solution of a boundary value problem. It is interesting to observe that the boundary conditions do not play any role in the results.

Exercise 2.1. Let V be a closed subspace of $W^{k,2}(\Omega)$ such that

- 1. we have: $\varphi \in C_0^{\infty}(\Omega)$, $v \in V \Longrightarrow \varphi v \in V$,
- 2. using local coordinates y, $v \in V^{\bullet} \Longrightarrow \Delta_h^{\tau} v \in V^{\bullet}$, with h sufficiently small.

Prove Theorem 2.2 directly. Compensation and regularizing operators are equal to zero.

Theorem 2.3. *Let us assume that the boundary value problem satisfies for $l \geq k$ the $(l+k)-$ regularity conditions. Then the equation $Au = f$ is satisfied almost everywhere in Ω, and all boundary conditions are satisfied in the sense of traces: we have $B_s u = B_s u_0 = g_{j_s}$ on $\partial\Omega$, $s = 1,2,\dots,\mu$, $C_t u = g_{i_t}$, $t = 1,2,\dots,k-\mu$ on $\partial\Omega$, where the operators C_t are the operators obtained by formal interpretation in 1.2.6.*

*We have the converse: let $u \in W^{k,2}(\Omega)$ be an arbitrary function such that $B_s u = g_{j_s}$, $s = 1,2,\dots,\mu$, $C_t u = g_{i_t}$, $t = 1,2,\dots,k-\mu$ on $\partial\Omega, Au = f$ in Ω. Then u is the unique solution of the corresponding problem. Altogether, the operator $(A, B_1, B_2, \dots, B_\mu,$
$C_1, C_2, \dots, C_{k-\mu})$ is an isomorphism of $W^{k+l,2}(\Omega)$ onto*

$$W^{l-k,2}(\Omega) \times W^{k+l-j_1-1/2,2}(\partial\Omega) \times \cdots \times W^{k+l-j_\mu-1/2,2}(\partial\Omega)) \times$$

$$W^{i_1+1/2+l-k,2}(\partial\Omega) \times \cdots \times W^{i_{k-\mu}+l-k+1/2,2}(\partial\Omega).$$

Proof. Using integration by parts, which is perfectly justified here, we have by Green's formula:

$$v \in V, \qquad ((v,u)) = \int_\Omega v\overline{Au}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{t=1}^{k-\mu} \frac{\partial^{i_t} v}{\partial n^{i_t}} \overline{C_t u}\,\mathrm{d}S. \tag{4.57a}$$

On the other hand,

$$v \in V, \qquad ((v,u)) = \int_\Omega v\overline{f}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{t=1}^{k-\mu} \frac{\partial^{i_t} v}{\partial n^{i_t}} \overline{g_{i_t}}\,\mathrm{d}S. \tag{4.57b}$$

If we use $\varphi \in C_0^\infty(\Omega)$, we obtain:

$$\int_\Omega (\overline{Au} - \overline{f})\varphi\,\mathrm{d}x = 0, \text{ hence } Au = f \text{ almost everywhere.}$$

If $v \in V$, we have

$$\int_{\partial\Omega} \sum_{t=1}^{k-\mu} \frac{\partial^{i_t} v}{\partial n^{i_t}} (\overline{C_t u} - \overline{g_{i_t}})\,\mathrm{d}S = 0.$$

Now let $\varphi_{i_t} \in W^{k-i_t-1/2,2}(\partial\Omega)$, $v \in W^{k,2}(\Omega)$ such that $\partial^{i_t} v/\partial n^{i_t} = \varphi_{i_t}$ on $\partial\Omega$, $t = 1,2,\dots k-\mu$, and $B_s v = 0$ on $\partial\Omega$, $s = 1,2,\dots\mu$. This is always possible: indeed we must have $\partial^{j_s} v/\partial n^{j_s} - F_s v = B_s v = 0$; if we know $\partial^{i_t} v/\partial n^{i_t}$ we also know $F_s v$, and we have $F_s v \in W^{k-j_s-1/2,2}(\partial\Omega)$. According to Theorem 2.5.8, there exists $v \in W^{k,2}(\Omega)$ such that $\partial^{i_t} v/\partial n^{i_t} = \varphi_{i_t}$, $\partial^{j_s} v/\partial n^{j_s} = F_s v$. It follows that $C_t u = g_{i_t}$ on $\partial\Omega$. Using again Lemma 2.3, the proof is complete. □

Remark 2.2. It is possible to give a precise interpretation of nonstable boundary conditions also for $k \le l < 2k$. In some sense we have

$$C_t u \quad \to \quad g_{i_t} \quad \text{in} \quad W^{l-k+i_t+1/2,2}(\partial\Omega),$$

cf. J.L. Lions, E. Magenes [5].

The solution $u \in W^{2k,2}(\Omega)$ of a boundary value problem for an operator of order $2k$ is called *a strong solution.*

We have the following lemmas:

Lemma 2.13. *Let $V \subset Q \subset L^2(\Omega)$, V a closed subspace of $W^{k,2}(\Omega)$, Q a reflexive Banach space. Then the functionals of type $< v, f >$, $f \in L^2(\Omega)$ are dense in Q'.*

Proof. If we assume that the conclusion is not true, there should exist $v \in Q$, $v \neq 0$, such that

$$f \in L^2(\Omega) \Longrightarrow \int_\Omega v f \, \mathrm{d}x = 0,$$

which is impossible. □

We prove analogously the following:

Lemma 2.14. *Let s be an integer, $0 \le s \le k-1$, $\Omega \in \mathfrak{N}^{2k,1}$. Then the functionals of type $\langle v, g \rangle_{\partial\Omega}$, $g \in W^{s+1/2,2}(\partial\Omega)$ are dense in $W^{s+1/2-k,2}(\partial\Omega)$.*

From Theorem 2.3.1 we deduce that if $\Omega \in \mathfrak{N}^0$, $W^{2k,2}(\Omega)$ is dense in $W^{k,2}(\Omega)$; consequently we have:

Theorem 2.4. *Suppose we are given a $2k$-regularizable problem, Q a normal Banach space as in Lemma 2.13, G the Green operator from 3.3.1. Then G is the continuous extension of $(A, B_1, \ldots B_\mu, C_1, \ldots C_{k-\mu})^{-1}$ introduced in Theorem 2.3.*

Remark 2.3. If $l \ge N/2 + k$, the solution of an $(l+k)$-regularizable problem is in $C^{2k}(\overline{\Omega})$; this is a consequence of Theorem 2.3.8. The solution is the classical solution. If $l = \infty$, $u \in C^\infty(\overline{\Omega})$.

Remark 2.4. Condition (4.20d) can be replaced by the V-ellipticity of $((v,u)) + \lambda(v,u)$, λ sufficiently large. If $Q \subset L^2(\Omega)$ algebraically and topologically, the conclusion of Theorem 2.2 follows with

$$|u|_{W^{k+l,2}(\Omega)} \le c\Big(|f|_{Q'} + |f| + |u_0|_{W^{k+l,2}(\Omega)} + \sum_{t=1}^{k-\mu} |g_{i_t}|_{W^{i_t+1/2+l-k,2}(\partial\Omega)} + |u|_{L^2(\Omega)}\Big). \tag{4.58}$$

Exercise 2.2. Prove (4.58).

4.2.9 Local Regularity in a Neighborhood of the Boundary

We prove a theorem concerning the local regularity in a neighborhood of the boundary.

A boundary value problem is $(l+k)$-times *regularizable in a neighborhood of a point* $y \in \partial\Omega$, $l \geq 1$, if there exists a neighborhood of G-type, cf. 1.2.4, containing y and such that $\partial\Omega \cap G$ can be described by a function from $C^{2k+l,1}(\overline{\Delta})$, the coefficients satisfying the conditions of $(l+k)$-regularity in $\overline{\Omega} \cap \overline{G}$, on $\partial\Omega \cap \overline{G}$ respectively, with $f \in Q'$, and for $l < k$ and $v \in V$ such that $\operatorname{supp} v \subset G \cap \overline{\Omega}$,

$$\sup_{|v|_{W^{k-l,2}(\Omega\cap G)} \leq 1} |fv| \equiv |f| < \infty,$$

for $l \geq k$,

$$Q \subset L^2(\Omega), \quad |f|_{W^{l-k,2}(\Omega\cap G)} \equiv |f| < \infty.$$

Concerning u_0, we assume $u_0 \in W^{k+l,2}(\Omega \cap G)$, and

$$g_{i_t} \in W^{i_t+1/2+l-k,2}(\partial\Omega \cap G).$$

We have

Theorem 2.5. *Let $u \in W^{k,2}(\Omega)$ be a solution of a problem $(l+k)$-regularizable in a neighborhood of $y \in \partial\Omega$. Then for every compact subset $K \subset G \cap \overline{\Omega}$, we have:*

$$|u|_{W^{k+l,2}(K)} \leq c(K)(|f|_{Q'} + |f| + |u_0|_{W^{k+l,2}(\Omega\cap G)} + \sum_{t=1}^{k-\mu} |g_{i_t}|_{W^{i_t+1/2+l-k,2}(\partial\Omega\cap G)}.$$

The proof is the same as in Lemma 2.8 proving that the functions $w_n = w\psi_n$ belong to $W^{k+l,2}(\Omega)$.

4.2.10 Dependence of the Solution on the Coefficients

In relation with theorems proved in Sect. 4.2, there are many open problems; in some sense the situation is the same as in Chap. 3, Sect. 3.4.

We shall prove a theorem concerning the dependence of the solution on the coefficients a_{ij}, $b_{i\alpha}$, $h_{s\alpha}$ and on f, g_{j_s}, g_{i_t}.

Theorem 2.6. *Suppose we are given a sequence of $(l+k)$-regularizable problems, $l \geq k$. We assume:*

$$B_{sv} = \frac{\partial^{j_s}}{\partial n^{j_s}} - \sum_{|\alpha|\leq j_s} h_{s\alpha}^{(v)} D^\alpha,$$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial\sigma_1^{\alpha_1} \dots \partial\sigma_{N-1}^{\alpha_{N-1}} \partial\sigma_N^{\alpha_N}},$$

where α_N is one of the indices i_t (the indices j_s, i_t don't depend on ν), $\nu = 1,2,\ldots$ We assume that $\lim_{\nu\to\infty} a_{ij}^{(\nu)} = a_{ij}$ *in* $C^{\alpha_i,1}(\overline{\Omega})$, $\alpha_i = \max(0,|i|+l-k-1)$, $\lim_{\nu\to\infty} a_{i^{(h)}i^{(h)}}^{(\nu)} = a_{i^{(h)}i^{(h)}}$ *in* $W^{k+l-1,\infty}(\Omega)$, $\lim_{\nu\to\infty} b_{i\alpha}^{(\nu)} = b_{i\alpha}$ *in* $C^{|\alpha|+l-k,1}(\partial\Omega)$ *for* $|\alpha| \geq k$, *and in* $C^{l-1,1}(\partial\Omega)$ *for* $|\alpha| < k$; $\lim_{\nu\to\infty} h_{s\alpha}^{(\nu)} = h_{s\alpha}$ *in* $C^{k-j_s-l,1}(\partial\Omega)$. *Finally we assume that* $\lim_{\nu\to\infty} f^{(\nu)} = f$ *in* $W^{l-k,2}(\Omega)$, $\lim_{\nu\to\infty} g_{j_s}^{(\nu)} = g_{j_s}$ *in* $W^{l-j_s-1/2}(\partial\Omega)$, *and* $\lim_{\nu\to\infty} g_{i_t}^{(\nu)} = g_{i_t}$ *in* $W^{i_s+1/2+l-k,2}(\partial\Omega)$.

Then if $u^{(\nu)}, u$ *are the solutions of the corresponding problems,* $\lim_{\nu\to\infty} u^{(\nu)} = u$ *weakly in* $W^{l+k,2}(\Omega)$, *and* $\lim_{\nu\to\infty} u^{(\nu)} = u$ *strongly in* $W^{l+k-1,2}(\Omega)$.

Proof. If $\nu \geq \nu_0$, $v \in V_\nu$, we have $|((v,v))_\nu| \geq c_1 |v|^2_{W^{k,2}(\Omega)}$, where c_1 does not depend on ν. If $v \in V$, $|((v,v))| \geq c_2 |v|^2_{W^{k,2}(\Omega)}$. In $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$ we have uniformly:

$$\lim_{\nu\to\infty} ((v,u))_\nu = ((v,u))$$

hence if $\nu \geq \nu_0$, $v \in V$,

$$|((v,v))|_\nu \geq \frac{c_2}{2} |v|^2_{W^{k,2}(\Omega)}. \tag{4.59}$$

Suppose now $v \in V_\nu$. There exists $w \in V$ such that

$$|v-w|_{W^{k,2}(\Omega)} \leq d_\nu |v|_{W^{k,2}(\Omega)}, \quad \lim_{\nu\to\infty} d_\nu = 0. \tag{4.59 bis}$$

To prove this, it is sufficient to verify:

$$|(B_s - B_{s\nu})v|_{W^{k-j_s-1/2,2}(\partial\Omega)} \leq \varepsilon_\nu |v|_{W^{k,2}(\Omega)}, \quad \lim_{\nu\to 0} \varepsilon_\nu = 0;$$

thus let $\omega \in W^{k,2}(\Omega)$ be a function such that $\partial^{i_t}\omega/\partial n^{i_t} = 0$ on $\partial\Omega$, $t = 1,2,\ldots, k-\mu$, $\partial^{j_s}\omega/\partial n^{j_s} = (B_s - B_{s\nu})v$. Using Theorem 2.5.8 we can construct a function such that

$$|\omega|_{W^{k,2}(\Omega)} \leq c_3 \sum_{s=1}^{\mu} |(B_s - B_{s\nu})v|_{W^{k-j_s-1/2,2}(\partial\Omega)}.$$

We obtain (4.59 bis) from (4.58), (4.59), with $w = v - \omega$, $\nu \geq \nu_1 \geq \nu_0$, $v \in V_\nu$:

$$|((v,v))_\nu| \geq \frac{c_2}{3} |v|^2_{W^{k,2}(\Omega)}.$$

According to Theorem 2.2 for $\nu \geq \nu_1$, we have:

$$|u_\nu|_{W^{k+l,2}(\Omega)} \geq c_4 \Big(|f_\nu|_{W^{k-l,2}(\Omega)} + \sum_{s=1}^{\mu} |g_{j_s}|_{W^{k+l-j_s-1/2,2}(\partial\Omega)} + \sum_{s=1}^{k-\mu} |g_{i_t}|_{W^{i_t+1/2+l-k,2}(\partial\Omega)}\Big). \tag{4.60}$$

We have $\lim_{\nu\to\infty} u_\nu = u$ weakly in $W^{k+l,2}(\Omega)$. If not, there would exist a subsequence u_{ν_r} such that $\lim_{r\to\infty} u_{\nu_r} = u^*$ weakly, $u^* \neq u$. Due to the compactness of the imbedding of $W^{k+l,2}(\Omega)$ in $W^{k+l-1,2}(\Omega)$, $\lim_{r\to\infty} u_{\nu_r} = u^*$ strongly in $W^{k+l-1,2}(\Omega)$. Clearly, we have $B_s u^* = g_{j_s}$. Let $v \in V$ be L; as above we can find $v_\nu \in V_\nu$ such that $\lim_{\nu\to\infty} v_\nu = v$ strongly in $W^{k,2}(\Omega)$. We have:

$$((v_\nu, u_\nu))_\nu = \langle v_\nu, \overline{f}_\nu \rangle + \overline{g}_\nu v_\nu;$$

we deduce,

$$((v, u^*)) = \langle v, \overline{f} \rangle + \overline{g} v,$$

thus, by uniqueness of the solution, $u = u^*$ and this is a contradiction to $u^* \neq u$. □

Theorem 2.3 gives us the existence of a classical solution of the boundary value problem satisfying all the given conditions. We can formulate the following:

Problem 2.1. Find all the boundary value problems such that there exists a sequence of $2k$-regularizable problems whose solutions converge, in some sense, to the solution of the initial problem.

We have seen in Chap. 3, Sect. 3.6, the possibility to construct such a sequence regularizing the domain, coefficients, and data. This is for example the case of the Dirichlet and Neumann problem.

Example 2.1. Let $N = 2$, Ω a triangle, $A = \triangle^2 + 1$. Let us put $f = 0$, $V = W^{2,2}(\Omega)$, $gv = v(x_0)$, where x_0 is a vertex of the triangle Ω. Let $u \in W^{2,2}(\Omega)$ be such that for all $v \in W^{2,2}(\Omega)$:

$$\int_\Omega \left(\frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 \overline{u}}{\partial x_1^2} + 2 \frac{\partial^2 v}{\partial x_1 \partial x_2} \frac{\partial \overline{u}}{\partial x_1 \partial x_2} + \frac{\partial^2 v}{\partial x_2^2} \frac{\partial^2 \overline{u}}{\partial x_2^2} + v\overline{u} \right) \mathrm{d}x = v(x_0).$$

There exists a unique solution to this problem; due to Theorem 3.2.1, and according to Theorem 2.5, we have using the same notation as in Chap. 1, Example 1.2.15 on the sides of the triangle, except the vertices, $Mu = 0$, $Tu = 0$. We have also $Au = 0$ in Ω. It is easy to see that a weak solution u is not determined by the values of non-stable boundary conditions on smooth parts of the boundary $\partial\Omega$, u being smooth almost everywhere.

4.3 Boundary Value Problems for Properly Elliptic Operators

4.3.1 The Operator A Properly Elliptic, B_s Cover A

The boundary value problems defined in 3.2.3 represent a large class of problems related to phenomenas in physics.

We can formulate and solve boundary value problems in the framework and with the hypotheses given in Sect. 3.3, Chap. 3. We shall prove particular results found in M. Schechter [2, 4]; we don't try to give a complete overview. For completeness we add some results and also some algebraic lemmas, cf. M. Schechter [2, 4], N. Aronszajn, A. Milgram [1], G. Geymonat, P. Grisvard [1]. For $\partial\Omega$ in any dimension we refer to cf. B. Ju. Sternin [1]. For simplicity here we assume that Ω and the coefficients are infinitely differentiable.

Example 3.1. Let Ω be a domain in $\mathfrak{N}^\infty$, $N=2$, $f\in L^2(\Omega)$ and let us find $u\in W^{4,2}(\Omega)$ such that $\triangle^2 u = f$ almost everywhere in Ω, $u=\partial^3 u/\partial n^3 = 0$ on $\partial\Omega$. This problem cannot be solved by the variational method and a decomposition in two auxiliary problems $\triangle u = v$, $\triangle v = f$ does not work. On the other hand $\triangle^2$ is properly and uniformly elliptic; indeed, for two linearly independent vectors ξ,η, we have:

$$(\xi_1+\tau\eta_1)^4+2(\xi_1+\tau\eta_1)^2(\xi_2+\tau\eta_2)^2+(\xi_2+\tau\eta_2)^4=((\xi_1+\tau\eta_1)^2+(\xi_2+\tau\eta_2)^2)^2,$$

this algebraic expression has two roots, each with multiplicity 2,

$$\tau_1=-\frac{\xi_1\eta_1+\xi_2\eta_2+i(\xi_2\eta_1-\xi_1\eta_2)}{\eta_1^2+\eta_2^2},\quad \tau_2=-\frac{\xi_1\eta_1+\xi_2\eta_2-i(\xi_2\eta_1-\xi_1\eta_2)}{\eta_1^2+\eta_2^2}.$$

The operators $B_1=1$, $B_3=\partial^3/\partial n^3$ form a covering of $\triangle^2$: we can change coordinates, the Laplace operator is invariant, and we can take $\xi_1=1$, $\eta_1=0$, $\xi_2=0$, $\eta_2=1$. By trivial computation the result follows.

It is natural to formulate the problem in following terms: Let $\Omega\in\mathfrak{N}^\infty$, $A=\sum_{|i|\le 2k} a_i D^i$ a uniformly elliptic operator in $\overline{\Omega}$, with coefficients in $C^\infty(\overline{\Omega})$. We assume A properly elliptic in $\overline{\Omega}$. Suppose we are given operators of order $m_s\le 2k-1$, $s=1,2,\dots,k$, $B_s=\sum_{|\alpha|\le m_s} b_{s\alpha}D^\alpha$ with coefficients in $C^\infty(\partial\Omega)$ covering A. Moreover assume $f\in L^2(\Omega)$, $g_s\in W^{2k-m_s-1/2,2}(\partial\Omega)$. We are looking for $u\in W^{2k,2}(\Omega)$ such that almost everywhere in Ω

$$Au=f, \tag{4.61a}$$

and on $\partial\Omega$ in the sense of traces

$$B_s u = g_s. \tag{4.61b}$$

Before finding sufficient conditions for the existence of a solution, let us observe:

Proposition 3.1. *Consider a $2k$-regularizable problem in the sense of the definition in 4.2.2, $N\ge 3$. Denote by $B_1,B_2,\dots B_\mu,C_1,C_2,\dots,C_{k-\mu}$ a system of boundary operators. Then the operator A considered is properly elliptic in $\overline{\Omega}$ and the boundary system covers A.*

Proof. By Theorem 2.2 we have the following inequality:

$$|u|_{W^{2k,2}(\Omega)} \leq c(|f|_{L^2(\Omega)} + \sum_{s=1}^{\mu} |g_{j_s}|_{W^{2k-j_s-1/2,2}(\partial\Omega)} + \sum_{t=1}^{k-\mu} |g_{i_t}|_{W^{2k-i_t-1/2,2}(\partial\Omega)}), \tag{4.61 bis}$$

thus the result follows from Theorem 3.4.5 and Remark 3.5.2.[1] □

In some sense we can see that the new definition of boundary value problems is a generalization of the definition from 3.2.2.

4.3.2 An Existence Theorem

A system of boundary operators B_s, $s = 1,2,\ldots,k$, of orders $m_s \leq 2k-1$, $B_s = \sum_{|\alpha|\leq m_s} b_{s\alpha} D^\alpha$ is called *normal* if for $s \neq r$ one has $m_s \neq m_r$ and if $\sum_{|\alpha|=m_s} b_{s\alpha} n^\alpha \neq 0$ for $x \in \partial\Omega$.

A normal system is called *canonical* if there exist indices $0 \leq j_1 < j_2 < \cdots < j_k \leq 2k-1$ such that in the local coordinates from 1.2.4,

$$B_s = \frac{\partial^{j_s}}{\partial t^{j_s}} - \sum_{|\alpha|\leq m_s} b_{s\alpha} D^\alpha,$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial\sigma_1^{\alpha_1}\partial\sigma_2^{\alpha_2}\ldots\partial\sigma_{N-1}^{\alpha_{N-1}}\partial t^{\alpha_N}}, \quad \alpha_N \neq j_l,\ l = 1,2,\ldots,k.$$

Let B_s, B_s^*, $s = 1,2,\ldots,k$, be two systems and of boundary operators; we say that these two systems *equivalent* if for $u \in W^{2k,2}(\Omega)$, $B_s u = 0 \Longleftrightarrow B_s^* u = 0$, $s = 1,2,\ldots,k$.

Lemma 3.1. *Let B_s, $s = 1,2,\ldots,k$ be a normal system. Then there exists another canonical normal system B_s^*, $s = 1,2,\ldots,k$, equivalent to B_s, $s = 1,2,\ldots,k$.*

Proof. Without loss of generality we can assume $m_1 < m_2 < \cdots < m_k$; B_1 is not characteristic with respect to the boundary, since it contains the coefficient $a\partial^{m_1}/\partial n^{m_1}$, $a \neq 0$ on $\partial\Omega$. We set $B_1^* = (1/a)B_1$. Let us assume that we have already constructed $B_1^*, B_2^*, \ldots, B_l^*$; the operator B_{l+1} is not characteristic, hence we can write in local coordinates:

$$(1/a)B_{l+1} = \frac{\partial^{m_l+1}}{\partial t^{m_{l+1}}} - \sum_{|\alpha|\leq m_{l+1}} h_{l+1,\alpha} \frac{\partial^{|\alpha|}}{\partial\sigma_1^{\alpha_1}\ldots.\partial\sigma_{N-1}^{\alpha_{N-1}}\partial t^{\alpha_N}},$$

$$a \neq 0 \text{ on } \partial\Omega.$$

[1]As in Theorem 3.5.4 we prove that (4.61 bis) implies A properly elliptic for $N \geq 2$. Thus Proposition 3.1 holds for $N \geq 2$.

Using $B_1^*,\dots,B_l^*$ we compute $\partial^{\alpha_N}/\partial t^{\alpha_N}$ for α_N equal to one of $m_1,m_2,\dots,m_l$ and we obtain B_{l+1}^*.

It is easy to see that all hypotheses of Lemma 3.1 are satisfied for $B_1^*,\dots,B_k^*$ obtained in this manner. □

Now, let B_s, $s=1,2,\dots,k$ be a normal canonical system, $\Omega\in\mathfrak{N}^\infty$, $u,v\in W^{4k,2}(\Omega)$, $B_s v=0$ on $\partial\Omega$, $s=1,2,\dots k$, $A=\sum_{|i|\le 2k} a_i D^i$ properly elliptic, with coefficients in $C^\infty(\overline{\Omega})$. By Green's formula, we obtain with $A^*=\sum_{|i|\le 2k}(-1)^{|i|}D^i(\overline{a}_i)$:

$$\int_\Omega v\overline{A^*Au}\,\mathrm{d}x=\int_{\partial\Omega}\left(\sum_{t=1}^k\frac{\partial^{\mu_t}v}{\partial n^{\mu_t}}\overline{B_t'Au}\right)\mathrm{d}S+\int_\Omega Av\overline{Au}\,\mathrm{d}x, \tag{4.62}$$

where the numbers μ_t are complement to the set of numbers $m_1,m_2,\dots,m_k$ with respect to the set $0,1,2,\dots,2k-1$.

At the same time let us define the *"adjoint" problem*: find $v\in W^{2k,2}(\Omega)$ such that we have almost everywhere in Ω,

$$A^*v=g\quad\text{in }\Omega,\quad g\in L^2(\Omega), \tag{4.63a}$$

$$B_t'v=0\quad\text{on }\partial\Omega,\quad t=1,2,\dots,k. \tag{4.63b}$$

Let us set

$$N=\{u\in W^{2k,2}(\Omega),\ B_s u=0\text{ on }\partial\Omega,\ s=1,2,\dots,k,\ Au=0\text{ in }\Omega\},$$
$$N^*=\{v\in W^{2k,2}(\Omega), B_t'v=0\text{ on }\partial\Omega,\ t=1,2,\dots,k,\ A^*v=0\text{ in }\Omega\}.$$

Theorem 3.1. *Let us assume $N^*=\{0\}$; this corresponds to the uniqueness of the solution of Problem (4.63). Hence (4.61) has a solution for each $f\in L^2(\Omega)$, $g_s=0$. The space N is finite dimensional and the solution u of (4.61) is defined modulo a linear combination of functions from N. If we choose u such that*

$$(v,u)=0,\quad v\in N, \tag{4.64}$$

then u is unique and we have:

$$|u|_{W^{2k,2}(\Omega)}\le c|f|_{L^2(\Omega)}. \tag{4.65}$$

Proof. If the system B_s, $s=1,2,\dots,k$ is not canonical, we can construct an equivalent canonical system, thus we assume B_s canonical. Due to Theorem 3.5.3, if λ is big enough, the sesquilinear form $\int_\Omega Av A\overline{u}\,\mathrm{d}x+\lambda\int_\Omega v\overline{u}\,\mathrm{d}x$ is V-elliptic with $V=\{v\in W^{2k,2}(\Omega), B_s v=0, s=1,2,\dots,k\}$. Let $f\in C_0^\infty(\Omega)$; we want to find $u\in W^{2k,2}(\Omega)$, $u\in V$ such that for $v\in V$, $\int_\Omega AvA\overline{u}\,\mathrm{d}x=(v,A^*f)$; this means that $A^*Au=A^*f$ in Ω, $B_s u=0$ on $\partial\Omega$, $s=1,2,\dots,k$ and formally $B_t'Au=0$ on $\partial\Omega$, $t=1,2,\dots,k$. This problem is selfadjoint, and by Theorem 3.3.1 or Theorem 1.6.1, the solution exists if and only if $(v,A^*f)=0$ for all v such that $Av=0$.

But if v is such a function, by Theorem 2.2 $v \in W^{4k,2}(\Omega)$ and by Theorem 2.3 the conditions $B'_t A v = 0$ are satisfied in the sense of traces. Let us set $Av = h$, so $h \in N^*$, i.e. $h \equiv 0$. We have $v \in N$, and $(v, A^* f) = (Av, f) = 0$. Moreover we have $Au - f \in N^*$, hence $Au = f$. Now we choose u such that (4.64) is satisfied. We can say that for $f \in C_0^\infty(\Omega)$ (4.65) is true: Assuming the converse, there exists a sequence $u_n \in W^{2k,2}(\Omega)$, $|u_n|_{W^{2k;2}(\Omega)} = 1$, such that $Au_n = f_n \in C_0^\infty(\Omega)$, $B_s u_n = 0$ on $\partial\Omega, 1 > n|f_n|_{L^2(\Omega)}$. Now by Theorem 3.5.3, we have:

$$|u_n|_{W^{2k,2}(\Omega)} \le c_1(|f_n|_{L^2(\Omega)} + |u_n|_{L^2(\Omega)}). \tag{4.66}$$

By Theorem 3.6.1 we can extract from the sequence u_n a subsequence converging to u in $L^2(\Omega)$. As a consequence of (4.66) $\lim_{n\to\infty} u_n = u$ weakly in $W^{2k,2}(\Omega)$; moreover we have $Au = 0$ in Ω and $B_s u = 0$ on $\partial\Omega$, hence $u \in N$ and $u \equiv 0$ by (4.64) which is a contradiction.

Let now $f \in L^2(\Omega)$; as $\overline{C_0^\infty(\Omega)} = L^2(\Omega)$ the result follows by continuous extension. □

Corollary 3.1. *Let the hypotheses of the previous theorem be satisfied and suppose also $g_s \in W^{2k-m_s-1/2,2}(\partial\Omega)$, $f \in L^2(\Omega)$. Then there exists a solution u of (4.61); $Au = f$ in Ω, $B_s u = g_s$ on $\partial\Omega$, which is unique modulo a linear combination of functions from N. If u satisfies (4.64), u is uniquely determined and we have:*

$$|u|_{W^{2k,2}(\Omega)} \le c(|f|_{L^2(\Omega)} + \sum_{s=1}^{k} |g_s|_{W^{2k-m_s-1/2,2}(\partial\Omega)}). \tag{4.67}$$

Proof. From Theorem 2.5.8 it follows that we can construct $u_0 \in W^{2k,2}(\Omega)$ such that $B_s u_0 = g_s$ on $\partial\Omega$ if we set $\partial^{m_s} u_0/\partial n^{m_s} = g_s$ on $\partial\Omega$, $s = 1,2,\ldots,k$, $\partial^{\mu_t} u_0/\partial n^{\mu_t} = 0$ on $\partial\Omega$, $t = 1,2,\ldots,k$, cf. (4.62). The problem can be solved as a homogeneous problem with right hand side $f - Au_0$. □

Remark 3.1. In the paper of S. Agmon, A. Douglis, L. Nirenberg [1], it is proved that problem (4.61) in the Dirichlet case has a unique solution if $N = \{0\}$. The converse is also true: if the Dirichlet problem has a solution for every $f \in L^2(\Omega)$, then $N = \{0\}$, cf. M. Schechter [2, 4]; this question is connected with the notion of index, cf. below.

Remark 3.2. M. Schechter in [4] has proved the existence of a system equivalent to B'_t which is normal and covers A^*. It follows by our previous results that $\dim N^* < \infty$ and the Problem (4.61) has a solution with $g_s = 0$ if and only if $(v, f) = 0$ for $v \in N^*$ and the Problem (4.63) has a solution if and only if $(u, g) = 0$ for $u \in N$. Then it is possible to consider the problem $A^*Au = f$ in Ω, $B_s u = 0$, $B'_t Au = 0$ on $\partial\Omega$. Let us set $W_B = \{u \in W^{2k,2}(\Omega), B_s u = 0\}$; $W_{B'} = \{v \in W^{2k,2}(\Omega), B'_t v = 0\}$; we obtain that AW_B is a closed subspace in $L^2(\Omega)$, $\dim(L^2(\Omega) - AW_B) \equiv \operatorname{codim} AW_B = \dim N^*$, for the same reason $A^*W_{B'}$ is closed in $L^2(\Omega)$, $\dim(L^2(\Omega) - A^*W_{B'}) \equiv \operatorname{codim} A^*W_{B'} = \dim N$. We define the *index* of A as $\operatorname{ind} A = \dim N - \operatorname{codim} AW_B$; if $\operatorname{ind} A = 0$ we say that A with B_s is of *Fredholm type*. For all these questions cf. M.S. Agranovich, L.R. Volevich, A.S Dynin [1, 2], L. Hörmander [1], A.I. Volpert [1].

4.3.3 Dependence with Respect to a Parameter

To find the solution of Problem (4.61) a “homotopical” approach of J. Schauder is useful, cf. C. Miranda [1], A.J. Koshelev [1], etc.:

Theorem 3.2. *Let $\Omega \in \mathfrak{N}^\infty$, A_λ, $B_{\lambda s}$, $0 \le \lambda \le 1$ a family of operators. We assume that all coefficients are continuous with respect to λ in the norms C^μ with μ arbitrary (or μ sufficiently large) and that independently of λ the following a priori estimates for $u \in W^{2k,2}(\Omega)$ hold:*

$$|u|_{W^{2k,2}(\Omega)} \le c(|A_\lambda u|_{L^2(\Omega)} + \sum_{s=1}^{k} |B_{\lambda s} u|_{W^{2k-m_s-1/2,2}(\partial\Omega)}). \tag{4.68}$$

Moreover, we assume that the operator $(A_0, B_{01}, B_{02}, \dots, B_{0k})$ is an isomorphism of $W^{2k,k}(\Omega)$ onto the product $L^2(\Omega) \times W^{2k-m_1-1/2,2}(\partial\Omega) \times \dots \times W^{2k-m_k-1/2,2}(\partial\Omega)$. Then the isomorphism holds for $0 \le \lambda \le 1$.

Proof. Let κ be the upper bound of $\lambda \in [0,1]$ such that $(A_\lambda, B_{\lambda 1}, B_{\lambda 2}, \dots, B_{\lambda k})$ is an isomorphism of $W^{2k,k}(\Omega)$ onto

$$M = L^2(\Omega) \times W^{2k-m_1-1/2,2}(\partial\Omega) \times \dots \times W^{2k-m_k-1/2,2}(\partial\Omega).$$

Using the perturbation method, we can see that $0 < \kappa \le 1$. Indeed if we assume $\kappa < 1$ then $(A_\kappa, B_{\kappa 1}, B_{\kappa 2}, \dots, B_{\kappa k})$ is such an isomorphism; by (4.68) the image of $W^{2k,2}(\Omega)$ by this transformation is closed in M.

The coefficients are continuous with respect to λ with the corresponding norms, the image of $W^{2k,2}(\Omega)$ is dense in M. Now, by perturbation we have a contradiction. Therefore $\kappa = 1$ and the last properties hold for $\kappa = 1$. □

Remark 3.3. Theorem 3.1 is based on the estimates obtained in Sect. 3.3, Chap. 3. If the domains, the coefficients are sufficiently smooth we obtain, without difficulty, using the method of Sect. 1.3, Chap. 1, the following estimates:

$$|u|_{W^{k+l,2}(\Omega)} \le c(|Au|_{W^{l-k,2}(\Omega)} + \sum_{s=1}^{k} |B_s u|_{W^{k+l-m_s-1/2,2}(\partial\Omega)} + |u|_{L^2(\Omega)}), \quad l \ge k. \tag{4.69}$$

Using the theory of multipliers, cf. for instance S.G. Mikhlin [1], B. Malgrange [2], P.I. Lizorkin [2], we obtain estimates of the following type:

$$|u|_{W^{k+l,p}(\Omega)} \le c(|Au|_{W^{l-k,p}(\Omega)} + \sum_{s=1}^{k} |B_s u|_{W^{k+l-m_s-1/p,p}(\partial\Omega)} + |u|_{L^p(\Omega)}), \tag{4.70}$$

$$p > 1, l \ge k,$$

cf. S. Agmon, A. Douglis, L. Nirenberg [1], F. Browder [3–5],.... By interpolation we can consider the case $l \geq k, l$ any real number, and by transposition we can modify (4.70) for the case $l \leq -k$ and using again the interpolation consider the case $-k \leq l \leq k$. There are exceptional values of l, the values for which $k+l-1/p$ is an integer, $p \neq 2$. Cf. works by J.L. Lions, E. Magenes $[1,2,\dots,8]$, and also M. Schechter [10].

Remark 3.4. In Sects. 4.2 and 4.3, we have proved results on the regularity of the solution in a neighborhood of a smooth boundary. It is of interest to know something about regularity of solutions for non-smooth boundaries. We will prove such results in Chaps. 5, 6 and 7, using a method introduced by the author and based on Rellich's equalities and using weights, and by methods analogous to the methods of E. De Giorgi [1], G. Stampacchia [2], G. Fichera [5].

Finally we mention that in the particular case $k=1$, $l=1$, Ω bounded and in some sense convex, the estimate (4.69) for the homogeneous Dirichlet problem was obtained by J. Kadlec [1].

4.4 Very Weak Solutions of Boundary Value Problems

4.4.1 Very Weak Solutions, the Homogeneous Case

For an elliptic operator of order $2k$ we have defined in 3.2.3 the weak solution for a boundary value problem. We use this notion hereafter. If the solution is in $W^{2k,2}(\Omega)$, we called this solution in 2.8 a strong solution. In the present section, we introduce the notion of a very weak solution for a boundary value problem: it is a distribution from $W^{-l,2}(\Omega)$, $l \geq -k+1$, l an integer.

In this section we use a well known approach based on "duality" or "transposition", as the methods used by M.I. Vishik, S.L. Sobolev [1], E. Magenes, G. Stampacchia [1], J.L. Lions [4], G. Fichera [3–5], etc.

Let us consider a boundary value problem as defined in 3.2.3 and let us assume the sesquilinear form $((v,u))$ is V-elliptic. Let v be the solution of the adjoint problem, cf. 3.2.4, with homogeneous boundary conditions, and with the right hand side $F \in W_0^{m,2}(\Omega)$. Let u be the solution corresponding to $f \in Q'$, with homogeneous boundary conditions. By definition we have $((v,u)) = \langle v, \overline{f} \rangle = \overline{((u,v))^*} = (F,u)$, hence

$$(F,u) = \langle v, \overline{f} \rangle. \tag{4.71}$$

Now we denote by G^* the Green operator from $[L^2(\Omega) \to W^{k,2}(\Omega)]$, associated with the adjoint problem. Then we have:

$$\langle F, \overline{u} \rangle = \langle G^* F, \overline{f} \rangle. \tag{4.72}$$

We assume the hypotheses of Theorem 1.2, i.e. $a_{ij} \in C^{\alpha_i,1}(\Omega)$ with $\alpha_i = \max(0,|i|+m-1)$. Let $\Omega' \subset \overline{\Omega'} \subset \Omega$ be a subdomain of Ω and let us consider the space $P = W^{2k+m,2}(\Omega') \cap Q$, with the topology defined by the norm

$$(|f|^2_{W^{2k+m,2}(\Omega')} + |f|^2_Q)^{1/2}.$$

We assume:

$$P = W^{2k+m,2}(\Omega') \cap Q \quad \text{is a normal space,} \tag{4.73}$$

$$Q' \text{ is dense in } P'. \tag{4.74}$$

Consider a boundary value problem satisfying the hypotheses mentioned, and let be $f \in P'$. A functional $u \in W^{-m,2}(\Omega)$ is called a *very weak solution* of the problem $Au = f$ in Ω with homogeneous boundary conditions if for every $F \in W_0^{m,2}(\Omega)$ we have (4.72).

The following is a consequence of Theorem 1.2:

Theorem 4.1. *There exists a unique very weak solution of the problem mentioned and the Green operator $G \in [Q' \to W^{k,2}(\Omega)]$ can be extended by continuity to a mapping from $[P' \to W^{-m,2}(\Omega)]$.*

4.4.2 Regularity of the Solution

We now describe some properties of very weak solutions:

Theorem 4.2. *Let K be an open set, $K \subset \Omega$, $\overline{K} \cap \overline{\Omega'} = \emptyset$. We keep the hypotheses of the previous theorem, and moreover let us assume $a_{ij} \in C^{k-1+m,1}(\Omega)$. Then $u \in W^{k,2}(K)$ and $|u|_{W^{k,2}(K)} \le c|f|_{P'}$.*

Proof. Let $f_n \in Q'$ be such that $\lim_{n\to\infty} f_n = f$ in P'. Let $\overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega''' \subset \overline{\Omega}''' \subset \Omega$. The very weak solution of the problem corresponding to f_n is a weak solution; then by Theorem 1.3 there exists a constant such that

$$|u_n|_{W^{k,2}(\Omega'''-\overline{\Omega}'')} \le c_1(|f_n|_{W^{-k,2}(\Omega-\overline{\Omega}')} + |u_n|_{W^{-m,2}(\Omega)}); \tag{4.75}$$

starting from this inequality and using Proposition 2.2.4 we get:

$$|u|_{W^{k,2}(\Omega'''-\overline{\Omega}'')} \le c_1(|f|_{W^{-k,2}(\Omega-\overline{\Omega}')} + |u|_{W^{-m,2}(\Omega)}), \tag{4.76}$$

obviously $|f|_{W^{-k,2}(\Omega-\overline{\Omega}')} \le c_2|f|_{P'}$.

If $\overline{K} \subset \Omega$, the theorem is proved. Let us consider the case $\partial K \cap \partial\Omega \ne \emptyset$. Let ψ_n be a function such that $\psi_n \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \Psi \cap \overline{\Omega}' = \emptyset$, $\psi(x) = 1$ in a neighborhood of $\partial\Omega$. Let us consider $u_n\psi$. We have $u_n\psi \in V$, and

$$((u_n\psi, u_n\psi)) = ((u_n|\psi|^2, u_n)) + ((u_n\psi, u_n\psi)) - ((u_n|\psi|^2, u_n)). \tag{4.77}$$

It follows from (4.75) that

$$|((u_n\psi, u_n\psi)) - ((u_n|\psi|^2, u_n))| \le c_3|f_n|_{P'}|u_n\psi|_{W^{k,2}(\Omega)}. \tag{4.78}$$

Moreover we have: $((u_n|\psi|^2, u_n)) = \langle u_n\psi\overline{\psi}, \overline{f_n}\rangle = \langle u_n\psi, \overline{f_n\psi}\rangle$; obviously $|\overline{f_n\psi}|_{Q'} \le c_4|f_n|_{P'}$, hence it follows from (4.77) and (4.78) due to the V-ellipticity of $((v,u))$:

$$|u_n\psi|^2_{W^{k,2}(\Omega)} \le c_5|u_n\psi|_{W^{k,2}(\Omega)}|f_n|_{P'}.$$

Now the results follows if $n \to \infty$. □

Moreover we have

Theorem 4.3. *Let $l \ge 1$ be an integer. We keep the hypotheses of the previous theorem and assume $a_{ij} \in C^{\beta_i,1}(\Omega)$ with $\beta_i = \max(0, |i| + l - k - 1)$. Let K be an open subset of $\forall \bar{K} \subset \Omega$, $\overline{K} \cap \overline{\Omega}' = \emptyset$, and assume $D^\alpha f \in W^{-k+1,2}(K)$ for $|\alpha| \le l-1$. Then if $K' \subset \overline{K}' \subset K$ we obtain:*

$$|u|_{W^{l+k,2}(K')} \le c(|f|_{P'} + \sum_{|\alpha|\le l-1} |D^\alpha f|_{W^{-k+1,2}(K)}).$$

Proof. As a consequence of the previous theorem, $u \in W^{k,2}(K)$. Let $\varphi \in C_0^\infty(K)$; we have:

$$\int_K \sum_{|i|,|j|\le k} \overline{a}_{ij} D^i\varphi D^j\overline{u}\,\mathrm{d}x = \langle \varphi, \overline{f}\rangle,$$

and the result follows from Theorem 1.2. □

4.4.3 Very Weak Solutions (Continuation)

Now we shall define the very weak solution of a boundary value problem with homogeneous boundary conditions for the case $P = Q \cap W^{s,2}(\Omega')$, $k < s < 2k$. We assume (4.73), (4.74), $a_{ij} \in C^{\alpha_i,1}(\Omega)$ with $\alpha_i = \max(0, |i| + s - 2k - 1)$; on P we put the topology associated with the norm $(|f|^2_{Q'} + |f|^2_{W^{s,2}(\Omega')})^{1/2}$.

Let us consider the boundary value problem 3.2.3 with $((v,u))$ V-elliptic. Let $f \in P'$. A function $u \in L^2(\Omega)$ is called a *very weak solution* of the problem $Au = f$ in Ω, with homogeneous boundary conditions, if for every $F \in L^2(\Omega)$ (4.72) holds.

Theorem 4.4. *There exists a unique very weak solution of the given problem and the Green operator $G \in [Q' \to W^{k,2}(\Omega)]$ can be extended by continuity to a mapping $G \in [P' \to L^2(\Omega)]$, and, if $a_{ij} \in C^{k,1}(\Omega)$, to $G \in [P' \to W^{2k-s,2}(\Omega)]$.*

Proof. First of all, by Theorem 1.2, v being the solution of the problem with homogeneous boundary conditions, we have: $A^*v = F$, $F \in L^2(\Omega)$, $v \in W^{s,2}(\Omega')$,

$|v|_{W^{s,2}(\Omega')} \le c_1 |F|_{L^2(\Omega)}$. This implies the existence and uniqueness of a solution in $L^2(\Omega)$. Now we apply Theorem 1.3 with $\kappa = k-1$, and we obtain for every $\Omega^* \subset \overline{\Omega}^* \subset \Omega$,

$$|u|_{W^{2k-s,2}(\Omega^*)} \le c_2(|f|_{P'} + |u|_{L^2(\Omega)}). \tag{4.79}$$

Then we obtain, as in Theorem 4.2, for every $\Omega' \subset \overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega$

$$|u|_{W^{k,2}(\Omega - \overline{\Omega}'')} \le c_3 |f|_{P'}. \tag{4.80}$$

Now the result follows from (4.79) and (4.80). □

By the same argument as in Theorem 4.3, we prove:

Theorem 4.5. *With the same hypotheses as in Theorem 4.4, let be $l \ge 1$, $a_{ij} \in C^{\beta_i,1}(\Omega)$, $\beta_i = \max(0, |i| + l - k - 1)$. Let K be an open set in Ω, $\overline{K} \subset \Omega$, $\overline{K} \cap \overline{\Omega}' = \emptyset$, and for $|\alpha| \le l-1$, $D^\alpha f \in W^{-k+1,2}(K)$. Then for $K' \subset \overline{K}' \subset K$,*

$$|u|_{W^{l+k,2}(K')} \le c(|f|_{P'} + \sum_{|\alpha| \le l-1} |D^\alpha f|_{W^{1-k,2}(K)}).$$

Exercise 4.1. Using the hypotheses given in Theorem 4.1, except the V-ellipticity of $((v,u))$ replaced by the V-ellipticity of $((v,u)) + \lambda(v,u)$, λ sufficiently large, and assuming that 0 is not an eigenvalue, prove Theorem 4.1.

Hint: Begin with the problem $(A+\lambda)u = f$ and then solve the problem $Au = -\lambda u$.

Example 4.1. We assume $k = 1$, $N \ge 3$, $1/q = 1/2 - 1/N$ and set $Q = L^q(\Omega)$, which is possible according to Proposition 3.2.5. Let us set $P = L^q(\Omega) \cap W^{s,2}(\Omega')$, $k+1 \le s$. Let us assume $\Omega' \in \mathfrak{N}^0$. Then P is normal: if $f \in P$, let $f_1 = f$ in $\Omega - \overline{\Omega}'$, $f_1 \equiv 0$ otherwise, $f_2 = f$ in Ω', $f_2 \equiv 0$ otherwise. Now we can find $f_1^n \in C_0^\infty(\Omega)$ with support in $\Omega - \overline{\Omega}'$ such that $\lim_{n\to\infty} f_1^n = f_1$ in $L^q(\Omega)$, hence $\lim_{n\to\infty} f_1^n = f$ in P. Using the same method as in the proof of Theorem 2.3.1 we can find for every $\varepsilon > 0$, $f_2^\varepsilon \in L^2(\Omega'') \cap W^{s,2}(\Omega'')$, $\overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega$ such that

$$|f_2^\varepsilon - f_2|_{L^q(\Omega') \cap W^{s,2}(\Omega')} < \varepsilon, \quad |f_2^\varepsilon|_{L^q(\Omega'' - \Omega')} < \varepsilon.$$

We set $f_2^\varepsilon = 0$ outside of Ω''; using a regularising process we construct $f_{2h}^\varepsilon \in C_0^\infty(\Omega)$ such that for h sufficiently small:

$$|f_2 - f_{2h}^\varepsilon|_{L^q(\Omega') \cap W^{s,2}(\Omega')} < \varepsilon, \quad |f_{2h}^\varepsilon|_{L^q(\Omega'' - \Omega')} < \varepsilon,$$

and the result follows. P as a closed subset of a reflexive space is also reflexive, hence Q' is dense in P'.

4.4.4 The Green Kernel

Let δ_y be the Dirac functional on $C^0(\overline{\Omega})$, $y \in \overline{\Omega}$, defined by $\delta_y v = v(y)$, and for fixed y, let $G(x,y)$ be the solution of the problem $AG = \delta_y$ in Ω with homogeneous boundary conditions; $G(x,y)$ is called *Green's kernel.*

Existence and fundamental properties of the kernel $G(x,y)$ are consequences of the results from 4.1–4.3. If we take into account the partial result given in Theorem 2.3.8 and if we assume $s > 2N$, $\Omega' \in \mathfrak{N}^{0,1}$, then algebraically and topologically $W^{s,2}(\Omega') \subset C^{0,\mu}(\Omega)$, $0 < \mu \leq 1$. Thus we get:

Corollary 4.1. *Let us consider a boundary value problem with $((v,u))$ V-elliptic and let $s = [N/2] + 1$. We have the following assumptions: for $s \leq k$, $a_{ij} \in L^\infty(\Omega)$; for $s > k$, $a_{ij} \in C^{\alpha_i,1}(\Omega)$, $\alpha_i = \max(0, |i| + s - 2k - 1)$. Let $y \in \Omega$, and $\Omega' \subset \overline{\Omega}' \subset \Omega$ a subdomain of Ω such that $y \in \Omega'$ and $\Omega' \in \mathfrak{N}^{0,1}$. Set $P = L^2(\Omega) \cap W^{s,2}(\Omega')$. Then $\delta_y \in P'$ and the Green kernel exists and is unique. We have $G(x,y)$ in $W^{2k-s,2}(\Omega)$ if $s > k$ (if $s < 2k$, we assume $a_{ij} \in C^{k,1}(\Omega)$) and in $W^{k,2}(\Omega)$ if $s \leq k$. Let $z \in \Omega'$. Then we get:*

$$|G(x,y) - G(x,z)|_{W^{2k-s,2}(\Omega)} \leq c|y-z|^\mu, \tag{4.81}$$

where μ is the Hölder exponent introduced by Theorem 2.3.8 for $p = 2$, $k = s$.

Proof. Indeed it is sufficient to prove (4.81); we have:

$$|G(x,y) - G(x,z)|_{W^{2k-s,2}(\Omega)} \leq c(\Omega')|\delta_y - \delta_z|_{P'} \leq c_1(\Omega') \sup_{|v|_{W^{s,2}(\Omega')} \leq 1} |\delta_y v - \delta_z v|$$

$$\leq c_2(\Omega') \sup_{|v|_{C^{0,\mu}(\overline{\Omega}')} \leq 1} |\delta_y v - \delta_z v| \leq c_2(\Omega') \sup_{|v|_{C^{0,\mu}(\overline{\Omega}')} \leq 1} |v(y) - v(z)| \leq c_2(\Omega')|y-z|^\mu.$$

□

We have the regularity result:

Corollary 4.2. *Let K be a subdomain of Ω such that $\overline{K} \cap \{y\} = \emptyset$. Under the hypotheses of Theorem 4.1 or Theorem 4.4 if $s > k$, we have $G(x,y) \in W^{k,2}(K)$ and for all $z \in \Omega'$, $z \notin \overline{K}$,*

$$|G(x,y) - G(x,z)|_{W^{k,2}(K)} \leq c|y-z|^\mu \tag{4.82}$$

holds, where the exponent μ is the same as in Corollary 4.1.

Corollary 4.3. *Let $l \geq 1$. Under the hypotheses of Theorem 4.3 ($s \geq 2k$) or Theorem 4.6 ($s < 2k$), and for K a subdomain of Ω such that $\overline{K} \subset \Omega$, $\overline{K} \cap \overline{\Omega'} = \emptyset$, we have that $G(x,y) \in W^{l+k,2}(K)$ and for $z \in \Omega'$:*

$$|G(x,y) - G(x,z)|_{W^{l+k,2}(K)} \leq c|y-z|^\mu, \tag{4.83}$$

where μ is the same as in Corollary 4.1.

Remark 4.1. It follows from Corollaries 4.2, 4.3 that the Green kernel is a continuous function on $\Omega \times \Omega$, $x \neq y$, if the coefficients a_{ij} of the operator are sufficiently smooth. In this case, we can compute $D_x^\alpha G(x,y)$ and we obtain the same result. But we can prove the existence of $D_y^\beta G(x,y)$ and also of $D_x^\alpha D_y^\beta G(x,y)$ for $x \neq y$, $x, y \in \Omega$, if a_{ij} are sufficiently smooth; this derivative is a continuous function on $\Omega \times \Omega$, $x \neq y$. In particular if $a_{ij} \in C^\infty(\Omega)$, $G(x,y)$ is infinitely differentiable in $\Omega \times \Omega$, $x \neq y$. We consider only the case $(\partial/\partial y_1)G(x,y)$: the general case is left to the reader, cf. L. Schwartz [3].

Lemma 4.1. *Let Ω be a domain such that $y \in \Omega$, and denote $y_h = y + H$, $H = (h,0,\dots,0)$. Then in $(C^1(\overline{\Omega}))'$ there exists* $\lim_{h\to 0}(1/h)(\delta_{y_h} - \delta_y)(v)$ *for $v \in C^1(\overline{\Omega})$ denoted $\partial\delta_y/\partial y_1$. We have $(\partial\delta_y/\partial y_1)(v) = (\partial v)/\partial x_1)(y)$. If $0 < \mu \leq 1$, we have:*

$$\lim_{h\to 0} \frac{1}{h}(\delta_{y_h} - \delta_y) = \frac{\partial \delta_y}{\partial y_1}$$

in the norm of $(C^{1,\mu}(\Omega))'$.

Proof. Let $v \in C^{1,\mu}(\overline{\Omega})$ we have

$$\lim_{h\to 0} \frac{1}{h}(\delta_{y_h} - \delta_y)v = \lim_{h\to 0} \frac{v(y_h) - v(y)}{h} = \frac{\partial v}{\partial y_1}(y).$$

Finally,

$$\sup_{|v|_{C^{1,\mu}(\overline{\Omega})}\leq 1} \frac{1}{h}(v(y_h) - v(y)) - \frac{\partial v}{\partial y_1}(y) \leq c|y_h - y|^\mu. \tag{4.84}$$

□

Corollary 4.4. *Let us consider a boundary value problem with $((v,u))$ V-elliptic, and assume that $a_{ij} \in C^\infty(\Omega)$, y, y_h as in Lemma 4.1. Then* $\lim_{h\to 0}(1/h)(G(x,y_h) - G(x,y))$ *exists in $W^{2k-s,2}(\Omega)$, where $s = [N/2] + 2$ if $s > k$, and in $W^{k,2}(\Omega)$, if $s \leq k$. Let us denote this limit by $\partial G/\partial y_1(x,y)$, which is a solution of the problem $Au = \partial\delta_y/\partial y_1$ in Ω with homogeneous boundary conditions. $\partial G/\partial y_1(x,y) \in W^{k,2}(\Omega - \Omega')$, where $\Omega' \subset \overline{\Omega}' \subset \Omega, y \in \Omega'$. $D_x^\alpha(\partial G/\partial y_1)(x,y)$ is a continuous function in $\Omega \times \Omega$, $x \neq y$, with $|\alpha| \geq 0$. If $s > k$, we again have:*

$$\left| \frac{1}{h}(G(x,y_h) - G(x,y)) - \frac{\partial G}{\partial y_1}(x,y) \right|_{W^{2k-s,2}(\Omega')} \leq C(\Omega')|h|^\mu, \tag{4.85}$$

and the inequalities corresponding to (4.82), (4.83) with μ as in Theorem 2.3.8 for $p = 2, k = s - 1$.

Proof: We set $P = L^2(\Omega) \cap W^{s,2}(\Omega')$. By Lemma 4.1, we have in P':

$$\lim_{h\to 0} \frac{1}{h}(\delta_{y_h} - \delta_y) = \frac{\partial \delta}{\partial y_1},$$

hence (4.84) implies (4.85); the other properties follow from the previous theorems. □

Exercise 4.2. Consider $D_y^\beta G(x,y)$ and prove the same result as in Corollary 4.4 for $s = [N/2] + 1 + |\beta|$.

4.4.5 The Green Operator and the Green Kernel

Now, we consider the relation between the Green operator and the Green kernel.

Theorem 4.6. *Under the same hypotheses as in Corollary 4.4, let $Q = L^2(\Omega)$ and $G \in [L^2(\Omega) \to W^{k,2}(\Omega)]$ be the Green operator. Let $\varphi \in C_0^\infty(\Omega)$ and let us define*

$$\int_\Omega G(x,y)\varphi(y)\,\mathrm{d}y = \lim_{h\to 0} \frac{1}{h^N} \sum_{i=1}^{l} G(x,y_i)\varphi(y_i), \tag{4.86}$$

in $W^{2k-s,2}(\Omega)$, if $s > k$, and in $W^{k,2}(\Omega)$, if $s \le k$, with $s = [N/2] + 1$. (The right hand side of (4.86) is a Riemann sum, the lattice is given by cubes of side h, y_i is a point in the corresponding cube). Then

$$\int_\Omega G(x,y)\varphi(y)\,\mathrm{d}y$$

exists, and

$$\int_\Omega G(x,y)\varphi(y)dy = G\varphi. \tag{4.87}$$

Proof. First of all let us consider the case $s \ge 2k$, and let $\psi \in C_0^\infty(\Omega)$. It is sufficient to prove:

$$\lim_{h\to 0}\left(\sup_{|\psi|_{W_0^{s-2k,2}(\Omega)}\le 1} \left| \frac{1}{h^N}\sum_{i=1}^{l} \langle G(x,y_i)\varphi(y_i),\psi\rangle - \langle G\varphi,\psi\rangle \right| \right) = 0. \tag{4.88}$$

From (4.72), it follows:

$$\frac{1}{h^N}\sum_{i=1}^{l} \langle G(x,y_i)\varphi(y_i),\psi\rangle = \frac{1}{h^N}\sum_{i=1}^{l} \varphi(y_i)(\overline{G^*\overline{\psi}})(y_i). \tag{4.89}$$

Indeed: Theorem 1.2 implies that for $\Omega' \subset \overline{\Omega}' \subset \Omega$,

$$|G^* \psi|_{W^{s,2}(\Omega')} \leq c_1(\Omega')|\psi|_{W^{s-2k,2}(\Omega)},$$

and applying Theorem 2.3.8, we obtain:

$$|G^* \psi|_{C^{0,\mu}(\overline{\Omega}')} \leq c_2(\Omega')|\psi|_{W^{s-2k,2}(\Omega)}.$$

We have $\langle G\varphi, \psi\rangle = \langle \varphi, \overline{G^*\overline{\psi}}\rangle$ and (4.89) follows.

Now let us consider the case $s < 2k$. We must prove:

$$\lim_{h\to 0} \left| \frac{1}{h^N} \sum_{i=1}^{l} G(x, y_i)\varphi(y_i) - G\varphi \right|_{W^{2k-s,2}(\Omega)} = 0 \quad \text{if } s > k,$$

and

$$\lim_{h\to 0} \left| \frac{1}{h^N} \sum_{i=1}^{l} G(x, y_i)\varphi(y_i) - G\varphi \right|_{W^{k,2}(\Omega)} = 0 \quad \text{if } s \leq k.$$

We know that

$$\frac{1}{h^N} \sum_{i=1}^{l} G(x, y_i)\varphi(y_i)$$

is the very weak solution of the problem with homogeneous boundary conditions and with the right hand side

$$\frac{1}{h^N} \sum_{i=1}^{l} \varphi(y_i)\delta_{y_i}.$$

Now let us set $Q = L^2(\Omega) \cap W^{s,2}(\Omega')$, where $\overline{\Omega'} \subset \Omega$, supp $\varphi \subset \Omega'$. Then we have in Q':

$$\lim_{h\to 0} \frac{1}{h^N} \sum_{i=1}^{l} \varphi(y_i)\delta_{y_i} = \varphi,$$

hence (4.88) follows for $s < 2k$. □

From our definition we have $G(x,y) \in W^{-m,2}(\Omega)$ with $m \geq -k$ for y fixed; then $G(x,y) \in \mathscr{D}'(\Omega)$. If we fix $x \in \Omega$, we can extend $G(x,y)$ in $\mathscr{D}'(\Omega)$ using Theorem 4.6 and if we define, by (4.86):

$$\int_{\Omega} G(x,y)\varphi(y)dy = \langle \varphi, G(x,.)\rangle.$$

Starting from this we have

Theorem 4.7. *For a boundary value problem 3.2.1 with $((v,u))$ V-elliptic and $a_{ij} \in C^\infty(\Omega)$, denote $G^*(x,y)$ the Green kernel for the adjoint problem. Then, for y fixed in Ω, $\overline{G^*(x,y)} = G(y,x)$ in $\mathscr{D}'(\Omega)$. If $((v,u))$ is a hermitian form, we have in this* *case:* $\overline{G(x,y)} = G(y,x)$*;* $G^*(x,y)$*,* $G(x,y)$ *are continuous in* $\Omega \times \Omega$*,* $x \neq y$ *(in fact infinitely continuously differentiable with respect to x and y) and* $\overline{G^*(x,y)} = G(y,x)$*.*

Proof. For $\varphi \in C_0^\infty(\Omega)$, y fixed in Ω:

$$\langle \varphi, G(y,x) \rangle = (G\varphi)(y) = \delta_y(G\varphi) = \langle G\varphi, \delta_y \rangle = \langle \varphi, \overline{G^*(x,y)} \rangle. \tag{4.90}$$

□

We can ask: If $a_{ij} \in C^\infty(\Omega)$, y fixed, does the Green kernel have a singularity for $x = y$ of the same type as the elementary solution? It is true, but we consider only the following example:

Example 4.2. Let us assume $N = 3$, $\Omega \in \mathfrak{N}^{0,1}$. Let us set for $x,y \in \Omega$, $x \neq y$, $E(x,y) = -(1/4\pi)|x-y|^{-1}$. It is easy to see, that in the sense of distributions $\triangle E(x,y) = \delta_y$. Let y be fixed, let $F(x,y)$ be the function in $W^{1,2}(\Omega)$, which solves the Dirichlet problem $\triangle F = 0$ in Ω, $F = E$ on $\partial\Omega$ in the sense of traces. Clearly $G(x,y) = E(x,y) - F(x,y)$; but $F \in C^\infty(\Omega)$, thus the singularity of E coincides with the singularity of G. From Corollary 4.4, we have for y fixed, $G(x,y) \in L^2(\Omega)$. But $F(x,y) \in W^{1,2}(\Omega)$ and $E(x,y) \in W^{1,p}(\Omega)$, $p < 3/2$, hence $G(x,y) \in W^{1,p}(\Omega)$, and by Theorem 2.3.4, $G(x,y) \in L^q(\Omega), q < 3$, etc.

4.5 Very Weak Solutions (Continuation)

In this section we shall use the results obtained in Sect. 4.2; of course we shall assume $\partial\Omega$ sufficiently smooth.

4.5.1 Very Weak Solutions, the Nonhomogeneous Case

We consider the boundary value problem 3.2.1 and assume the sesquilinear form $((v,u))$ V-elliptic. Let $F \in W_0^{m,2}(\Omega)$, m a nonnegative integer, and v the solution of the adjoint problem $A^*v = F$ in Ω with homogeneous boundary conditions. We assume for this problem $(l+k)$-regularity with $l = k+m$, $m \geq 0$. Now let u be the weak solution of the problem: $Au = f$ in Ω, $f \in Q'$, $B_s u = B_s u_0 = g_{j_s}$, $C_t u = g_{i_t}$ on $\partial\Omega$. By definition, we have:

$$((v,u)) = \langle v, \overline{f} \rangle + \overline{g}v. \tag{4.91}$$

On the other hand, by integration by parts, here completely justified, using Theorem 2.2, we obtain:

$$((u,v))^* = \int_\Omega u\overline{F}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{i=1}^{k-1} \frac{\partial^i u}{\partial n^i} \overline{N_i v}\,\mathrm{d}S. \tag{4.91 bis}$$

On $\partial\Omega$ we have for $s = 1,2,\ldots,\mu$: $B_s u = \partial^{j_s} u/\partial n^{j_s} - F_s u = g_{j_s}$, thus it follows from (4.91 bis):

$$((u,v))^* = \int_\Omega u\overline{F}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{s=1}^{\mu} g_{j_s}\overline{N_{j_s} v}\,\mathrm{d}S + \int_{\partial\Omega} \sum_{t=1}^{k-\mu} \frac{\partial^{i_t} u}{\partial n^{i_t}} \overline{M_{i_t} v}\,\mathrm{d}S.$$

As in Theorem 2.3 we get that $M_{i_t} v = 0$ on $\partial\Omega$ for $t = 1,2,\ldots,k-\mu$, hence we obtain

$$(F,u) = \langle v,\overline{f}\rangle + \overline{g}v - \int_{\partial\Omega} \sum_{s=1}^{\mu} N_{j_s} v\overline{g_{j_s}}\,\mathrm{d}S. \tag{4.92}$$

We define a *very weak solution with nonhomogeneous boundary conditions* as follows: Consider Ω, $((v,u))$ V-elliptic, the problem satisfying the $(k+m)$-regularity. Let Q be a normal space, $V \subset Q$ algebraically and topologically, and P another normal space such that $V \cap W^{2k+m,2}(\Omega) \subset P \subset Q$ algebraically and topologically. We assume Q' dense in P'. (It holds for P reflexive.) Let us consider $f \in P'$, $g_{j_s} \in W^{-j_s-1/2-m,2}(\partial\Omega)$, $s = 1,2,\ldots,\mu$, $g_{i_t} \in W^{i_t+1/2-2k-m,2}(\partial\Omega)$, $t = 1,2,\ldots,k-\mu$. A functional $u \in W^{-m,2}(\Omega)$ is a *very weak solution* of the boundary value problem $Au = f$ in Ω, $B_s u = g_{j_s}$, $C_t u = g_{i_t}$ on $\partial\Omega$, if for every solution v of the adjoint problem $A^* v = F$ in Ω, $F \in W_0^{m,2}(\Omega)$, $B_s v = 0$, $C_t^* v = 0$ on $\partial\Omega$, we have:

$$\langle F,\overline{u}\rangle = \langle v,\overline{f}\rangle - \sum_{s=1}^{\mu} \langle N_{j_s} v, \overline{g}_{j_s}\rangle_{\partial\Omega} + \sum_{t=1}^{k-\mu} \left\langle \frac{\partial^{i_t} v}{\partial n^{i_t}}, \overline{g}_{i_t} \right\rangle_{\partial\Omega}. \tag{4.93}$$

Theorem 5.1. *There exists a unique very weak solution of the problem and the Green operator:*

$$G \in [Q' \times W^{k-j_1-1/2,2}(\partial\Omega) \times \ldots \times W^{k-j_\mu-1/2,2}(\partial\Omega)$$
$$\times W^{i_1+1/2-k,2}(\partial\Omega) \times \ldots \times W^{i_{k-\mu}+1/2-k,2}(\partial\Omega) \to W^{k,2}(\Omega)],$$

can be extended to an operator from

$$[P' \times W^{-m-j_1-1/2,2}(\partial\Omega) \times \cdots \times W^{-m-j_\mu-1/2,2}(\partial\Omega)$$
$$\times W^{i_1+1/2-2k-m,2}(\partial\Omega) \times \ldots \times W^{i_{k-\mu}+1/2-2k-m,2}(\partial\Omega) \to W^{-m,2}(\Omega)].$$

Proof. The result is a direct consequence of (4.93) and Theorem 2.3 and of the property of density of

$$Q' \times W^{k-j_1-1/2,2}(\partial\Omega) \times \ldots \times W^{k-j_\mu-1/2,2}(\partial\Omega)$$
$$\times W^{i_1+1/2-k,2}(\partial\Omega) \times \ldots \times W^{i_{k-\mu}+1/2-k,2}(\partial\Omega)$$

in

$$P' \times W^{-m-j_1-1/2,2}(\partial\Omega) \times \ldots \times W^{-m-j_\mu-1/2,2}(\partial\Omega)$$
$$\times W^{i_1+1/2-2k-m,2}(\partial\Omega) \times \ldots \times W^{i_{k-\mu}+1/2-2k-m,2}(\partial\Omega).$$

□

Moreover, using Theorem 1.3, if f is locally regular, we obtain theorems about the regularity in Ω.

Exercise 5.1. Let $\Omega' \subset \overline{\Omega}' \subset \Omega'' \subset \overline{\Omega}'' \subset \Omega$, $f \in W^{s,2}(\Omega'')$, $s \geq 0$ an integer. Let us assume $a_{ij} \in C^\infty(\Omega)$. Then the very weak solution u of problem mentioned belongs to $W^{2k+s,2}(\Omega')$, and we have the estimate:

$$|u|_{W^{2k+s,2}(\Omega')} \leq c(|f|_{W^{s,2}(\Omega'')} + |u|_{W^{-m,2}(\Omega)}).$$

Remark 5.1. Moreover one can prove, cf. E. Magenes [3], a "trace theorem": let $D(P')$ be the subspace of $L^2(\Omega)$ of functions u such that $Au \in P'$ equipped with the graph norm $(|u|^2_{L^2(\Omega)} + |Au|^2_{P})^{1/2}$ ($m = 0$, $\partial\Omega, a_{ij}$ sufficiently smooth). Let us assume $C^\infty(\overline{\Omega})$ dense in P'. The mapping $[C^\infty(\overline{\Omega}) \to [C^\infty(\partial\Omega)]^k]$ defined by

$$\left[B_1, B_2, \ldots B_\mu, C_1, C_2, \ldots C_{k-\mu}\right],$$

can be extended by continuity onto the mapping:

$$[D(P') \to W^{-j_1-1/2,2}(\partial\Omega) \times \cdots \times W^{-j_\mu-1/2,2}(\partial\Omega)$$
$$\times W^{i_1+1/2-2k,2}(\partial\Omega) \times \ldots \times W^{i_{k-\mu}+1/2-2k,2}(\partial\Omega)].$$

Cf. also L. Hörmander [6]. Taking this point of view we can say as in Theorem 2.3, that the operator $\left(A, B_1, B_2, \ldots B_\mu, C_1, C_2, \ldots C_{k-\mu}\right)$ is an isomorphism of $D(P')$ onto $W^{-j_1-1/2,2}(\partial\Omega) \times \ldots$; the same result holds for $m > 0$.

Example 5.1. Let us assume $k = 1, N \geq 3$. Let us take $Q = L^q(\Omega)$, $1/q = 1/2 - 1/N$, let us consider $m > N/2 - 2$, $P = L^{q_1}(\Omega)$ with arbitrary $q_1 < \infty$. This works for instance in the case of the Neumann problem. In the case of the Dirichlet problem we must have $W^{2+m,2}(\Omega) \cap W_0^{1,2}(\Omega) \subset P$. If $m > N/2 - 1$, we have for every $q \geq 2$: $W^{2+m,2}(\Omega) \subset W^{1,q}(\Omega)$ and due to Theorem 2.4.10 $W^{1,q}(\Omega) \cap W_0^{1,2}(\Omega) = W_0^{1,q}(\Omega)$. We can take $P = W_0^{1,q}(\Omega)$.

Example 5.2. We are looking for the solution of the problem $\triangle u = 0$ in Ω, $u = g$ on $\partial\Omega$, with $g \in L^2(\partial\Omega)$. We have $g \in W^{-1/2,2}(\partial\Omega)$, we take $m = 0$ and obtain a unique solution in $L^2(\Omega)$. We get again a result given in G. Cimmino [1]. It follows from Theorem 2.5.5 for $N \geq 3$ for instance that $L^p(\partial\Omega) \subset W^{-1/2,2}(\partial\Omega)$, $p = 2 - 2/N$. Hence we can take $g \in L^p(\partial\Omega)$, etc.

Remark 5.2. Every distribution on $\partial\Omega$ ($\Omega \in \mathfrak{N}^\infty$) is in $W^{-\lambda+1/2,2}(\partial\Omega)$ with λ sufficiently large. Theorem 5.1 guaranties the existence of the solution for the boundary value problem with any given distributions on $\partial\Omega : g_{j_s}, g_{i_t}$. In this case, by Theorem 1.3, we have $u \in C^\infty(\Omega)$ if $f \in C^\infty(\Omega)$ ($a_{ij} \in C^\infty(\Omega)$).

In a series of papers by J.L. Lions, E. Magenes [1–8], more precise results have been obtained: the solution is found in $W^{k,p}(\Omega)$, $p > 1$, k any real number. They use the estimates (4.70), the dual process and interpolation methods (the details can be found in papers by E. Magenes [3] or [4] and references in these papers). We obtain Theorem 5.1 for any real m and $p > 1$ (i.e. we work in spaces $W^{-m-j_1-1/p,p}(\partial\Omega)\dots$). For $p \neq 2$, there are some exceptions; we must eliminate the cases where $m - 1/p$ is an integer; in these cases the results are weaker.

4.5.2 Very Weak Solutions, the Nonhomogeneous Case (Continuation)

Let us consider a boundary value problem as in 4.5.1 and let us assume that the conditions of $2k$-regularity are satisfied, cf. 4.2.1. Let $k < n < 2k$, Q, P two normal spaces such that $V \cap W^{n,2}(\Omega) \subset P \subset Q$ algebraically and topologically. Let us assume that Q' is dense in P' (for reflexive P it holds). Let

$$f \in P', \quad g_{j_s} \in W^{2k-j_s-1/2-n,2}(\partial\Omega), \quad s = 1,2,\dots,\mu,$$
$$g_{i_t} \in W^{i_t+1/2-n,2}(\partial\Omega), \quad t = 1,2,\dots,k-\mu.$$

A function $u \in L^2(\Omega)$ is a *very weak solution* of the boundary value problem $Au = f$ in Ω, $B_s u = g_{js}$, $s = 1,2,\dots,\mu$, $C_t u = g_{i_t}$, $t = 1,2,\dots,k-\mu$, on $\partial\Omega$, if for each solution v of the adjoint problem $A^*v = F$ in Ω, $B_v v = 0$, $s = 1,2,\dots,\mu$, $C_t^* v = 0$, $t = 1,2,\dots,k-\mu$, on $\partial\Omega$, with $F \in L^2(\Omega)$, we have (4.93).

For this given problem we intend to find the solution $u \in W^{2k-n,2}(\Omega)$; existence and uniqueness were proved in Theorem 5.1. To prove that the solution is in $W^{2k-n,2}(\Omega)$ we shall assume:

$$B_s, C_t^* \text{ are a canonical normal system.} \tag{4.94}$$

Theorem 5.2. *Let us assume (for simplicity) $\partial\Omega$, a_{ij}, $b_{s\alpha}$, $h_{s\alpha}$ infinitely continuously differentiable and (4.94). Then the Green operator, corresponding to the given problem as:*

$$G \in [Q' \times W^{k-j_1-1/2,2}(\partial\Omega) \times \cdots \times W^{k-j_\mu-1/2,2}(\partial\Omega)$$
$$\times W^{i_1-k+1/2,2}(\partial\Omega) \times \cdots \times W^{i_{k-\mu}+1/2-k,2}(\partial\Omega) \to W^{k,2}(\Omega)]$$

can be extended by continuity to

$$G \in [P' \times W^{2k-j_1-1/2-n,2}(\partial\Omega) \times \dots \times W^{2k-j_\mu-1/2-n,2}(\partial\Omega) \times \dots$$
$$\times W^{i_1+1/2-n,2}(\partial\Omega) \times \dots \times W^{i_{k-\mu}+1/2-n,2}(\partial\Omega) \to W^{2k-n,2}(\Omega)],$$
$$when \quad k < n < 2k.$$

Proof. The solution of this problem is very weak belonging to $L^2(\Omega)$. Let $\omega \in W^{2k,2}(\Omega)$ be a function such that $B_s\omega = 0$, $s = 1,2,\dots,\mu_s$, $C'_t\omega = 0$, $t = 0,1,2,....,k-\mu$, on $\partial\Omega$. It results from (4.93) that for u we have:

$$(u, A^*\omega) = \langle\overline{\omega}, f\rangle + \sum_{t=1}^{k-\mu}\left\langle\frac{\partial^{i_t}\omega}{\partial n^{i_t}}, g_{i_t}\right\rangle_{\partial\Omega} - \sum_{s=1}^{\mu}\langle\overline{N_{j_s}\omega}, g_{j_s}\rangle_{\partial\Omega}. \tag{4.95}$$

Let $w \in W^{2k,2}(\Omega)$ be the solution of the problem $A^*w = u$ in Ω, $B_sw = 0$, $C^*_tw = 0$ on $\partial\Omega$. This solution exists and is unique by Theorem 2.2. Using (4.95) it follows that

$$(A^*\omega, A^*w) = <\omega, \overline{f}> + \sum_{t=1}^{k-\mu}\left\langle\frac{\partial^{i_t}\omega}{\partial n^{i_t}}, g_{i_t}\right\rangle_{\partial\Omega} - \sum_{s=1}^{\mu} < \overline{N_{j_s}\omega}, g_{j_s} >_{\partial\Omega}. \tag{4.96}$$

The differential operators N_{j_s} are of orders not bigger than $2k-1-j_s$. We can find μ indices, say $i_{k-\mu+1}, i_{k-\mu+2}, \dots, i_k$, such that with complementary indices $j_1, j_2, \dots, j_\mu, j_{\mu+1}, \dots, j_k$ to the set $\{0,1,2,\dots,2k-1\}$, the conditions $B_s\omega = 0$, $C^*_t\omega = 0$ can be written in local coordinates as

$$\frac{\partial^{j_s}\omega}{\partial t^{j_s}} = \sum_{|\alpha|\le j_s} h_{s\alpha}\frac{\partial^{|\alpha|}\omega}{\partial\sigma_1^{\alpha_1}\dots\partial\sigma_{N-1}^{\alpha_{N-1}}\partial t^{\alpha_N}}, \tag{4.97}$$

with $\alpha_N = i_t$ for t well chosen, $t = 1,2,\dots,k$. Using (4.96) in N_{j_s} to compute $\partial^{j_s}\omega/\partial n^{j_s}$ $(n = -t)$ and after an integration by parts over $\partial\Omega$, we get:

$$\sum_{s=1}^{\mu} < \overline{N_{j_s}\omega}, g_{j_s} > = \sum_{t=1}^{k}\left\langle\overline{\frac{\partial^{i_t}\omega}{\partial n^{i_t}}}, e_t\right\rangle,$$

where $e_t \in W^{-n+i_t+1/2,2}(\partial\Omega,)$ and

$$\sum_{t=1}^{k}|e_t|_{W^{-n+i_t+1/2,2}(\partial\Omega)} \le c_1\sum_{s=1}^{\mu}|g_{j_s}|_{W^{2k-j_s-1/2-n,2}(\partial\Omega)}. \tag{4.98}$$

If we denote $W = \{v \in W^{2k,2}(\Omega),\ B_sv = 0,\ C^*_tv = 0 \text{ on } \partial\Omega\}$, the sesquilinear form $(A^*\omega, A^*w)$, $\omega, w \in W$, is W-elliptic, according to Theorem 2.2. We can use

Theorem 2.2 to solve the problem $AA^*w = f$ in Ω, $w \in W$, with non stable, non homogeneous boundary conditions, given in (4.96). All conditions of $(4k-n)$-regularity are satisfied and by (4.97), we have:

$$|w|_{W^{4k-n,2}(\Omega)} \le c_2\big(|f|_{P'} + \sum_{s=1}^{\mu} |g_{j_s}|_{W^{2k-j_s-1/2-n,2}(\partial\Omega)} + \sum_{t=1}^{k-\mu} |g_{i_t}|_{W^{i_t-n+1/2,2}(\partial\Omega)}\big).$$

□

Example 5.3. Let Ω be a domain in $\mathbb{R}^2$ with $\partial\Omega$ sufficiently smooth, $f \in W^{1,2}(\partial\Omega)$, $g \in L^2(\partial\Omega)$ and let us find the very weak solution of the problem $\triangle^2 u = 0$, in Ω; $u = f$, $\frac{\partial u}{\partial n} = g$ on $\partial\Omega$. We have $f \in W^{1/2,2}(\partial\Omega)$, $g \in W^{-1/2,2}(\partial\Omega)$, hence a unique solution u exists and belongs to $W^{1,2}(\Omega)$. It follows from results of J.L. Lions, E. Magenes [2], that $u \in W^{3/2,2}(\Omega)$.

Example 5.4. Let

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + 1,$$

with the followings hypotheses: $\Omega \in \mathfrak{N}^\infty$, a_{ij} real functions, $a_{ij} \in C^\infty(\overline{\Omega})$,

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \ge c\sum_{i=1}^{N} \xi_i^2.$$

For $g \in W^{-1/2,2}(\partial\Omega)$ there exists a unique solution of the problem $Au = 0$ in Ω,

$$\sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_j} n_i = g \text{ on } \partial\Omega.$$

Due to Theorems 2.2, 5.1, 5.2, if $g \in W^{-1/2+n,2}(\partial\Omega)$, n an integer ($n < 0$ is possible) then a unique solution exists in $W^{1+n,2}(\Omega)$.

Example 5.5. Let us consider

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

with the same hypotheses as in the previous example. Let us assume $g \in W^{-1/2+n,2}(\partial\Omega)$, $n \ge -1$, $\langle 1, g\rangle_{\partial\Omega} = 0$. Let u_1 be the solution of the problem from Example 5.4, $n \ge -1$. We have $\langle 1, u_1\rangle = 0$; indeed, by (4.93) for $F \equiv 1$, we have $v = 1$, $\langle 1, u_1\rangle = \langle 1, g\rangle_{\partial\Omega} = 0$. Then, using 3.2.1 we find $u_2 \in W^{1,2}(\Omega)$ such that $Au_2 = u_1$ in Ω,

$$\sum_{i,j=1}^{N} a_{ij}\frac{\partial u_2}{\partial x_j} n_i = 0 \text{ on } \partial\Omega.$$

We have $u_1 \in W^{1+n,2}(\Omega)$, which implies $u_2 \in W^{3+n,2}(\Omega)$ and $u = u_1 + u_2$ solves the problem $Au = 0$ in Ω,

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_j} n_i = g \text{ on } \partial\Omega.$$

Exercise 5.2. Keep the hypotheses of Example 5.5 except for $n \geq -1$; take an integer $n < -1$. Prove the existence of a very weak solution. To prove it, we consider the adjoint problem $A^* v = F$ in $\Omega, F \in W_0^{-n-1,2}(\Omega), (1,F) = 0$, and

$$\frac{\partial v}{\partial \nu^*} = \sum_{i,j=1}^{N} \overline{a}_{ji} \frac{\partial v}{\partial x_j} n_i = 0 \text{ on } \partial\Omega.$$

Let $M = \{v \in W_0^{-n-1,2}(\Omega),\ (1,v) = 0\}$. Prove the isomorphism of M' and the quotient space $W^{1+n,2}(\Omega)/\text{const}$.

4.5.3 The Green and Poisson Kernels

We can refine the theorems concerning the Green kernel when we replace Theorem 1.2 by Theorem 2.2.

Corollary 5.1. *Let us consider $s = [N/2] + 1$, $\Omega \in \mathfrak{N}^\infty$, a_{ij}, $b_{i\alpha}$, $h_{s\alpha}$ infinitely continuously differentiable, $((v,u))$ V-elliptic. Let us fix $y \in \Omega$. Then $G(x,y) \in W^{2k-s,2}(\Omega)$ is a continuous function on $\Omega \times \Omega$, $x \neq y$ (indeed infinitely continuously differentiable) and if $y,z \in \overline{\Omega}$, we have:*

$$|G(x,y) - G(x,z)|_{W^{2k-s,2}(\Omega)} \leq c|y-z|^\mu$$

(cf. Corollary 4.1).

Let us keep the hypotheses given in Corollary 5.1, and let $u \in C^\infty(\overline{\Omega})$ be the solution of the problem: $Au = 0$ in Ω, $B_s u = g_{j_s}$, $C_t u = g_{i_t}$ on $\partial\Omega$, $g_{j_s}, g_{i_t} \in C^\infty(\partial\Omega)$, and φ_n such that $\lim_{n\to\infty} \varphi_n = \delta_y$, $y \in \Omega$, the convergence taking place in $(C(\overline{\Omega}))'$ and v_n the solutions of $A^* v_n = \varphi_n$ in Ω, $B_s v_n = 0, C_t^* v_n = 0$ on $\partial\Omega$. By (4.93) we have:

$$(u, \varphi_n) = \sum_{t=1}^{k-\mu} \left\langle \overline{\frac{\partial^{i_\tau} v_n}{\partial n^{i_\tau}}}, g_{i_t} \right\rangle_{\partial\Omega} - \sum_{s=1}^{\mu} \langle \overline{N_{j_s} v_n}, g_{j_s} \rangle_{\partial\Omega}. \tag{4.99}$$

Let $K \subset \Omega$ be a domain such that $\text{dist}(K,y) > 0$. According to Theorem 1.3,

$$\lim_{n\to\infty} v_n = G^*(x,y)$$

in $W^{k+l,2}(K)$, with l any real number, thus $G^*(x,y) \in W^{k+l,2}(K)$, and by (4.99) it follows (y fixed):

$$\begin{aligned} u(y) &= \sum_{t=1}^{k-\mu} \left\langle \overline{\frac{\partial^{i_t} G^*}{\partial n^{i_t}}}, g_{i_t} \right\rangle_{\partial\Omega} - \sum_{s=1}^{\mu} \langle \overline{N_{j_s} G^*}, g_{j_s} \rangle_{\partial\Omega} \\ &= \sum_{t=1}^{k-\mu} \int_{\partial\Omega} \overline{\frac{\partial^{i_t} G^*(x,y)}{\partial n^{i_t}}} g_{i_t} \, \mathrm{d}S_x - \sum_{s=1}^{\mu} \int_{\partial\Omega} \overline{N_{j_s} G^*(x,y)} g_{j_s} \, \mathrm{d}S_x. \end{aligned} \tag{4.100}$$

We obtain the so called *Poisson kernels*: $\partial^{i_t} G^*(x,y)/\partial n^{i_t}$, $\overline{N_{j_s} G^*(x,y)}$.

If we use the results obtained in Theorems 5.1, 5.2 and construct the Green kernel setting $P = V \cap W^{s,2}(\Omega)$ with $s = [N/2]+1$, $f = \delta_y$, we obtain regularity results for $G(x,y)$ and then for the Poisson kernels in $\overline{\Omega}$. If we consider a particular Poisson kernel, for its existence, it is sufficient that the condition of $\max(s,\kappa)$-regularity of the problem holds for $\kappa = i_t + 1$ or $j_s + 1$.

Remark 5.3. We define the *Dirac functional* $\delta_{\partial\Omega,y}$ *on* $C^0(\partial\Omega)$ by $\delta_{\partial\Omega,y} v = v(y)$. It is easy to see (proved in Exercise 5.3) that $\partial^{i_t} G^*(x,y)/\partial n^{i_t}$ and $\overline{N_{j_s} G^*(x,y)}$ are, for fixed x, the solutions of the problem $A^* u = 0$ in $\Omega, C^*_{t_1} u = \delta_{\partial\Omega,x}$, $B_s u = 0$, $C^*_t u = 0$, $t \neq t_1$, and respectively $B_{s_1} u = \delta_{\partial\Omega,x}$, $C^*_t u = 0$, $B_s u = 0$, $s \neq s_1$.

Example 5.6. Let $\Omega = K(1)$ be the unit disc and let us try to find the Poisson kernel $P(y,x)$ for the Laplacian $\triangle$ with the Dirichlet condition. For this we must take $l = 3$. For a fixed x, we have $P(y,x) \in W^{-1,2}(\Omega), P(y,x) \in C^\infty(\Omega)$,

$$P(y,x) = (1/2\pi)(1-|y|^2)[1+|y|^2 - 2(y,x)]^{-1}.$$

Indeed: let $\varphi \in C_0^\infty(\Omega)$, $-\triangle v = \varphi$ in Ω, $v = 0$ on $\partial\Omega$; we have:

$$\begin{aligned} \frac{1}{2\pi} \int_{K(1)} \frac{1-|y|^2}{1+|y|^2-2(y,x)} \varphi(y) \, \mathrm{d}y &= -\lim_{\rho\to 1} \frac{1}{2\pi} \int_{K(\rho)} \frac{1-|y|^2}{1+|y|^2-2(y,x)} \triangle v(y) \, \mathrm{d}y \\ &= -\lim_{\rho\to 1} \frac{1}{2\pi} \int_{\partial K(\rho)} \frac{1-|y|^2}{1+|y|^2-2(y,x)} \frac{\partial v}{\partial n}(y) \, \mathrm{d}S_y \\ &= -\lim_{\rho\to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-\rho^2)\rho}{1+|\rho|^2 - 2\rho^2 \cos(\alpha-\beta)} \frac{\partial v}{\partial \rho}(\rho,\alpha) \, \mathrm{d}\alpha \end{aligned}$$

(we use polar coordinates). It is easy to see that on $\partial K(1)$:

$$\lim_{\rho\to 1} \frac{1}{2\pi} \frac{(1-\rho^2)\rho}{1+|\rho|^2 - 2\rho^2\cos(\alpha-\beta)} = \delta_\beta.$$

Chapter 5
Applications of Rellich's Equalities and Their Generalizations to Boundary Value Problems

The results obtained in Chap. 4 dealing with the existence of very weak solutions of non-homogeneous boundary value problems are based on regularity theorems, which are true only if $\partial\Omega$, the boundary of the domain Ω, is smooth enough; if this is not the case, then the corresponding estimates do not hold. To eliminate this hypothesis, it turns out that the Rellich equality is very useful; F. Rellich proved this type of equality for the Laplace operator in [1], and a generalization of this equality for second-order operators was given by Hörmander [5]. A technique is extended to the second order systems, which can be found in L.E. Payne, H.F. Weinberger [2]. All results are of the type of *a priori* estimates which must be proved for real solutions. Using these results and proving a new Rellich equality for operators of fourth order, the author has obtained various regularity theorems and, by the dual method, also existence theorems; cf. J. Nečas [1–4, 6, 8, 13]. A generalization of theses results for the second order Dirichlet problem was obtained by J. Kadlec [1].

For connected questions cf. also G. Adler [1–3], P. Doktor [1], [2], G. Fichera [5, 8, 9], E. Magenes [5], L.G. Magnaradze [1], C. Miranda [2], J. Nečas [10], B. Pini [1–5], C. Pucci [1, 3], G. Cimmino [1, 2].

5.1 The Rellich Equality for a Second Order Equation

5.1.1 The Rellich Equality

Let us consider $\Omega \in \mathfrak{N}^\infty$, A a second order operator,

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + d.$$

J. Nečas et al., *Direct Methods in the Theory of Elliptic Equations*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-10455-8_5,

Let the coefficients a_{ij}, b_i, d be real functions satisfying $a_{ij}, b_i \in C^{0,1}(\overline{\Omega})$, $d \in L^\infty(\Omega)$. Let $v \in W^{2,2}(\Omega)$ be a real function and $h_i \in C^\infty(\overline{\Omega})$, $i = 1, 2, \ldots, N$, N real functions.

In this chapter we shall use the usual convention: We omit the symbol Σ; every summation is taken over indices which appear *twice*.

Let us denote $A' = -\partial/\partial x_i (a_{ij} \partial/\partial x_j)$. We have almost everywhere in Ω the following identity:

$$\frac{\partial}{\partial x_k}[h_k a_{ij} - h_i a_{kj} - h_j a_{ik}] \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} = b_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + 2 h_i \frac{\partial v}{\partial x_i} A' v, \tag{5.1}$$

where

$$b_{ij} = \frac{\partial h_k}{\partial x_k} a_{ij} - \frac{\partial h_i}{\partial x_k} a_{kj} - \frac{\partial h_j}{\partial x_k} a_{ik} + h_k \frac{\partial a_{ij}}{\partial x_k} - h_j \frac{\partial a_{ik}}{\partial x_k} + h_j \frac{\partial a_{ki}}{\partial x_k}.$$

Applying Green's formula we get the *Rellich equality*

$$\int_{\partial\Omega} (h_k a_{ij} - h_i a_{kj} - h_j a_{ik}) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} n_k \, \mathrm{d}S = \int_\Omega b_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} \, \mathrm{d}x + 2 \int_\Omega h_i \frac{\partial v}{\partial x_i} A' v \, \mathrm{d}x. \tag{5.2}$$

5.1.2 Lemmas, Regularity of the Dirichlet Problem

Hereafter we shall work with domains of the type $\mathfrak{N}^{0,1}$, cf. 2.1.1.

We shall say that $\Omega \in \mathfrak{M}$ if $\Omega \in \mathfrak{N}^{0,1}$ and there exists a sequence of subdomains $\Omega_s \subset \Omega$, $\Omega_s \in \mathfrak{N}^\infty$ such that $\lim_{s\to\infty} \Omega_s = \Omega$ in $\mathfrak{N}^{0,1}$, cf. 2.4.2. The author proved in [4] that $\mathfrak{N}^{0,1} \equiv \mathfrak{M}$;

We have the following lemma:

Lemma 1.1. *Let $\Omega \in \mathfrak{M}$, $g \in W^{1,2}(\partial\Omega)$. Then there exists a linear operator extending g to Ω, $g \in W^{1,2}(\Omega)$ such that*

$$|g|_{W^{1,2}(\Omega)} \le c |g|_{W^{1,2}(\partial\Omega)}, \tag{5.3}$$

$$|g|_{W^{1,2}(\partial\Omega_s)} \le c |g|_{W^{1,2}(\partial\Omega)}. \tag{5.4}$$

Indeed: using a partition of unity as in 1.2.4 and setting in V_r

$$g_r(x'_r, x_{rN}) = g(x'_r, a_r(x'_r)) \varphi_r(x),$$

then obviously

$$g(x) = \sum_{r=1}^m g_r(x) \tag{5.5}$$

is the desired extension with properties (5.3, 5.4). □

Clearly we have

Lemma 1.2. *Let* $\Omega \in \mathfrak{N}^{\infty}$, $g \in W^{1,2}(\partial\Omega)$. *Then* $C^{\infty}(\partial\Omega)$ *is dense in* $W^{1,2}(\partial\Omega)$.

We formulate a fundamental lemma:

Lemma 1.3. *Let* $\Omega \in \mathfrak{N}^{\infty}$, A^* *be the adjoint operator of* A, $f \in L^2(\Omega)$, $g \in W^{1,2}(\partial\Omega)$. *We assume that* $\lambda = 0$ *is not an eigenvalue for the operator* A^* *with Dirichlet conditions. Let* G^* *be the Green operator corresponding to the Dirichlet problem* $A^*u = f$ *in* Ω, $u = g$ *on* $\partial\Omega$, $G^* \in [L^2(\Omega) \times W^{1,2}(\partial\Omega) \to W^{1,2}(\Omega)]$. *If* $g \in C^{\infty}(\partial\Omega)$, *then we have* $G^*(f,g) = v \in W^{2,2}(\Omega)$, $\partial v/\partial x_i \in L^2(\partial\Omega)$, $i = 1,2,\dots,N$, *and the following inequality holds:*

$$\left(\sum_{i=1}^{N}\left|\frac{\partial v}{\partial x_i}\right|^2_{L^2(\partial\Omega)}\right)^{1/2} \le c_1(|f|^2_{L^2(\Omega)} + |g|^2_{W^{1,2}(\partial\Omega)})^{1/2}. \tag{5.6}$$

The operator T *defined by*

$$T(f,g) = \frac{\partial v}{\partial \nu^*} = a_{ij}\frac{\partial v}{\partial x_i}n_j, f \in L^2(\Omega), g \in C^{\infty}(\partial\Omega),$$

has an extension to $[L^2(\Omega) \times W^{1,2}(\partial\Omega) \to L^2(\partial\Omega)]$, *and*

$$|T(f,g)|_{L^2(\partial\Omega)} \le c_2(|f|^2_{L^2(\Omega)} + |g|^2_{W^{1,2}(\partial\Omega)})^{1/2}. \tag{5.7}$$

If $\partial\Omega$ *is described in the system of charts* (x'_r, x_{rN}), $r = 1,2,\dots,m$, *by* $x_{rN} = a_r(x'_r)$ *on* $\overline{\Delta}_r \equiv |x_{ri}| \le \alpha$, $i = 1,2,\dots,N-1$, *then* c_1 *depends only on*

$$|a_{ij}|_{C^{0,1}(\overline{\Omega})},\ |b_i|_{C^{0,1}(\overline{\Omega})},\ |d|_{L^{\infty}(\Omega)},\ |G^*|_{[L^2(\Omega)\times W^{1,2}(\partial\Omega)\to L^2(\partial\Omega)]},\ |a_r|_{C^{0,1}(\overline{\Delta}_r)}.$$

Proof. By Theorems 3.2.1, 3.3.1, 3.4.1, the Green operator G^* exists. Using Theorem 4.2.2 we prove $(f,g) \in L^2(\Omega) \times C^{\infty}(\partial\Omega) \Rightarrow v \in W^{2,2}(\Omega)$. Now we construct a vector $h = (h_1, h_2, \dots h_N)$ putting $h^{(r)} = (0,\dots,0,-\varphi_r)$ in the chart (x'_r, x_{rN}) and $h = \sum_{r=1}^{m} h^{(r)}$. We have

$$(n, h^{(r)}) = \varphi_r\left(1 + \sum_{i=1}^{N-1}\left(\frac{\partial a_r}{\partial x_{ri}}\right)^2\right)^{-1/2} \ge c_2\varphi_r,$$

where $c_2 > 0$ depends only on $|a_r|_{C^{0,1}(\overline{\Delta}_r)}$. Then we get $(n,h) \ge c_2 > 0$. Now according to (5.2), we obtain:

$$\begin{aligned}&\int_{\partial\Omega}(h_k a_{ij} - h_i a_{kj} - h_j a_{ik})\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}n_k\,\mathrm{d}S\\ &= \int_{\Omega} b^*_{ij}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}\,\mathrm{d}x - 2\int_{\Omega} h_i\frac{\partial v}{\partial x_i}\left(A^*v + \frac{\partial}{\partial x_i}(b_i v) - dv\right)\mathrm{d}x.\end{aligned} \tag{5.8}$$

Moreover we have

$$\begin{aligned}&\int_{\partial\Omega}(h_k a_{ij}-h_i a_{kj}-h_j a_{ik})\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}n_k\,\mathrm{d}S\\ &=-\int_{\partial\Omega}[(h_i a_{jk}-h_k a_{ji})+(h_j a_{ki}-h_k a_{ji})+h_k a_{ji}]\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}n_k\,\mathrm{d}S.\end{aligned}\tag{5.9}$$

For j (resp. i) fixed, the vector $(h_i a_{jk}-h_k a_{ij})n_k$ (resp. $(h_j a_{ki}-h_k a_{ji})n_k$) is orthogonal to the normal vector n. Then $(h_i a_{jk}-h_k a_{ij})n_k(\partial v/\partial x_i)$ and $(h_j a_{ki}-h_k a_{ji})n_k(\partial v/\partial x_i)$ are the derivatives of v in the tangent plane. As $n_k h_k \geq c_1 > 0$ it follows from (5.8), (5.9) and Theorem 3.4.5 (this theorem guarantees the existence of $c_3>0$ such that $\xi\in\mathbb{R}^N \Rightarrow a_{ij}\xi_i\xi_j \geq c_3|\xi|^2$):

$$\sum_{i=1}^N\left|\frac{\partial v}{\partial x_i}\right|^2_{L^2(\partial\Omega)} \leq c_4\left[|g|_{W^{1,2}(\partial\Omega)}\left(\sum_{i=1}^N\left|\frac{\partial v}{\partial x_i}\right|^2_{L^2(\partial\Omega)}\right)^{1/2}+|f|^2_{L^2(\Omega)}+|g|^2_{W^{1,2}(\partial\Omega)}\right].\tag{5.10}$$

Now using the inequality $2ab\leq \varepsilon^2a^2+(1/\varepsilon^2)b^2$, $a\geq 0$, $b\geq 0$, $\varepsilon>0$ we obtain

$$\sum_{i=1}^N\left|\frac{\partial v}{\partial x_i}\right|^2_{L^2(\partial\Omega)} \leq c_5(|f|^2_{L^2(\Omega)}+|g|^2_{W^{1,2}(\partial\Omega)}).\tag{5.11}$$

□

Now, let us consider $\Omega\in\mathfrak{M}$, and let Ω_s be the corresponding sequence such that $\lim_{s\to\infty}\Omega_s=\Omega$. By Lemma 3.1.2 we have for Ω_s, Ω independently of s and for $h\in L^2(\partial\Omega_s)$ or $h\in L^2(\partial\Omega)$:

$$c_1|h|_{L^2(\partial\Omega_s)} \leq \left(\sum_{r=1}^m\int_{\Delta_r}|h(x'_r,a_{rs}(x'_r))|^2\,\mathrm{d}x'_r\right)^{1/2} \leq c_2|h|_{L^2(\partial\Omega_s)}.\tag{5.12}$$

This last inequality allows us to identify $L^2(\partial\Omega_s)$ and $L^2(\partial\Omega)$ with a closed subspace of $\prod_{r=1}^m L^2(\Delta_r)$. Hereafter in this chapter we shall always use this identification. We have

Lemma 1.4. *Let $\Omega\in\mathfrak{M}$, $A=-(\partial/\partial x_i)(a_{ij}\partial/\partial x_j)+b_i\partial/\partial x_i+d$ be an elliptic operator with $a_{ij},b_i\in C^{0,1}(\Omega)$, $d\in L^\infty(\Omega)$. We assume that $u\in W^{1,2}(\Omega)$, the uniquely determined solution of the Dirichlet problem $Au=0$ in Ω with $u=0$ on $\partial\Omega$ satisfies $u\equiv 0$. Let us suppose that f, g are such that $f\in L^2(\Omega)$, $g\in W^{1,2}(\partial\Omega)$. We extend g into Ω as in Lemma 1.1 and let Ω_s be a sequence such that* $\lim_{s\to\infty}\Omega_s=\Omega$ *(cf. the definition of $\mathfrak{M}$). Then for $s\geq s_0$, there exists a unique solution v_s of the problem $A^*v_s=f$ in Ω_s, $v_s=g$ on $\partial\Omega_s$ and a unique solution v of the problem $A^*v=f$ in Ω, $v=g$ on $\partial\Omega$ such that*

$$|v_s|_{W^{1,2}(\Omega_s)} \leq c_1(|f|^2_{L^2(\Omega_s)} + |g|^2_{W^{1,2}(\partial\Omega_s)})^{1/2}, \tag{5.13}$$

$$|v|_{W^{1,2}(\Omega)} \leq c_1(|f|^2_{L^2(\Omega)} + |g|^2_{W^{1,2}(\partial\Omega)})^{1/2}. \tag{5.14}$$

Let $\partial v_s/\partial \nu^* = a_{ij}(\partial v_s/\partial x_i)n_j$ on $\partial\Omega$ as in Lemma 1.3. Then

$$\left|\frac{\partial v_s}{\partial \nu^*}\right|_{L^2(\partial\Omega_s)} \leq c_2(|f|^2_{L^2(\Omega_s)} + |g|^2_{W^{1,2}(\partial\Omega_s)})^{1/2}, \tag{5.15}$$

where c_2 does not depend on s, and $\lim_{s\to\infty} \partial v_s/\partial \nu^* = w$ weakly in $\prod_{r=1}^m L^2(\Delta_r)$.

Proof. According to Theorems 3.2.1, 3.3.1, 3.4.1, the Green operator

$$G^* \in [L^2(\Omega) \times W^{1,2}(\partial\Omega) \to W^{1,2}(\Omega)]$$

exists. It is also the case for $s \geq s_0$; if not, then there would exist a sequence, $v_{s_t} \in W_0^{1,2}(\Omega_{s_t})$, $t \to \infty$, such that

$$\int_{\Omega_{s_t}} a_{ij} \frac{\partial v_{s_t}}{\partial x_i} \frac{\partial v_{s_t}}{\partial x_j}\, \mathrm{d}x = 1$$

and $A^* v_{s_t} = 0$ in Ω_{s_t}. From the sequence v_{s_t} we could extract a subsequence called for simplicity v_{s_t} again, such that $\lim_{t\to\infty} v_{s_t} = v$ weakly in $W_0^{1,2}(\Omega)$. By Theorem 2.6.1, v_{s_t} converges strongly to v in $L^2(\Omega)$. Then we have:

$$1 = \lim_{t\to\infty} \int_{\Omega_{s_t}} \left(-v_{s_t} b_i \frac{\partial v_{s_t}}{\partial x_i} - d v_{s_t}^2\right) \mathrm{d}x = \int_{\Omega} \left(-v b_i \frac{\partial v}{\partial x_i} - d v^2\right) \mathrm{d}x.$$

v is obviously a weak solution of the problem $A^* v = 0$ in Ω with $v = 0$ on $\partial\Omega$, and

$$\int_{\Omega} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\, \mathrm{d}x = 1.$$

Then v is an eigenfunction, which is impossible. In the same way we can conclude that $\limsup_{s\to\infty} |G_s^*| < \infty$. Then (5.13) follows.

Now let us construct h as in the proof of Lemma 1.3. If s is big enough, we have on $\partial\Omega_s$, $h_i n_i \geq c_3$, c_3 independent of s, hence (5.15) follows from Lemma 1.3. Now from Lemma 1.1 we have:

$$\left|\frac{\partial v_s}{\partial \nu^*}\right|_{L^2(\partial\Omega_s)} \leq c_4(|f|^2_{L^2(\Omega)} + |g|^2_{W^{1,2}(\partial\Omega)})^{1/2}, \tag{5.16}$$

where c_4 does not depend on s. Let $\psi \in C_0^\infty(\Delta_r)$, $w \in C_0^\infty(U_r)$, cf. 1.2.4, such that $\psi(x'_r) w(x'_r, a_{rs}(x'_r)) = \psi(x'_r)$, for s big enough.

We denote $\psi w = h$. By Theorem 3.6.7 v_s can be extended to g, and converges to $v \in W^{1,2}(\Omega)$. Indeed, by Theorem 2.6.1, we easily get $\lim_{s\to\infty} v_s = v$ in $L^2(\Omega)$; if we consider the sesquilinear form $((v,u)) + \lambda(v,u)$, λ big enough, we are in the setting of the hypotheses of Theorem 3.6.7. Then we have:

$$\int_{\Omega_s}\left(a_{ji}\frac{\partial h}{\partial x_i}\frac{\partial v_s}{\partial x_j} + b_i\frac{\partial h}{\partial x_i}v_s + dhv_s\right)\mathrm{d}x$$
$$= \int_{\Delta_r} a_{ji}\frac{\partial v_s}{\partial x_j}n_i\psi\left(1+\sum_{i=1}^{N-1}\left(\frac{\partial a_{rs}}{\partial x_{ri}}\right)^2\right)^{1/2}\mathrm{d}x'_r + \int_{\partial\Omega_s} b_i n_i h v_s\,\mathrm{d}S + \int_{\Omega_s} hf\,\mathrm{d}x. \tag{5.17}$$

Now using Theorem 2.4.5, we get the existence of the limit

$$\lim_{s\to\infty}\int_{\Delta_r} a_{ji}\frac{\partial v_s}{\partial x_j}n_i\psi\left(1+\sum_{i=1}^{N-1}\left(\frac{\partial a_{rs}}{\partial x_{ri}}\right)^2\right)^{1/2}\mathrm{d}x'_r,$$

and also the existence of the limit

$$\lim_{s\to\infty}\int_{\Delta_r} a_{ji}\frac{\partial v_s}{\partial x_j}n_i\psi\left(1+\sum_{i=1}^{N-1}\left(\frac{\partial a_r}{\partial x_{ri}}\right)^2\right)^{1/2}\mathrm{d}x'_r.$$

But the set of functions

$$\psi\left(1+\sum_{i=1}^{N-1}\left(\frac{\partial a_{rs}}{\partial x_{ri}}\right)^2\right)^{1/2}$$

is dense in $L^2(\Delta_r)$, and the conclusion of the lemma follows from (5.16) □

We denote by w_r the corresponding weak limit in $L^2(\Delta_r)$ and define:

$$\frac{\partial v}{\partial \nu^*} \equiv \sum_{r=1}^{m}\varphi_r w_r. \tag{5.18}$$

We have the following regularity theorem for the Dirichlet problem in domains of the type $\mathfrak{M}$:

Theorem 1.1. *Let $\Omega \in \mathfrak{M}$, A be the operator given in Lemma 1.4, $f \in L^2(\Omega)$, $g \in W^{1,2}(\partial\Omega)$, v the solution of the problem $A^*v = f$ in Ω, $v = g$ on $\partial\Omega$. Let T be the mapping of $L^2(\Omega)\times W^{1,2}(\partial\Omega)$ into $L^2(\partial\Omega)$ defined by (5.18): $T(f,g) = \partial v/\partial\nu^*$. Then T is a linear and bounded mapping, and we have for $u \in W^{1,2}(\Omega)$:*

$$\int_\Omega uf\,\mathrm{d}x = -\int_{\partial\Omega} u\left(\frac{\partial v}{\partial \nu^*} + b_i n_i v\right)\mathrm{d}S + \int_\Omega\left(a_{ji}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + b_i\frac{\partial u}{\partial x_i}v + duv\right)\mathrm{d}x. \tag{5.19}$$

The mapping T for which (5.19) holds is unique.

Proof. Let $u \in W^{1,2}(\Omega)$ and set $u_r = u\varphi_r$. With the same notations as in the previous lemma, with s big enough, we have

$$\int_\Omega \left(a_{ji} \frac{\partial u_r}{\partial x_i} \frac{\partial v_s}{\partial x_j} + b_i \frac{\partial u_r}{\partial x_i} v_s + d u_r v_s \right) \mathrm{d}x = \int_{\partial\Omega_s} u_r \frac{\partial v_s}{\partial \nu^*} \,\mathrm{d}S + \int_{\partial\Omega_s} u_r b_i n_i v_s \,\mathrm{d}S + \int_{\Omega_s} u_r f \,\mathrm{d}x; \tag{5.20}$$

but we can write:

$$\int_{\partial\Omega} \frac{\partial v_s}{\partial \nu^*} u_r \,\mathrm{d}S = \int_{\Delta_r} \frac{\partial v_s}{\partial \nu^*} u\varphi_r \left(1 + \sum_{i=1}^{N-1} \left(\frac{\partial a_{rs}}{\partial x_{ri}} \right)^2 \right)^{1/2} \mathrm{d}x'_r,$$

and according to Lemma 1.4:

$$\lim_{s\to\infty} \int_{\partial\Omega_s} \frac{\partial v_s}{\partial \nu^*} u_r \,\mathrm{d}S = \int_{\Delta_r} w_r u\varphi_r \left(1 + \sum_{i=1}^{N-1} \left(\frac{\partial a_{rs}}{\partial x_{ri}} \right)^2 \right)^{1/2} \mathrm{d}x'_r.$$

Using again Theorem 3.5.7, letting $s \to \infty$ in (5.20) and taking the summation of (5.10) for $r = 1,2,\dots,m$, we get (5.19). The uniqueness of T is a direct consequence of Theorem 2.4.9. □

Remark 1.1. We can prove Theorem 1.1, cf. Nečas [8], under the condition $f \in L^{2N/(N+1)}(\Omega)$. Also we can show that

$$|v|_{W^{1,2N/(N+1)}(\Omega)} \le c\big(|f|_{L^{2N/(N+1)}(\Omega)} + |g|_{W^{1,2}(\partial\Omega)}\big).$$

We can give an example proving that $\partial v/\partial \nu^* \notin L^p(\partial\Omega)$, $p > 2$. We can also introduce on $\partial\Omega$ logarithmic weights denoted by σ, and show that (cf. O. Horáček [1]):

$$\frac{\partial v}{\partial \nu^*} \in L^2_\sigma(\partial\Omega), \quad L^2_\sigma(\partial\Omega) = \{f, \int_{\partial\Omega} |f|^2 \sigma \,\mathrm{d}S < \infty\}.$$

5.1.3 Very Weak Solutions

Let $u \in W^{1,2}(\Omega)$ be a weak solution of the equation $Au = 0$ in Ω. It follows from Theorem 1.1 that for v defined by $A^* v = f$ with $g \equiv 0$, we have:

$$\int_\Omega u f \,\mathrm{d}x = -\int_{\partial\Omega} u \left(\frac{\partial v}{\partial \nu^*} + b_i n_i v \right) \mathrm{d}S. \tag{5.21}$$

Formally, this is the formula (4.5.2) for the Dirichlet problem with $k = 1$. According to Theorem 1.1, we get the inequality:

$$|u|_{L^2(\Omega)} \leq c|u|_{L^2(\partial\Omega)}. \tag{5.22}$$

Taking into account Theorems 2.4.9, 2.5.7, and (5.22) we get

Theorem 1.2. *Let $\Omega \in \mathfrak{M}$, A be the operator defined in Lemma 1.4. Then the Green operator $G : W^{1/2,2}(\partial\Omega) \to W^{1,2}(\Omega)$ which determines the solution of the problem $Au = 0$ in Ω, $u = g$ on $\partial\Omega$, can be extended by continuity as a mapping from* $[L^2(\partial\Omega) \to L^2(\Omega)]$.

This gives a *very weak solution* of the second order Dirichlet problem with a boundary condition in $L^2(\partial\Omega)$.

Remark 1.2. It possible to construct a counterexample showing that for $\Omega \in \mathfrak{N}^{0,1}$ we cannot replace $L^2(\partial\Omega)$ by $L^p(\partial\Omega)$, $p < 2$, in the Dirichlet boundary conditions.

Exercise 1.1. Construct the example just mentioned in the case $\Omega \subset \mathbb{R}^2$, $\Omega = \{x, 0 < r < 1, |\varphi| < \pi - \varepsilon, \varepsilon > 0\}$ using a conformal mapping.

It follows from Theorem 4.1.2 that for $\Omega' \subset \overline{\Omega}' \subset \Omega$ the solution u obtained in Theorem 1.2 belongs to $W^{2,2}(\Omega')$; if $a_{ij} \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

Results concerning the role of boundary data on $\partial\Omega$ are proved in a paper of the author [3], cf. also Chap. 6.

We say that u is a *classical solution* of the Dirichlet problem if $u \in C^2(\Omega)$ and $Au = 0$ in Ω. Prove:

Exercise 1.2. Let $\Omega \in \mathfrak{M}$, A as in Lemma 1.4, u the classical solution of the Dirichlet problem. Then $u = Gu$, G as in Theorem 1.2. Hint: Apply Lemma 1.4.

Remark 1.3. In J. Nečas [8] it is proved that for $g \in L^2(\partial\Omega)$, $u = Gg \in L^{2N/(N-1)}(\Omega)$, etc.

Exercise 1.3. Let $\Omega \in \mathfrak{M}$, A the operator from Lemma 1.4, $u \in W^{1,2N/(N+1)}(\Omega)$ a weak solution of the equation $Au = 0$ in Ω. Let $\lim_{s\to\infty} |u|_{L^2(\partial\Omega_s)} = 0$. Then $u \equiv 0$. Hint: Use Lemma 1.4.

Example 1.1. Let Ω be as in Exercise 1.1, $p < 4/3$. Let $\varepsilon > 0$ be such that $\alpha = \pi/2(\pi - \varepsilon) < (2/p) - 1$. Set $u_0 = r^{-\alpha}\cos\alpha\varphi$ in polar coordinates. We have $\triangle u_0 = 0$ in Ω. Let $u \in W^{1,2}(\Omega)$ be a weak solution of $\triangle u = 0$ in Ω such that $u = u_0$ on $\partial\Omega$: such a solution is uniquely determined. $u_0 \notin W^{1,2}(\Omega)$, hence $u - u_0 \neq 0$. We have $u - u_0 \in W_0^{1,p}(\Omega)$.

Corollary 1.1. *The hypotheses of Theorem 1.2 being satisfied, let W be the closed subspace of $W^{1,2}(\Omega)$ of weak solutions of the equation $Au = 0$ in Ω. Let u_n, $n = 1, 2, \ldots$, be a basis of W. Let us denote by u_n the trace of u_n on $\partial\Omega$. Let ω_n be the sequence obtained by the orthonormalization procedure with the scalar product $\int_{\partial\Omega} vu\,\mathrm{d}S$. Let $g \in L^2(\partial\Omega)$,*

$$g = \sum_{n=1}^{\infty} g_n \omega_n,$$

its Fourier series. Then in Ω

$$Gg = u = \sum_{n=1}^{\infty} g_n \omega_n.$$

Remark 1.4. Let $\Omega \in \mathfrak{M}$ and consider the Dirichlet problem $Au = 0$ in Ω, $u = g$ on $\partial\Omega$, $g \in W^{1,2}(\partial\Omega)$. We can justify the approach used by M. Picone and compute $\partial u/\partial \nu \in L^2(\partial\Omega)$. Then we obtain u using a fundamental solution. Cf. P. Doktor [1].

Theorem 1.3. *Let $\Omega \in \mathfrak{M}$, A be as in Lemma 1.4. Let us consider the operator $T(0,g)$, $g \in W^{1,2}(\partial\Omega)$, corresponding to the Dirichlet problem $Au = 0$ in Ω, $u = g$ on $\partial\Omega$, and the conormal derivative $\partial u/\partial \nu = a_{ij}(\partial u/\partial x_j)n_i$ as in Theorem 1.1. Then T can be extended continuously to a mapping from $[L^2(\partial\Omega) \to W^{-1,2}(\partial\Omega)]$.*

Proof. The space $W^{1,2}(\partial\Omega)$ is dense in $L^2(\partial\Omega)$. Let $g \in W^{1,2}(\partial\Omega)$, u given by the theorem, and $h \in W^{1,2}(\partial\Omega), v$ the solution of the problem $A^*v = 0$ in Ω, $v = h$ on $\partial\Omega$. According to Theorem 1.1, we have by (5.19)

$$\int_{\partial\Omega} g\left(\frac{\partial v}{\partial \nu^*} + b_i n_i v\right) \mathrm{d}S = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} h \,\mathrm{d}S,$$

hence $|\partial u/\partial \nu|_{W^{-1,2}(\partial\Omega)} \le c_1 |g|_{L^2(\partial\Omega)}$. □

Problem 1.1. This doesn't make sense that for $\Omega \in \mathfrak{M}$ (in fact $\Omega \in \mathfrak{N}^{0,1}$), G as in Theorem 1.2 maps $L^2(\partial\Omega)$ in $W^{1/2,2}(\partial\Omega)$; moreover if we denote $M = \{u \in W^{1/2,2}(\Omega),\ Au = 0 \text{ in } \Omega\}$, is it possible to define traces on $\partial\Omega$ of functions from M? Is G an isomorphism of $L^2(\partial\Omega)$ onto M?

In the paper of the author [3], there is given another point of view concerning the regularity of a very weak solution. In the case of convex domains, the solution of the problem $Au = 0$ in Ω, $u = g$ on $\partial\Omega$ with $g \in L^2(\partial\Omega)$ is such that

$$\int_{\Omega} \left[\sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2\right] \rho \,\mathrm{d}x < \infty,$$

where ρ is the distance $\rho(x) = \mathrm{dist}(x, \partial\Omega)$. If

$$M = \{u \in L^2(\Omega), \quad \int_{\Omega} \left[\sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2\right] \rho \,\mathrm{d}x < \infty, \quad Au = 0 \text{ in } \Omega\},$$

then $u \in M$ has a trace on $\partial\Omega$ in $L^2(\partial\Omega)$. Cf. also Chap. 6.

5.2 The Neumann and Newton Problems

5.2.1 Lemmas, Regularity of the Solution

Lemma 2.1. *Let $\Omega \in \mathfrak{M}$, $g \in L^2(\partial\Omega)$. Suppose the sequence $g_s \in L^2(\partial\Omega_s)$ is constructed by the following procedure:*

$$h_r(x'_r, a_r(x'_r)) = h_{rs}(x'_r, a_{rs}(x'_r)) = g(x'_r, a_r(x'_r))\varphi(x'_r, a_r(x'_r)),\ g_s = \sum_{r=1}^m h_{rs}.$$

Then $\lim_{s\to\infty} g_s = g$ *in* $\prod_{r=1}^m L^2(\Delta_r)$.

Proof. Let us consider h_{rs} and let us prove that $\lim_{s\to\infty} h_{rs} = h_r$ in $\prod_{i=1}^m L^2(\Delta_i)$. In $L^2(\Delta_r)$ we obviously have $h_{rs} = h_r$. Let $i \neq r$. According to (5.12) and by Lemma 3.1.1 applied to the domain Ω_s, we get

$$|h_{rs}|_{L^2(\Delta_i)} \leq c_1 |h_{rs}|_{L^2(\Delta_r)}. \tag{5.23}$$

If g is continuous on $\partial\Omega$, the conclusion is clear. Then approximating g by a continuous function, (5.23) gives the result. □

If $\Omega \in \mathfrak{N}^\infty$, $C^\infty(\partial\Omega)$ is obviously dense in $L^2(\partial\Omega)$.

We can prove the following lemma which is analogous to Lemma 1.3:

Lemma 2.2. *Let $\Omega \in \mathfrak{N}^\infty$, A be the differential operator $A = -(\partial/\partial x_i)(a_{ij}\partial/\partial x_j) + b_i\partial/\partial x_i + d$, $a_{ij}, b_i \in C^{0,1}(\overline{\Omega})$, $d \in L^\infty(\Omega)$, a_{ij}, b_i, d real functions, $a_{ij} = a_{ji}$, A elliptic: $a_{ij}\xi_i\xi_j \geq c|\xi|^2$, $\xi \in \mathbb{R}^N$. Let $\sigma \in C^{0,1}(\partial\Omega)$ (σ can be $\equiv 0$) be a real function. The sesquilinear form $A(v,u) + \lambda(v,u) + \int_{\partial\Omega} \sigma v u \, \mathrm{d}S$ is $W^{1,2}(\Omega)$-elliptic for λ big enough, and it is assumed that $\lambda = 0$ is not an eigenvalue for A with the boundary condition $\frac{\partial u}{\partial \nu} + \sigma u = 0$ on $\partial\Omega$.[1] Let v be the solution of the problem $A^*v = f$ in Ω, $\partial v/\partial \nu + b_i n_i v + \sigma v = g$ on $\partial\Omega$, $f \in L^2(\Omega)$, $g \in C^\infty(\partial\Omega)$. Then $v \in W^{2,2}(\Omega)$, $\partial v/\partial x_i \in L^2(\partial\Omega)$, and*

$$\left(\sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^2_{L^2(\partial\Omega)}\right)^{1/2} \leq c_1\left(|f|^2_{L^2(\Omega)} + |g|^2_{L^2(\partial\Omega)}\right)^{1/2}; \tag{5.24}$$

more precisely $v \in W^{1,2}(\partial\Omega)$, and by continuous extension, we find the solution v of $A^*v = f$ in Ω, $\partial v/\partial \nu + \sigma v + b_i n_i v = g$ on $\partial\Omega$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, and we have the following estimate:

$$|v|_{W^{1,2}(\partial\Omega)} \leq c\left(|f|^2_{L^2(\Omega)} + |g|^2_{L^2(\partial\Omega)}\right)^{1/2}. \tag{5.25}$$

The dependence of c_2 on the coefficients is the same as in Lemma 1.3.

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[1] This is true if $b_i \equiv 0$, $d \geq 0$, $\sigma \geq 0$ except the case where we should have at the same time $d \equiv 0$, $\sigma \equiv 0$. The case $b_i \equiv 0$, $d \equiv 0$, $\sigma \equiv 0$ will be considered later.

Proof. By Theorems 3.2.1, 3.3.1, 3.4.1 the problem has a uniquely determined solution in $W^{1,2}(\Omega)$. It follows from Theorem 4.2.2 that for $f \in L^2(\Omega)$, $g \in C^\infty(\partial\Omega)$ we have $v \in W^{2,2}(\Omega)$. Let h be as in the proof of Lemma 1.3. We have (5.8), and

$$\frac{\partial v}{\partial \nu} = a_{ij}\frac{\partial v}{\partial x_i}n_j,$$

hence

$$h_i a_{jk}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}n_k = h_i\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial \nu}, \quad h_j a_{ki}\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j}n_k = h_j\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial \nu},$$

and from (5.8) we get the inequality (5.24) as in Lemma 1.3. □

We now have

Lemma 2.3. *Let $\Omega \in \mathfrak{M}$, A be the operator from Lemma 2.2, $\sigma \in C^{0,1}(\overline{\Omega})$. Let us assume that $\lambda = 0$ is not an eigenvalue of A with boundary condition $\partial u/\partial \nu + \sigma u = 0$ on $\partial\Omega$. We consider the following class of regularity for the data: $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$. We define g_s on $\partial\Omega_s$ as in Lemma 2.1. Let v be the solution of the problem $A^*v = f$ in Ω, $\partial v/\partial \nu + b_i n_i v + \sigma v = g$ on $\partial\Omega$, and v_s the solution of the problem $A^*v_s = f$ in Ω_s, $\partial v_s/\partial \nu + b_i n_i v_s + \sigma v_s = 0$ on $\partial\Omega_s$. The existence and uniqueness are clear if s is big enough. Then $v_s \in W^{1,2}(\partial\Omega_s)$, and we have*

$$|v_s|_{W^{1,2}(\partial\Omega_s)} \le c(|f|^2_{L^2(\Omega_s)} + |g_s|^2_{L^2(\partial\Omega_s)})^{1/2}, \tag{5.26}$$

where c does not depend on s; v_s converges weakly to v in $\prod_{i=1}^m W^{1,2}(\Delta_i)$.

Proof. There exist extension operators P_s from $W^{1,2}(\Omega_s)$ onto $W^{1,2}(\Omega)$, with $|P_s| < c_2$, and which satisfy (3.6.38). It is sufficient to apply Theorem 2.3.9. Let G^* be the Green operator $G^* : L^2(\Omega) \times L^2(\partial\Omega) \to W^{1,2}(\Omega)$ associated with the problem $A^*v = f$ in Ω, $\partial v/\partial \nu + b_i n_i v + \sigma v = g$ on $\partial\Omega$. Such an operator exists for $s \ge s_0$ with $|G_s^*| \le c_3$. If not there will exist a sequence $v_{s_t} \in W^{1,2}(\Omega_{s_t})$ satisfying $A^*v_{s_t} = 0$ in Ω_{s_t}, $\partial v_{s_t}/\partial \nu + b_i n_i v_{s_t} + \sigma v_{s_t} = 0$ on $\partial\Omega_{s_t}$, and such that

$$\int_{\Omega_{s_t}} a_{ij}\frac{\partial v_{s_t}}{\partial x_i}\frac{\partial v_{s_t}}{\partial x_j}\,\mathrm{d}x + \int_{\Omega_{s_t}} v_{s_t}^2\,\mathrm{d}x = 1.$$

If we set $v_{s_t} = P_{s_t}v_{s_t}$ outside of Ω_{s_t} in Ω, we have:

$$1 \le \int_{\Omega} a_{ij}\frac{\partial v_{s_t}}{\partial x_i}\frac{\partial v_{s_t}}{\partial x_j}\,\mathrm{d}x + \int_{\Omega} v_{s_t}^2\,\mathrm{d}x \le c_4.$$

We can extract from v_{s_t} a subsequence converging weakly in $W^{1,2}(\Omega)$; we denote this subsequence again v_{s_t}. Let $\lim_{t\to\infty} v_{s_t} = v$. We have

$$\int_{\Omega_{s_t}} \left(a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v_{s_t}}{\partial x_j} \, \mathrm{d}x + v v_{s_t} \right) \mathrm{d}x = \int_{\Omega_{s_t}} v v_{s_t} \, \mathrm{d}x - \int_{\Omega_{s_t}} b_i \frac{\partial v}{\partial x_i} v_{s_t} \, \mathrm{d}x$$
$$- \int_{\Omega_{s_t}} d v v_{s_t} \, \mathrm{d}x - \int_{\partial\Omega_{s_t}} \sigma v v_{s_t} \, \mathrm{d}S.$$

On the other hand we have $\lim_{t\to\infty} v_{s_t} = v$ in $L^2(\Omega)$ due to Theorem 2.6.1 and it follows by Theorems 2.4.5 and 2.6.2, that $\lim_{t\to\infty} v_{s_t} = v$ in $L^2(\partial\Omega)$. Thus

$$1 = \int_\Omega v^2 \, \mathrm{d}x - \int_\Omega b_i \frac{\partial v}{\partial x_i} v \, \mathrm{d}x - \int_\Omega d v^2 \, \mathrm{d}x - \int_{\partial\Omega} \sigma v^2 \, \mathrm{d}S = \int_\Omega \left(a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + v^2 \right) \mathrm{d}x.$$

But v is clearly an eigenfunction of the operator A^* with the boundary condition $\partial v/\partial \nu + b_i n_i v + \sigma v = 0$ on $\partial\Omega$, which is impossible. With the same proof we can also show that $|G_s^*| \le c_4$. Now we use the function h introduced in Lemma 1.3 and we get the inequality (5.26) according to the previous lemma. The sequence $P_s v_s$ converges weakly to v in $W^{1,2}(\Omega)$. (Theorem 3.6.8 implies that $\lim_{s\to\infty} P_s v_s = v$ strongly in $W^{1,2}(\Omega)$.)

We get $\lim_{s\to\infty} v_s = v$ in $\prod_{i=1}^m L^2(\Delta_i)$. □

We also have the following consequence:

Theorem 2.1. *Let $\Omega \in \mathfrak{M}$, A be the operator from Lemma 2.2, $\sigma \in C^{0,1}(\partial\Omega)$. We consider for A the same hypotheses as in Lemma 2.3. Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, v the solution of the problem $A^* v = f$ in Ω, with $\partial v/\partial \nu + b_i n_i v + \sigma v = g$ on $\partial\Omega$. Then $v \in W^{1,2}(\partial\Omega)$, and the following inequality holds:*

$$|v|_{W^{1,2}(\partial\Omega)} \le c_2 (|f|^2_{L^2(\Omega)} + |g|^2_{L^2(\partial\Omega)})^{1/2}. \qquad (5.27).$$

Remark 2.1. Under the hypotheses of Theorem 2.1, cf. J. Nečas [8], we can prove the inequalities

$$|v|_{W^{1,2}(\partial\Omega)} \le c (|f|^2_{L^{2N/(N+1)}(\Omega)} + |g|^2_{L^2(\partial\Omega)})^{1/2},$$

and

$$|v|_{W^{1,2N/(N+1)}(\Omega)} \le c (|f|^2_{L^{2N/(N+1)}(\Omega)} + |g|^2_{L^2(\partial\Omega)^{1/2}}.$$

It follows from Theorem 2.1 that the solution of the Neumann-Newton problem with $g \in L^2(\partial\Omega)$ is a solution of the Dirichlet problem with the boundary value in $W^{1,2}(\partial\Omega)$.

5.2.2 Lemmas, Regularity of the Solution (Continuation)

Now we can consider the case of the operator

$$A = -\frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial}{\partial x_j}\right), \tag{5.28}$$

with

$$a_{ij} = a_{ji}, \quad a_{ij} \in C^{0,1}(\overline{\Omega}), \quad \xi \in \mathbb{R}^N \Longrightarrow a_{ij}\xi_i\xi_j \geq c|\xi|^2. \tag{5.29}$$

We have

Theorem 2.2. *Let $\Omega \in \mathfrak{M}$, A be the operator defined by (5.28), (5.29) and let*

$$f \in L^2(\Omega), \quad g \in L^2(\partial\Omega), \quad \int_\Omega f\,\mathrm{d}x + \int_{\partial\Omega} g\,\mathrm{d}S = 0.$$

Then there exists a unique solution of the problem $A^*v = f$ in Ω, $\partial v/\partial \nu = g$ on $\partial\Omega$ such that $\int_\Omega v\,\mathrm{d}x = 0$ and

$$|v|_{W^{1,2}(\Omega)} \leq c_1(|f|^2_{L^2(\Omega)} + |g|^2_{L^2(\partial\Omega)})^{1/2}. \tag{5.30}$$

Indeed: According to Theorem 1.6.1, the solution v is defined and is unique and we have

$$|v|_{W^{1,2}(\Omega)} \leq c_2(|f|^2_{L^2(\Omega)} + |g|^2_{L^2(\partial\Omega)})^{1/2}. \tag{5.31}$$

We have $A^*v + v = f + v$; $f + v$ satisfies the hypotheses of Theorem 2.1, hence (5.31) implies (5.30). □

We can also consider Lemma 2.3 for the operator (5.28), (5.29), cf. Nečas [8].

Exercise 2.1. Prove the following proposition: Let $\Omega \in \mathfrak{M}$, A from (5.28), (5.29). Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $\int_\Omega f\,\mathrm{d}x + \int_{\partial\Omega} g\,\mathrm{d}S = 0$. There exists a unique solution of the problem $Av = f$ in Ω, $\partial v/\partial\nu = g$ on $\partial\Omega$, $\int_\Omega v\,\mathrm{d}x = 0$. Let us define the sequence $f_s = f + c_s$ such that $\int_{\Omega_s} f_s\,\mathrm{d}x + \int_{\partial\Omega_s} g_s\,\mathrm{d}S = 0$, g_s as in Lemma 2.1. Then there exists a unique solution v_s of the problem $Av_s = f_s$ in Ω_s, $\partial v_s/\partial\nu = g_s$ on $\partial\Omega_s$, $\int_{\Omega_s} v_s\,\mathrm{d}x = 0$. Moreover if we extend v_s by $P_s v_s$ on Ω, we have $\lim_{s\to\infty} v_s = v$ in $W^{1,2}(\Omega)$, $v_s \in W^{1,2}(\partial\Omega_s)$, and $\lim_{s\to\infty} v_s = v$ in $\prod_{i=1}^m W^{1,2}(\Delta_i)$.

Problem 2.1. Let $\Omega \in \mathfrak{M}$, v given by Theorems 2.1 or 2.2. Does v belong to $W^{3/2,2}(\Omega)$, satisfying the estimate $|v|_{W^{3/2,2}(\Omega)} \leq c(|f|^2_{L^2(\Omega)} + |g|^2_{L^2(\partial\Omega)})^{1/2}$?

5.2.3 Very Weak Solutions

Now we consider the operator

$$A = -\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + b_i\frac{\partial}{\partial x_i} + d, \tag{5.32}$$

with the following assumptions:

$$a_{ij}, b_i \in C^{0,1}(\overline{\Omega}), \quad d \in L^\infty(\Omega), \quad a_{ij} = a_{ji}, \quad \xi \in \mathbb{R}^N \Longrightarrow a_{ij}\xi_i\xi_j \geq c|\xi|^2. \tag{5.33}$$

If, moreover, v satisfies

$$v \in W^{1,2}(\Omega), \quad Av = 0 \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} + \sigma v = 0 \text{ on } \partial\Omega \text{ with } \quad \sigma \in C^{0,1}(\partial\Omega), \tag{5.34}$$

then $v = 0$.

Using duality, we can prove

Theorem 2.3. *Let $\Omega \in \mathfrak{M}$, A be the operator defined by (5.32)–(5.34), u the solution of the problem $Au = 0$ in Ω, $\partial u/\partial \nu + \sigma u = h$ on $\partial\Omega$ with $h \in L^2(\partial\Omega)$. Then*

$$|u|_{L^2(\Omega)} \leq c|h|_{W^{-1,2}(\partial\Omega)}. \tag{5.35}$$

Moreover, the Green operator $G : L^2(\partial\Omega) \to W^{1,2}(\Omega)$ can be extended by continuity to $G \in [W^{-1,2}(\partial\Omega) \to L^2(\Omega)]$.

Proof. Let $f \in L^2(\Omega), v$ be the solution of the problem $A^*v = f$ in Ω, $\frac{\partial v}{\partial \nu} + b_i n_i v + \sigma v = 0$ on $\partial\Omega$. We have:

$$\int_\Omega uf \,\mathrm{d}x = \int_{\partial\Omega} hv \,\mathrm{d}S, \tag{5.36}$$

hence, according to Theorem 2.1, we obtain (5.35). The space $L^2(\partial\Omega)$ is dense in $W^{-1,2}(\partial\Omega)$. □

The more general class of functions g such that the solution of the considered problem is given by Theorem 3.2.1 reads: $W^{-1/2,2}(\partial\Omega)$, $\Omega \in \mathfrak{N}^{0,1}$. We have obtained a *very weak solution* of the Neumann-Newton problem given on $\partial\Omega$ with $g \in W^{-1,2}(\partial\Omega)$, cf. Remark 4.5.3.

Example 2.1. Let $N = 2$, $A = -\triangle + 1$, $g = \delta_{y,\partial\Omega}$, with $y = (1,1)$, Ω is the square $|x_i| < 1$. Theorem 2.3 gives a solution of the problem $-\triangle u + u = 0$ in Ω, $\partial u/\partial n = \delta_{y,\partial\Omega}$ on $\partial\Omega$.

In a paper by the author [8], it was also proved

$$|u|_{L^{2N/(N-1)}(\Omega)} \leq c|h|_{W^{-1,2}(\partial\Omega)}.$$

5.2.4 Uniqueness Theorems

Without difficulty we obtain

Corollary 2.1. *Let $N : L^2(\partial\Omega) \to L^2(\partial\Omega)$ be the mapping defined by $Nh = u$ with the notations as in Theorem 2.3. Then we have*

$$|u|_{L^2(\partial\Omega)} \le c|h|_{W^{-1,2}(\partial\Omega)} \tag{5.37}$$

and this mapping can be extended by continuity to $N \in [W^{-1,2}(\partial\Omega) \to L^2(\partial\Omega)]$.

Let v be the solution of the problem $A^*v = 0$ in Ω, $\partial v/\partial\nu + b_i n_i v + \sigma v = g$ on $\partial\Omega$, $g \in L^2(\partial\Omega)$, and let u be the solution of the problem $Au = 0$ in Ω, $\partial v/\partial\nu + b_i n_i v + \sigma v = h$ on $\partial\Omega$, $h \in L^2(\partial\Omega)$. We have $\int_{\partial\Omega} ug\,\mathrm{d}S = \int_{\partial\Omega} hv\,\mathrm{d}S$, hence according to inequality (5.27) this implies (5.37). □

A very weak solution of the Neumann-Newton problem is also a weak solution of the Dirichlet problem; it concerns the solution of the problem $Au = 0$ in Ω. We have also the converse as a consequence of Theorem 1.3. As in Sect. 1, it is possible to prove uniqueness theorems:

Exercise 2.2. Let us consider $A = -\partial/\partial x_i(a_{ij}\partial/\partial x_j) + 1$ satisfying (5.29), $u \in W^{2,2}(\Omega')$ for every $\Omega' \subset \overline{\Omega}' \subset \Omega$, $Au = 0$ in Ω, $\lim_{s\to\infty} |\partial u/\partial\nu|_{W^{-1,2}(\partial\Omega_s)} = 0$. Then $u \equiv 0$. Hint: use Lemma 2.3.

Problem 2.2. Let $\Omega \in \mathfrak{M}$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, v be the solution of the problem $Av = f$ in Ω, $\partial v/\partial\nu + b_i n_i v + \sigma v = g$, on $\partial\Omega$, with A satisfying (5.32)–(5.34). By Theorem 4.2.2, we have $v \in W^{2,2}_{loc}(\Omega)$, thus $\partial v/\partial\nu \in L^2(\partial\Omega_s)$. Is it true that $\lim_{s\to\infty} \partial v_s/\partial\nu = g$ in $\prod_{i=1}^m L^2(\Delta_i)$?

Problem 2.3. In Theorem 2.3, is it possible to remove the hypothesis $a_{ij} = a_{ji}$?

5.2.5 *Very Weak Solutions (Continuation)*

We will apply the duality method for the operator given by (5.28), (5.29). Without difficulty we prove that $L^{2,\perp}(\partial\Omega)$, the space of functions on $\partial\Omega$ orthogonal to 1, is dense in $W^{-1,2,\perp}(\partial\Omega)$, the space of distributions which vanish at 1.[2] Applying Theorem 2.2 to $g = 0$ we obtain immediately

Theorem 2.4. *Let* $\Omega \in \mathfrak{M}$, *A be the operator as in (5.28), (5.29), u the solution of the problem* $Au = 0$ *in* Ω, $\partial u/\partial\nu = h$ *on* $\partial\Omega$, $h \in L^{2,\perp}(\partial\Omega)$, *such that* $\int_\Omega u\,\mathrm{d}x = 0$. *Then*

$$|u|_{L^2(\Omega)} \le c|h|_{W^{-1,2}(\partial\Omega)}. \tag{5.38}$$

Moreover, the Green operator $G : L^{2,\perp}(\partial\Omega) \to W^{1,2}(\Omega) \cap L^{2,\perp}(\Omega)$ *can be continuously extended to* $G \in [W^{-1,2,\perp}(\partial\Omega) \to L^{2,\perp}(\Omega)]$.

Example 2.2. Let $N = 2$, $A = -\triangle$, $\Omega = \{|x_i| < 1,\ i = 1,2\}$, $g = \delta_{(1,1),\partial\Omega} - \delta_{(-1,-1),\partial\Omega}$. Then there exists a unique function $u \in L^2(\Omega)$, $\int_\Omega u\,\mathrm{d}x = 0$, such that $\triangle u = 0$ in Ω, $\partial u/\partial\nu = \delta_{(1,1),\partial\Omega} - \delta_{(-1,-1),\partial\Omega}$ on $\partial\Omega$.

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[2] We proceed as usual, using the reflexivity of $W^{1,2}(\partial\Omega)$.

Finally, let us remark that according to Theorem 4.1.3 a very weak solution from Theorems 2.3, 2.4 belongs to $W^{2,2}_{loc}(\Omega)$; if a_{ij}, b_i $d \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

Problem 2.4. For a second order operator let us consider a mixed problem: for $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\operatorname{dist}(\Gamma_1, \Gamma_2) = 0$, suppose that $u = g_1$ on Γ_1, $\partial u/\partial \nu = g_2$ on Γ_2, $Au = f$ in Ω, $f \in L^2(\Omega)$. In some sense which should be precised, is it true that

$$\left|\frac{\partial u}{\partial \nu}\right|_{L^2(\partial\Omega)} + |u|_{W^{1,2}(\partial\Omega)} \le c(|g_1|_{W^{1,2}(\Gamma_1)} + |g_2|_{L^2(\Gamma_2)} + |f|_{L^2(\Omega)})?$$

5.3 Second Order Strongly Elliptic Systems

5.3.1 Definitions

The generalized Rellich equalities can be used also for second order systems. We shall restrict ourselves to the Dirichlet problem and consider the system:

$$A_{rs} = -\frac{\partial}{\partial x_i}\left(a^{rs}_{ij}\frac{\partial}{\partial x_j}\right) + b^{rs}, \quad i,j = 1,2,\dots,N, \quad r,s = 1,2,\dots,M, \tag{5.39}$$

$$a^{rs}_{ij} \in C^{0,1}(\overline{\Omega}), \quad b^{rs} \in L^\infty(\Omega) \quad a^{rs}_{ij} = a^{rs}_{ji}, \quad a^{rs}_{ij} = a^{sr}_{ij}.\text{[3]} \tag{5.40}$$

Let us observe once more that in this chapter, we consider only real functions. The system (5.39) is assumed to be strongly elliptic, cf. (3.181a):

$$\xi \in \mathbb{R}^N, \quad \eta \in \mathbb{R}^M \Longrightarrow a^{rs}_{ij}\xi_i\xi_j\eta_r\eta_s \ge e(\sum_{i=1}^N \xi_i^2)(\sum_{r=1}^M \eta_r^2). \tag{5.41}$$

For simplicity we assume:

$$\eta \in \mathbb{R}^M \Longrightarrow b^{rs}\eta_r\eta_s \ge 0, \tag{5.42}$$

$$\varphi_r, \varphi_s \in C_0^\infty(\Omega) \Longrightarrow \int_\Omega \left(a^{rs}_{ij}\frac{\partial\varphi_r}{\partial x_i}\frac{\partial\varphi_s}{\partial x_j} + b^{rs}\varphi_r\varphi_s\right)\mathrm{d}x \ge c_1\sum_{r=1}^M |\varphi_r|^2_{W^{1,2}(\Omega)}, \tag{5.43}$$

then we have $\left[W_0^{1,2}(\Omega)\right]^M$-ellipticity.

Let us observe that if the coefficients are constant, then (5.43) follows from Theorem 3.7.3 and from (5.41), (5.42), and that Theorem 3.7.4 together with (5.41) imply the inequality:

[3] This condition is very restrictive. Nevertheless, it is valid also in the more general case of non homogeneous, anisotropic media, for the system of elasticity.

$$A(\varphi,\varphi)+\lambda(\varphi,\varphi)\geq c_1\sum_{r=1}^{M}|\varphi_r|^2_{W^{1,2}(\Omega)}, \tag{5.44}$$

with λ big enough. The reader can investigate as an exercise, for the case when $A(\mathbf{v},\,\mathbf{u})$ is a hermitian form, the Fredholm theory for the Dirichlet problem and adapt the hypothesis that $\lambda=0$ is not an eigenvalue.

5.3.2 Regularity of the Solution, Ω Smooth

Let us consider $\Omega\in\mathfrak{N}^{0,1}$ $\mathbf{g}\in\left[L^2(\partial\Omega)\right]^M$, $\mathbf{f}\in\left[L^2(\Omega)\right]^M$. According to Theorems 3.7.2, 2.4.10, there exists a unique solution $\mathbf{u}\in\left[W^{1,2}(\Omega)\right]^M$ of the problem $A_{rs}u_s=f_r$ in Ω, $r=1,2,\dots,M$, $\mathbf{u}=\mathbf{g}$ on $\partial\Omega$ in the trace sense.

To complete the study of systems (5.39)–(5.43), we must consider the dependence on the domain and prove a regularity theorem as in Theorem 4.2.2.[4]

According to the proof of Theorem 3.6.7 we obtain

Lemma 3.1. *Let Ω be a bounded domain and Ω_n be a sequence of domains $\Omega_n\subset\Omega$, $\lim_{n\to\infty}\Omega_n=\Omega$, such that for every compact $K\subset\Omega$, there exists n_0 such that $n\geq n_0\Longrightarrow K\subset\Omega_n$. Let $\mathbf{f}\in\left[W^{-1,2}(\Omega)\right]^M$, $\mathbf{u_0}\in\left[W^{1,2}(\Omega)\right]^M$, and $\mathbf{u}_n\in\left[W^{1,2}(\Omega_n)\right]^M$ the solution of the Dirichlet problem $A_{rs}u_{ns}=f_r$ in Ω_n, $\mathbf{u_n}=\mathbf{u_0}$ on $\partial\Omega_n$, where A_{rs} are the operators defined by (5.39)–(5.43).[4] We extend $\mathbf{u_n}$ outside of Ω_n by $\mathbf{u_0}$. Then $\lim_{n\to\infty}\mathbf{u_n}=\mathbf{u}$ in $\left[W^{1,2}(\Omega)\right]^M$, where $\mathbf{u}$ is the solution of the problem in Ω.*

Changing slightly the proofs of Theorems 4.1.1, 4.2.1 we obtain, cf. J. Nečas [13]:

Lemma 3.2. *Let $\Omega\in\mathfrak{N}^\infty$, A_{rs} be the system (5.39)–(5.43)[4] $\mathbf{g}\in\left[W^{3/2,2}(\partial\Omega)\right]^M$, $\mathbf{f}\in\left[L^2(\Omega)\right]^M$, $\mathbf{u}$ the solution of the problem $A_{rs}u_s=f_r$ in Ω, $r=1,2,\dots,M$, $\mathbf{u}=\mathbf{g}$ on $\partial\Omega$. Then*

$$|\mathbf{u}|_{\left[W^{2,2}(\Omega)\right]^M}\leq c\left(|\mathbf{f}|^2_{\left[L^2(\Omega)\right]^M}+|\mathbf{g}|^2_{\left[W^{3/2,2}(\partial\Omega)\right]^M}\right)^{1/2}. \tag{5.45}$$

Proof. Let us consider a partition of unity and the transformation in charts (y',y_N). Then we prove as in the theorems mentioned above

$$\left|\frac{\partial^2u_s}{\partial y_iy_j}\right|_{L^2(K_+)}\leq c\left(|\mathbf{f}|^2_{\left[L^2(\Omega)\right]^M}+|\mathbf{g}|^2_{\left[W^{3/2,2}(\partial\Omega)\right]^M}\right)^{1/2},\quad \begin{array}{l} i,j=1,2,\dots N,\\ s=1,2,\dots,M,\end{array}$$

[4] Here the condition $a^{rs}_{ij}=a^{rs}_{ji}$, $a^{rs}_{ij}=a^{sr}_{ij}$ is not necessary.

except in the case $i = j = N$. It follows from (5.39)–(5.43) written in charts (y', y_N), that the condition (5.41) is the same after this transformation:

$$a^{rs}_{NN}\frac{\partial^2 u_s}{\partial x_N^2} = h_r, \quad h_r \in L^2(\Omega), \quad |h_r|_{L^2(\Omega)} \le c\left(|\mathbf{f}|^2_{[L^2(\Omega)]^M} + |\mathbf{g}|^2_{[W^{3/2,2}(\partial\Omega)]^M}\right)^{1/2},$$

then by (5.41) we have $\det(a^{rs}_{NN}) \neq 0$ in $\overline{\Omega}$. □

5.3.3 The Rellich Equality

Let $\Omega \in \mathfrak{N}^\infty$, $\mathbf{v} \in \left[W^{2,2}(\Omega)\right]^M$, A_{rs} be the system of operators given by (5.39)–(5.43). Let $h = (h_1, h_2, \dots, h_N)$ be a vector with components in $C^\infty(\Omega)$. Let us denote $A'_{rs} = -\partial/\partial x_i(a^{rs}_{ij}\partial/\partial x_j)$. Almost everywhere in Ω we have the identity:

$$\frac{\partial}{\partial x_k}\left[(h_k a^{rs}_{ij} - h_i a^{rs}_{kj} - h_j a^{rs}_{ik})\frac{\partial v_r}{\partial x_i}\frac{\partial v_s}{\partial x_j}\right] = b^{rs}_{ij}\frac{\partial v_r}{\partial x_i}\frac{\partial v_s}{\partial x_j} + 2h_i\frac{\partial v_r}{\partial x_i}A'_{rs}v_s, \tag{5.46}$$

where

$$b^{rs}_{ij} = \frac{\partial h_k}{\partial x_k}a^{rs}_{ij} + h_k\frac{\partial a^{rs}_{ij}}{\partial x_k} - 2\frac{\partial h_i}{\partial x_k}a^{rs}_{kj}.$$

The Green formula gives

$$\begin{aligned}\int_{\partial\Omega}&(h_k a^{rs}_{ij} - h_i a^{rs}_{kj} - h_j a^{rs}_{ik})\frac{\partial v_r}{\partial x_i}\frac{\partial v_s}{\partial x_j}n_k\,\mathrm{d}S\\ &= \int_\Omega\left(b^{rs}_{ij}\frac{\partial v_r}{\partial x_i}\frac{\partial v_s}{\partial x_j} + 2h_i\frac{\partial v_r}{\partial x_i}A'_{rs}v_s\right)\mathrm{d}x.\end{aligned} \tag{5.47}$$

We can prove for systems a result analogous to that of Lemma 1.3:

Lemma 3.3. *Let $\Omega \in \mathfrak{N}^\infty$, A_{rs} be the system (5.39)–(5.43), $\mathbf{f} \in \left[L^2(\Omega)\right]^M$, $\mathbf{g} \in [C^\infty(\partial\Omega)]^M$, $\mathbf{v}$ the solution of the problem $A_{rs}v_s = f_r$ in Ω, $r = 1,2,\dots,M$, $\mathbf{v} = \mathbf{g}$ on $\partial\Omega$. Then we have $\mathbf{v} \in \left[W^{2,2}(\Omega)\right]^M$, $\partial v_r/\partial x_i \in L^2(\partial\Omega)$, $i = 1,2,\dots,N$, $r = 1,2,\dots,M$, and the following inequality:*

$$\left|\frac{\partial\mathbf{v}}{\partial\nu}\right|_{[L^2(\partial\Omega)]^M} \le c_1\left(|\mathbf{f}|^2_{[L^2(\Omega)]^M} + |\mathbf{g}|^2_{[W^{1,2}(\partial\Omega)]^M}\right)^{1/2}, \tag{5.48}$$

where

$$\frac{\partial\mathbf{v}}{\partial\nu} \equiv \left(a^{1s}_{ij}\frac{\partial v_s}{\partial x_j}n_i,\ a^{2s}_{ij}\frac{\partial v_s}{\partial x_j}n_i, \dots,\ a^{Ms}_{ij}\frac{\partial v_s}{\partial x_j}n_i\right) \equiv \left(\left(\frac{\partial\mathbf{v}}{\partial\nu}\right)_1, \dots, \left(\frac{\partial\mathbf{v}}{\partial\nu}\right)_M\right).$$

Let us define T by $T(\mathbf{f},\ \mathbf{g}) = \partial\mathbf{v}/\partial\nu$, $\mathbf{f} \in \left[L^2(\Omega)\right]^M$, $\mathbf{g} \in [C^\infty(\partial\Omega)]^M$; we can extend T continuously to $T : \left[\left[L^2(\Omega)\right]^M \times \left[W^{1,2}(\partial\Omega)\right]^M \to \left[L^2(\Omega)\right]^M\right]$. In (5.48) c_1 depends only on $\partial\Omega$ as in Lemma 1.3.

Proof. Theorem 3.7.2 gives the existence and uniqueness of the solution. By Lemma 3.2, $\mathbf{v} \in \left[W^{2,2}(\Omega)\right]^M$; let us take a function h as in the proof of Lemma 3.1. The vector $h_k n_k a_{ij}^{rs} - h_i n_k a_{kj}^{rs}$ with r, s, j fixed is orthogonal to the normal vector; as in the proof of Lemma 3.1, we obtain the inequality:

$$\int_{\partial\Omega} a_{ij}^{rs} \frac{\partial v_r}{\partial x_i} \frac{\partial v_s}{\partial x_j}\, \mathrm{d}S \le c_1 \left(|\mathbf{f}|^2_{\left[L^2(\Omega)\right]^M} + |\mathbf{g}|^2_{\left[W^{1,2}(\partial\Omega)\right]^M} \right), \tag{5.49}$$

where c_1 depends on the data as in Lemma 3.1.

Let us consider the inverse matrix to the matrix of coefficients $a_{ij}^{rs} n_i n_j$; let d^{rs} be the coefficients of this inverse. Then inequality (5.41) implies

$$c_2 \sum_{r=1}^{M} \eta_r^2 \ge d^{rs} \eta_r \eta_s \ge c_3 \sum_{r=1}^{M} \eta_r^2,$$

the constants c_2, c_3 depend only on $\partial\Omega$ and c_1. We have

$$\begin{aligned} d^{rs} \left(\frac{\partial\mathbf{v}}{\partial\nu}\right)_r \left(\frac{\partial\mathbf{v}}{\partial\nu}\right)_s - a_{ij}^{rs} \frac{\partial v_r}{\partial x_i} \frac{\partial v_s}{\partial x_j} &= d^{rs} a_{kl}^{rt} \frac{\partial v_t}{\partial x_l} n_k a_{mn}^{s\tau} \frac{\partial v_\tau}{\partial x_n} n_m - a_{nl}^{t\tau} \frac{\partial v_t}{\partial x_n} \frac{\partial v_\tau}{\partial x_l} \\ &= (d^{rs} a_{kl}^{rt} n_k a_{mn}^{s\tau} n_m - a_{nl}^{t\tau}) \frac{\partial v_t}{\partial x_n} \frac{\partial v_\tau}{\partial x_l}. \end{aligned}$$

For t, τ, l fixed the vector $(d^{rs} a_{kl}^{rt} n_k a_{mn}^{s\tau} n_m - a_{nl}^{t\tau})$ is orthogonal to the normal vector; using (5.49), we obtain the following inequality:

$$\begin{aligned} &\int_{\partial\Omega} \sum_{r=1}^{M} \left[\left(\frac{\partial\mathbf{v}}{\partial\nu} \right)_r \right]^2 \mathrm{d}S \le \\ &c_4 \left(|\mathbf{f}|^2_{\left[L^2(\Omega)\right]^M} + |\mathbf{g}|^2_{\left[W^{1,2}(\partial\Omega)\right]^M} + |\mathbf{g}|_{\left[W^{1,2}(\partial\Omega)\right]^M} \sum_{r=1}^{M} \sum_{i=1}^{N} \left| \frac{\partial v_r}{\partial x_i} \right|_{L^2(\partial\Omega)} \right). \end{aligned} \tag{5.50}$$

For r, s, j fixed the vector $(a_{ij}^{rs} - n_k a_{kj}^{rs} n_i)$ is orthogonal to the normal vector; we get:

$$(a_{ij}^{rs} n_j - n_k a_{kj}^{rs} n_i n_j) \frac{\partial v_s}{\partial x_i} = \left(\frac{\partial\mathbf{v}}{\partial\nu} \right)_r - (a_{kj}^{rs} n_k n_j) \frac{\partial v_s}{\partial n},$$

thus

$$\sum_{r=1}^{M} \left| \frac{\partial v_r}{\partial n} \right|^2_{L^2(\partial\Omega)} \le c_5 \left(\sum_{r=1}^{M} \left| \left(\frac{\partial\mathbf{v}}{\partial\nu} \right)_r \right|^2_{L^2(\partial\Omega)} + |\mathbf{g}|^2_{\left[W^{1,2}(\partial\Omega)\right]^M} \right).$$

Let us now consider the piece of $\partial\Omega$ described by the function $a(x')$, cf. the definition of $\mathfrak{N}^{0,1}$, and let us set $w_r(x') = v_r(x', a(x'))$. For $i \le N-1$, we have $\partial w_r/\partial x_i = \partial v_r/\partial x_i + (\partial v_r/\partial x_N)(\partial a/\partial x_i)$; on the other hand using the coordinates (x', x_N), we have

$$\frac{\partial v_r}{\partial n} = \left(\frac{\partial v_r}{\partial x_i}\frac{\partial a}{\partial x_i} - \frac{\partial v_r}{\partial x_N}\right)\left(1 + \sum_{i=1}^{N-1}\left(\frac{\partial a}{\partial x_i}\right)^2\right)^{-1/2},$$

thus

$$\sum_{r=1}^{M}\sum_{i=1}^{N}\left|\frac{\partial v_r}{\partial x_i}\right|_{L^2(\partial\Omega)} \le c_6\left(\sum_{r=1}^{M}\left|\left(\frac{\partial \mathbf{v}}{\partial \nu}\right)_r\right|_{L^2(\partial\Omega)} + |\mathbf{g}|_{\left[W^{1,2}(\partial\Omega)\right]^M}\right). \tag{5.51}$$

It follows from (5.50), (5.51) that

$$\sum_{r=1}^{M}\sum_{i=1}^{N}\left|\frac{\partial v_r}{\partial x_i}\right|_{L^2(\partial\Omega)} \le c_7(|\mathbf{f}|^2_{\left[L^2(\Omega)\right]^M} + |\mathbf{g}|^2_{\left[W^{1,2}(\partial\Omega)\right]^M})^{1/2},$$

thus in particular (5.48) holds. □

5.3.4 Regularity of the Solution, Ω Non-smooth

We can prove a lemma analogous to Lemma 1.4:

Lemma 3.4. *Let $\Omega \in \mathfrak{M}$, A_{rs} be the system (5.39)–(5.43), $\mathbf{f} \in \left[L^2(\Omega)\right]^M$, $\mathbf{g} \in \left[W^{1,2}(\partial\Omega)\right]^M$, $\mathbf{v}$ the unique solution of the problem $A_{rs}v_s = f_r$ in Ω, $r = 1, 2, \ldots, M$, $\mathbf{v} = \mathbf{g}$ on $\partial\Omega$. We extend $\mathbf{g}$ to Ω using Lemma 1.1. Let Ω_n be a sequence of subdomains, $\lim_{n\to\infty}\Omega_n = \Omega$ as in the definition of $\mathfrak{M}$. Let $\mathbf{v}_n$ be the solution of the problem $A_{rs}v_{sn} = f_r$ in Ω_n, $\mathbf{v}_n = \mathbf{g}$ on $\partial\Omega_n$. Let $(\partial\mathbf{v}_n/\partial\nu)_r = a_{ij}^{rs}(\partial v_{sn}/\partial x_j)n_i$ on $\partial\Omega_n$. Then $\partial\mathbf{v}_n/\partial\nu$ converges weakly in $\prod_{i=1}^m\left[L^2(\Delta_i)\right]^M$. Denoting by $\mathbf{w}_i$ this limit in $\left[L^2(\Delta_i)\right]^M$, and setting*

$$\frac{\partial \mathbf{v}}{\partial \nu} = \sum_{i=1}^{m}\varphi_i \mathbf{w}_i. \tag{5.52}$$

we get

$$\left|\frac{\partial \mathbf{v}}{\partial \nu}\right|_{\left[L^2(\partial\Omega)\right]^M} \le c\left(|\mathbf{f}|^2_{\left[L^2(\Omega)\right]^M} + |\mathbf{g}|^2_{\left[W^{1,2}(\partial\Omega)\right]^M}\right)^{1/2}. \tag{5.53}$$

Proof. According to (5.43) we have the existence and uniqueness of the solution. Let us construct h as in Lemma 1.3, and by Lemma 1.1 and the previous lemma, we get:

$$\left|\frac{\partial v_n}{\partial \nu}\right|_{L^2(\partial\Omega_s)} \le c_1 \left(|\mathbf{f}|^2_{\left[L^2(\Omega)\right]^M} + |\mathbf{g}|^2_{\left[W^{1,2}(\partial\Omega)\right]^M} \right)^{1/2}, \tag{5.54}$$

where c_1 does not depend on s. Using Lemma 3.1 we obtain the result using the same approach as in the proof of Lemma 1.3. □

We also have

Theorem 3.1. *Let $\Omega \in \mathfrak{M}$, A_{rs} be the system (5.39)–(5.43), $\mathbf{f} \in \left[L^2(\Omega)\right]^M$, $\mathbf{g} \in \left[W^{1,2}(\partial\Omega)\right]^M$, $\mathbf{v}$ the solution of the problem $A_{rs}v_s = f_r$ in Ω, $r = 1,2,\dots,M$, $\mathbf{v} = \mathbf{g}$ on $\partial\Omega$. Let us define $T : \left[L^2(\Omega)\right]^M \times \left[W^{1,2}(\partial\Omega)\right]^M \to \left[L^2(\partial\Omega)\right]^M$ by (5.52). Then $T \in [[L^2(\Omega)]^M \times [W^{1,2}(\partial\Omega)]^M \to [L^2(\partial\Omega)]^M]$ and for $\mathbf{u} \in \left[W^{1,2}(\Omega)\right]^M$:*

$$\int_\Omega u_r f_r \, \mathrm{d}x = -\int_{\partial\Omega} u_r \left(\frac{\partial \mathbf{v}}{\partial \nu}\right)_r \mathrm{d}S + A(\mathbf{v},\mathbf{u}). \tag{5.55}$$

The mapping T satisfying (5.55) is uniquely determined.

The proof is a straightforward modification of the proof given in Theorem 1.1.

5.3.5 Very Weak Solutions

Using the duality method we obtain the *very weak* solutions for the Dirichlet problem.

Let $\mathbf{u} \in \left[W^{1,2}(\Omega)\right]^M$ be a weak solution of the equations $A_{rs}u_s = 0$ in Ω, $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$, $\mathbf{g} \in \left[W^{1/2,2}(\partial\Omega)\right]^M$. It follows from Theorems 2.4.10, 3.7.2, that there exists a unique solution of this problem. We can prove easily

Theorem 3.2. *Let $\Omega \in \mathfrak{M}$, A_{rs} be the operators (5.39)–(5.43). Then the Green operator $G : \left[W^{1/2,2}(\partial\Omega)\right]^M \to \left[W^{1,2}\Omega)\right]^M$ giving the solution of the problem $A_{rs}u_s = 0$ in Ω, $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ can be extended continuously to $G \in [[L^2(\partial\Omega)]^M \to [L^2(\Omega)]^M]$.*

Proof. Let $\mathbf{u}$ be from the theorem and $\mathbf{v}$ be the solution of the problem $A_{rs}v_s = f_r$ in Ω, $\mathbf{v} = 0$ on $\partial\Omega$, with $\mathbf{f} \in \left[L^2(\Omega)\right]^M$. By (5.55) we get

$$\int_\Omega u_r f_r \, \mathrm{d}x = -\int_{\partial\Omega} u_r (\partial \mathbf{v}/\partial \nu)_r \, \mathrm{d}S,$$

then according to Theorem 3.1, $|\mathbf{u}|_{\left[L^2(\Omega)\right]^M} \le c_1 |\mathbf{g}|_{\left[L^2(\partial\Omega)\right]^M}$. Using Theorem 2.4.9 the result follows. □

Exercise 3.1. Let $\Omega \in \mathfrak{M}$, A_{rs} be the operators (5.39)–(5.43).
Let $\mathbf{u} \in [W^{1,2N/(N+1)}(\Omega)]^M$ be a very weak solution of the equations $A_{rs}u_s = 0$ in Ω. Let $\lim_{s\to\infty} |\mathbf{u}|_{[L^2(\partial\Omega_s)]^M} = 0$. Then $\mathbf{u} \equiv 0$. Hint: Use Lemma 3.4.

Example 3.1. Let $\Omega = (-1,1)\times(-1,1)\times(-h,h)$,

$$A_{rs}u_s = -\mu\triangle u_r - (\lambda+\mu)(\partial/\partial x_r)\,\mathrm{div}\,\mathbf{u},$$

$M = 3$, $\mu > 0$, $\lambda \geq 0$, cf. (3.201). We prescribe $u_r = g_r$ on $\partial\Omega, g_r \equiv 0$ for $x_3 = \pm h$; We assume, for instance, g_r smooth enough on the faces without conditions of compatibility on the different faces. In general, we do not have $g_r \in W^{1/2,2}(\partial\Omega)$, the method from Chap. 3 cannot be used. Nevertheless a very weak solution exists according to Theorem 3.2.

It follows by results of A. Douglis, L. Nirenberg [1] that this solution is in $[C^\infty(\Omega)]^3$.

Problem 3.1. Keep the hypotheses of Theorem 3.2, and let $\mathbf{g} \in [C^0(\partial\Omega)]^M$. Is $\mathbf{u} \in [C^\infty(\overline{\Omega})]^M$?

As in Sect. 5.1, we obtain the following result: for a very weak solution from Theorem 3.3, we can define $\partial\mathbf{u}/\partial\nu \in [W^{-1,2}(\partial\Omega)]^M$ such that $|\partial\mathbf{u}/\partial\nu|_{[W^{-1,2}(\partial\Omega)]^M} \leq c|\mathbf{u}|_{[L^2(\partial\Omega)]^M}$; we proceed as in the proof of Theorem 1.3.

Problem 3.2. In Theorem 3.1, is the condition $a_{ij}^{rs} = a_{ij}^{sr}$ necessary?

5.4 A Fourth Order Equation, the Dirichlet Problem

5.4.1 Definition

Now we consider a fourth order operator which is written in the following form, different from that used in Chap. 3:

$$\begin{aligned} A = {} & \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ijkl}\frac{\partial^2}{\partial x_k \partial x_l}\right) - \frac{\partial}{\partial x_i}\left(b_{ikl}\frac{\partial^2}{\partial x_k \partial x_l}\right) + c_{kl}\frac{\partial^2}{\partial x_k \partial x_l} + \\ & + \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ijk}\frac{\partial}{\partial x_k}\right) - \frac{\partial}{\partial x_i}\left(e_{ij}\frac{\partial}{\partial x_j}\right) + f_i\frac{\partial}{\partial x_i} + \frac{\partial^2}{\partial x_i \partial x_j}(g_{ij}) - \frac{\partial}{\partial x_i}(h_i) + q \\ = {} & A' + A'', \end{aligned} \tag{5.56}$$

where

$$A' = \frac{\partial^2}{\partial x_i \partial x_j}\left(a_{ijkl}\frac{\partial^2}{\partial x_k \partial x_l}\right).$$

Concerning the coefficients we assume:

$$\begin{cases} a_{ijkl} \in C^{1,1}(\overline{\Omega}), & b_{ikl} \in C^{0,1}(\overline{\Omega}), \quad c_{kl} \in L^\infty(\Omega), \\ d_{ijk} \in C^{1,1}(\overline{\Omega}), & e_{ij} \in C^{0,1}(\overline{\Omega}), \quad f_i \in L^\infty(\Omega), \\ g_{ij} \in C^{1,1}(\overline{\Omega}), & h_i \in C^{0,1}(\overline{\Omega}), \quad q \in L^\infty(\Omega). \end{cases} \tag{5.57}$$

Let us consider a symmetric real matrix with coefficients ξ_{ij}. We assume

$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c \sum_{i,j=1}^{N} \xi_{ij}^2, \quad a_{ijkl} = a_{jikl} = a_{ijlk}. \tag{5.58}$$

Now let $v \in W^{3,2}(\Omega)$. We have almost everywhere the following identity:

$$\begin{aligned} &\frac{\partial}{\partial x_m}\left[(h_i a_{mjkl} + h_k a_{ijml} - h_m a_{ijkl})\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_k \partial x_l}\right] \\ &\quad = h_i a_{mjlk}\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^3 v}{\partial x_k \partial x_l \partial x_m} + h_k a_{ijml}\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_m}\frac{\partial^2 v}{\partial x_k \partial x_l} + \\ &\quad\quad + \frac{\partial}{\partial x_m}(h_i a_{mjkl} + h_k a_{ijml} - h_m a_{ijkl})\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_k \partial x_l}. \end{aligned} \tag{5.59}$$

Using condition (5.58), we have due to Theorem 3.4.2 for λ big enough and for $v \in W_0^{2,2}(\Omega)$:

$$A(v,v) + \lambda(v,v) \geq c|v|^2_{W^{2,2}(\Omega)}. \tag{5.60}$$

Moreover we assume:

$$v \in W_0^{2,2}(\Omega), \quad Av = 0 \text{ weakly in } \Omega \Longrightarrow v \equiv 0. \tag{5.61}$$

5.4.2 The Second Order Rellich Inequality

We shall prove the Rellich equality and the associated inequality. We call this inequality *second order Rellich inequality.*

Lemma 4.1. *Let $\Omega \in \mathfrak{M}$, A be the operator defined in (5.56)–(5.58), (5.61). Let G^* be the Green operator associated with the adjoint A^* and the Dirichlet problem $A^*v = f$ in Ω, $v = \partial v/\partial n = 0$ on $\partial\Omega$, $G^* \in [W^{-1,2}(\Omega) \to W^{2,2}(\Omega)]$; $f \in W^{-1,2}(\Omega)$. Then $v \in W^{3,2}(\Omega)$, where $\partial^2 v/\partial x_i \partial x_j \in L^2(\partial\Omega)$, and we have the inequality:*

$$\sum_{i,j=1}^{N} \left|\frac{\partial^2 v}{\partial x_i \partial x_j}\right|_{L^2(\partial\Omega)} \leq c_1 |f|_{W^{-1,2}(\Omega)}, \tag{5.62}$$

where c_1 depends only on the data as in Lemma 1.1.

Proof. The existence of the Green operator is a consequence of Theorems 3.2.1, 3.3.1; let $f \in L^2(\Omega)$; then Theorem 4.2.2 implies $v \in W^{4,2}(\Omega)$.

Using the function h as in the proof of Lemma 1.3, due to (5.59) we get, together with the fact that $\partial v/\partial x_i = 0$ on $\partial\Omega$,

$$\begin{aligned}
&\int_{\partial\Omega}(h_i a_{mjkl} + h_k a_{ijml} - h_m a_{ijkl})\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_k \partial x_l} n_m \,\mathrm{d}S \\
&= \int_\Omega \left(h_i a_{mjlk}\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^3 v}{\partial x_k \partial x_l \partial x_m} + h_k a_{ijml}\frac{\partial^3 v}{\partial x_i \partial x_j \partial x_m}\frac{\partial^2 v}{\partial x_k \partial x_l}\right)\mathrm{d}x \\
&\quad + \int_\Omega \frac{\partial}{\partial x_m}(h_i a_{mjlk} + h_k a_{ijml} - h_m a_{ijkl})\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_k \partial x_l}\,\mathrm{d}x \\
&= -2\int_\Omega h_i \frac{\partial v}{\partial x_i} A' v \,\mathrm{d}x + B(v,v),
\end{aligned} \tag{5.63}$$

where

$$B(v,v) = \int_\Omega \sum_{|\alpha|,|\beta|\le 2} b_{\alpha\beta} D^\alpha v D^\beta v \,\mathrm{d}x,$$

with α, β multi-indices, $b_{\alpha\beta} \in L^\infty(\Omega)$, and

$$|b_{\alpha\beta}|_{L^\infty(\Omega)} \le c_1\Big(\sum_{i,j,k,l=1}^N |a_{ijkl}|_{C^{1,1}(\overline{\Omega})} + 1\Big)\Big(1 + \sum_{i=1}^N |h_i|_{C^{1,1}(\overline{\Omega})}\Big). \tag{5.64}$$

The operator A'' from (5.56) is of third order, hence it follows from (5.63),

$$\begin{aligned}
&\int_{\partial\Omega}(h_i a_{mjkl} + h_k a_{ijml} - h_m a_{ijkl})\frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 v}{\partial x_k \partial x_l} n_m \,\mathrm{d}S \\
&\qquad\qquad = -2\int_\Omega h_i \frac{\partial v}{\partial x_i} A^* v \,\mathrm{d}x + B'(v,v),
\end{aligned} \tag{5.65}$$

with $b'_{\alpha\beta} \in L^\infty(\Omega)$. The vector $(h_i a_{mjkl} - h_m a_{ijkl})n_m$ for j,k,l fixed is orthogonal to the normal vector as well as the vector $(h_k a_{ijml} - h_m a_{ijkl})n_m$ for i,j,l fixed. We have $(h_i a_{mjkl} - h_m a_{ijkl})n_m \partial^2 v/\partial x_i \partial x_j = 0$, $(h_k a_{ijml} - h_m a_{ijkl})n_m \partial^2 v/\partial x_k \partial x_l = 0$ on $\partial\Omega$, then using (5.65), (5.58) and the existence of the Green operator, we deduce (5.62) for $f \in L^2(\Omega)$. But $L^2(\Omega)$ is dense in $W^{-1,2}(\Omega)$; on the other hand, by Theorem 4.2.2, $f \in W^{-1,2}(\Omega) \Longrightarrow v \in W^{3,2}(\Omega)$. □

Problem 4.1. We keep the hypotheses of Lemma 4.1, but we consider the nonhomogeneous Dirichlet problem: $A^* v = f$, $f \in W^{-1,2}(\Omega)$, $v = g_0$, $\partial v/\partial n = g_1$ on $\partial\Omega$. We assume that there exists $v_0 \in W^{2,2}(\Omega)$ such that in the trace sense $v = v_0$, $\partial v/\partial n = \partial v_0/\partial n$ on $\partial\Omega$; using the data g_0, g_1 we can formally compute $\partial v/\partial x_i$ on $\partial\Omega$, $i = 1,2,\ldots,N$. We assume $|\partial v/\partial x_i|_{W^{1,2}(\partial\Omega)} < \infty$. Determine whether the following inequality holds:

$$\sum_{i,j=1}^{N}\left|\frac{\partial^2 v}{\partial x_i \partial x_j}\right|_{L^2(\partial\Omega)} \le c(|f|^2_{W^{-1,2}(\Omega)} + \sum_{i=1}^{N}\left|\frac{\partial v_0}{\partial x_i}\right|^2_{W^{1,2}(\partial\Omega)} + |v_0|^2_{W^{1,2}(\Omega)})^{1/2}.$$

5.4.3 The Third Order Rellich Inequality

Let us assume that the operator A^* can be decomposed in the following form

$$A^* = \frac{\partial}{\partial x_i}\left[A_{ij}\frac{\partial}{\partial x_j}\left(C_{kl}\frac{\partial^2}{\partial x_k \partial x_l}\right)\right] + \frac{\partial}{\partial x_i}\left(B_{ikl}\frac{\partial^2}{\partial x_k \partial x_l}\right) + D_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + E_i\frac{\partial}{\partial x_i} + F, \tag{5.66}$$

with

$$A_{ij}, C_{kl} \in C^{1,1}(\overline{\Omega}), \quad B_{ikl} \in C^{0,1}(\overline{\Omega}), \quad D_{ij}, E_i, F \in L^\infty(\Omega), \tag{5.67}$$

$$\xi \in \mathbb{R}^N \Longrightarrow A_{ij}\xi_i\xi_j \ge c|\xi|^2, \quad C_{kl}\xi_k\xi_l \ge c|\xi|^2. \tag{5.68}$$

Let us observe that in the decomposition (5.66) of A^* the first operator is a product of two second order elliptic operators; if $N = 2$ this is always the case except perhaps the regularity condition (5.67). Sufficient conditions for (5.66) are given in the paper of the author [2]; for instance, (5.66) holds if a_{ijkl} are constant. If $N \ge 3$, (5.66) is not satisfied in general.

We prove the *third order Rellich inequality*.

Lemma 4.2. *Let $\Omega \in \mathfrak{N}^\infty, A$ be the operator satisfying (5.56)–(5.58), (5.61), (5.66)–(5.68). Let G^* be the Green operator corresponding to the problem $A^*v = f$ in Ω, $f \in L^2(\Omega)$, $v = \partial v/\partial n = 0$ on $\partial\Omega$, $G^* \in [L^2(\Omega) \to W^{2,2}(\Omega)]$. Then $v \in W^{4,2}(\Omega)$, where $\partial^3 v/\partial x_i \partial x_j \partial x_k \in L^2(\partial\Omega)$, and we have the inequality*

$$\left|A_{ij}\frac{\partial}{\partial x_j}\left(C_{kl}\frac{\partial^2 v}{\partial x_k \partial x_l}\right)n_i\right|_{W^{-1,2}(\partial\Omega)} \le c|f|_{L^2(\Omega)}, \tag{5.69}$$

where c depends only on the data as in Lemma 1.1.

Proof. Let h be the solution of the Dirichlet problem $-\partial/\partial x_i(A_{ij}\partial h/\partial x_j) = 0$ in Ω, $h \in C^\infty(\partial\Omega)$. We have $h \in W^{2,2}(\Omega)$ by Theorem 4.2.2 and it follows that $v \in W^{4,2}(\Omega)$. We can use the Green formula and obtain

$$\int_{\Omega} hf\,\mathrm{d}x = \int_{\partial\Omega} hA_{ij}\frac{\partial}{\partial x_j}\left(C_{kl}\frac{\partial^2 v}{\partial x_k \partial x_l}\right) n_i\,\mathrm{d}S + \int_{\partial\Omega} hB_{ikl}\frac{\partial^2 v}{\partial x_k \partial x_l} n_i\,\mathrm{d}S$$
$$- \int_{\Omega} A_{ij}\frac{\partial h}{\partial x_i}\frac{\partial}{\partial x_j}\left(C_{kl}\frac{\partial^2 v}{\partial x_k \partial x_l}\right)\mathrm{d}x + C(h,v), \tag{5.70}$$

where

$$C(h,v) = \int_{\Omega} \sum_{|\alpha|\le 1, |\beta|\le 2} C_{\alpha\beta} D^{\alpha} h D^{\beta} v\,\mathrm{d}x,$$

$$|C_{\alpha\beta}| \le c_1\Big(\sum_{i,k,l=1}^{N} |B_{ikl}|_{L^\infty(\Omega)} + \sum_{i,j=1}^{N} |D_{ij}|_{L^\infty(\Omega)} + \sum_{i=1}^{N} |E_i|_{L^\infty(\Omega)} + |F|_{L^\infty(\Omega)}\Big).$$

We have also

$$\int_{\Omega} A_{ij}\frac{\partial h}{\partial x_i}\frac{\partial}{\partial x_j}\left(C_{kl}\frac{\partial^2 v}{\partial x_k \partial x_l}\right)\mathrm{d}x = \int_{\partial\Omega} A_{ij}\frac{\partial h}{\partial x_i} C_{kl}\frac{\partial^2 v}{\partial x_k \partial x_l} n_j\,\mathrm{d}S$$
$$- \int_{\Omega} \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial h}{\partial x_i}\right) C_{kl}\frac{\partial^2 v}{\partial x_k \partial x_l}\,\mathrm{d}x = \int_{\partial\Omega} A_{ij}\frac{\partial h}{\partial x_i} C_{kl}\frac{\partial^2 v}{\partial x_k \partial x_l} n_j\,\mathrm{d}S,$$

thus by (5.70) according to (5.68), (5.6), it follows:

$$\left|\int_{\partial\Omega} hA_{ij}\frac{\partial}{\partial x_j}\left(C_{kl}\frac{\partial^2 v}{\partial x_k \partial x_l}\right) n_i\,\mathrm{d}S\right| \le c_2 |f|_{L^2(\Omega)} |h|_{W^{1,2}(\partial\Omega)},$$

and using the density $\overline{C^\infty(\partial\Omega)} = W^{1,2}(\partial\Omega)$ the result follows. □

Let us remark that we can obtain a stronger result: for instance if $N \ge 3$, then we have $L^q(\Omega) \subset W^{-1,2}(\Omega)$, $1/q = 1/2 + 1/N$, and thus

$$\left|\int_{\Omega} hf\,\mathrm{d}x\right| \le c|h|_{W^{1,2}(\Omega)} |f|_{L^q(\Omega)},$$

since $|h|_{L^{q'}(\Omega)} \le c_1 |h|_{W^{1,2}(\Omega)}$, $1/q' = 1 - 1/q$. In (5.69) we can replace $|f|_{L^2(\Omega)}$ by $|f|_{L^q(\Omega)}$.

Problem 4.2. Is it possible to have an inequality of type (5.69) without assumption (5.66)?

5.4.4 Dependence on the Domain

We want to investigate the dependence of the solution of the problem $A^* v = f$ in Ω, $v = \partial v/\partial n = 0$ on $\partial\Omega$, with respect to Ω. These considerations are strongly connected to questions in Sect. 3.6, Chap. 3.

Lemma 4.3. *Let $\Omega \in \mathfrak{M}$,*[5] *A be an operator satisfying (5.56)–(5.58), (5.61). Let $f \in L^2(\Omega)$, Ω_s be the sequence from the definition of $\mathfrak{M}$. Then there exists for $s \geq s_0$ a unique solution v_s of the problem $A^* v_s = f$ in Ω_s, $v_s = \partial v_s / \partial n = 0$ on $\partial \Omega_s$, $G_s^* \in [L^2(\Omega) \to W^{2,2}(\Omega_s)]$, $|G_s^*| < c$. If we extend v_s by zero outside of Ω_s we have* $\lim_{s\to\infty} v_s = v$ *in* $W^{2,2}(\Omega)$.

Proof. Let us assume that $\lim_{s\to\infty} G_s^*$ does not exist; then we can find a subsequence of weak solutions of $A^* v_{s_t} = 0$, $v_{s_t} \in \overset{\circ}{W}{}^{2,2}_0(\Omega_s) \subset W_0^{2,2}(\Omega)$ such that

$$\int_\Omega a_{ijkl} \frac{\partial^2 v_{s_t}}{\partial x_i \partial x_j} \frac{\partial^2 v_{s_t}}{\partial x_k \partial x_l} \, \mathrm{d}x = 1.$$

We can extract a subsequence denoted again v_{s_t} such that $\lim_{s_t\to\infty} v_{s_t} = v$ weakly in $W_0^{2,2}(\Omega)$. We have:

$$\int_{\Omega_s} a_{ijkl} \frac{\partial^2 v_{s_t}}{\partial x_i \partial x_j} \frac{\partial^2 v_{s_t}}{\partial x_k \partial x_l} \, \mathrm{d}x \equiv B(v_{s_t}, v_{s_t}) = \int_{\Omega_s} \sum_{\substack{|\alpha|\leq 2, |\beta| \leq 2 \\ |\alpha|+|\beta|<4}} b_{\alpha\beta} D^\alpha v_{s_t} D^\beta v_{s_t} \, \mathrm{d}x,$$

$b_{\alpha\beta} \in L^\infty(\Omega)$. By Theorem 2.6.1, $\lim_{s_t\to\infty} D^\alpha v_{s_t} = D^\alpha v$ strongly in $L^2(\Omega)$ if $|\alpha| \leq 1$, then:

$$\int_\Omega \sum_{\substack{|\alpha|\leq 2, |\beta| \leq 2 \\ |\alpha|+|\beta|<4}} b_{\alpha\beta} D^\alpha v D^\beta v \, \mathrm{d}x = 1;$$

on the other hand, we also have for $\varphi \in W_0^{2,2}(\Omega)$:

$$\int_\Omega a_{ijkl} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_k \partial x_l} \, \mathrm{d}x = \int_\Omega \sum_{\substack{|\alpha|\leq 2, |\beta| \leq 2 \\ |\alpha|+|\beta|<4}} b_{\alpha\beta} D^\alpha \varphi D^\beta v \, \mathrm{d}x,$$

hence v is an eigenfunction satisfying $A^* v = 0$ in Ω, $v \in W_0^{2,2}(\Omega)$, such that

$$\int_\Omega a_{ijkl} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_k \partial x_l} \, \mathrm{d}x = 1,$$

which is not possible. Now let us assume that for a choice of indices s_r, $r \to \infty$, $\lim |G_{s_r}^*| = \infty$. This implies that there exist $f_{s_r} \in L^2(\Omega_{s_r})$, $\lim_{r\to\infty} |f_{s_r}|_{L^2(\Omega_{s_r})} = 0$ such that the corresponding solutions v_{s_r} satisfy

$$\int_\Omega a_{ijkl} \frac{\partial^2 v_{s_r}}{\partial x_i \partial x_j} \frac{\partial^2 v_{s_r}}{\partial x_k \partial x_l} \, \mathrm{d}x = 1.$$

[5] We can actually take a more general condition.

Using the same approach as above, we have:

$$\int_{\Omega_{s_r}} a_{ijkl} \frac{\partial^2 v_{s_r}}{\partial x_i \partial x_j} \frac{\partial^2 v_{s_r}}{\partial x_k \partial x_l} \, dx = \int_{\Omega_{s_r}} v_{s_r} f_{s_r} \, dx + B(v_{s_r}, v_{s_r}),$$

and we get (after extraction of a subsequence denoted again v_{s_r}) $\lim_{r\to\infty} v_{s_r} = v$ weakly in $W_0^{2,2}(\Omega)$; from this we deduce as above that v is an eigenfunction, which is impossible. Finally, let $f \in L^2(\Omega)$, and v_s obtained in this lemma; we have $|v_s|_{W^{2,2}(\Omega)} \leq c_1 |f|_{L^2(\Omega)}$. Then $\lim_{s\to\infty} v_s = v$ weakly in $W_0^{2,2}(\Omega)$. If not, it will be possible to find two subsequences v_{s_r}, v_{s_t} such that $\lim_{r\to\infty} v_{s_r} = v_1$, $\lim_{t\to\infty} v_{s_t} = v_2$, $v_1 \neq v_2$, v_1, v_2 solutions of our problem which is impossible. Now we have according to Theorem 2.6.1:

$$\begin{aligned}
\int_{\Omega} a_{ijkl} \frac{\partial^2 (v-v_s)}{\partial x_i \partial x_j} \frac{\partial^2 (v-v_s)}{\partial x_k \partial x_l} \, dx &= \int_{\Omega} v f \, dx + B(v,v) + \int_{\Omega} v_s f \, dx + B(v_s, v_s) \\
- \int_{\Omega} a_{ijkl} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 v_s}{\partial x_k \partial x_l} \, dx &- \int_{\Omega} a_{ijkl} \frac{\partial^2 v_s}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_k \partial x_l} \, dx \to \\
2 \int_{\Omega} v f \, dx + 2B(v,v) &- 2 \int_{\Omega} v f \, dx - 2B(v,v) = 0.
\end{aligned}$$

Using also (5.58) the conclusion of the lemma follows. □

Let us observe that Lemma 4.3 is true under weaker hypotheses.

Exercise 4.1. Prove Lemma 4.3 supposing that Ω is a bounded domain and that $\lim_{s\to\infty} f_s = f$ in $W^{-2,2}(\Omega)$. The other hypotheses remain, there exists a sequence Ω_s, $\lim_{s\to\infty} \Omega_s = \Omega$ in the usual sense: for every compact $K \subset \Omega$, there exists s_0 such that $s \geq s_0 \Longrightarrow \Omega_s \supset K$.

5.4.5 A Density Lemma

We shall prove the following important density lemma:

Lemma 4.4. *Let $\Omega \in \mathfrak{M}$. Let us consider the subset, denoted by M, $M \subset W^{1,2}(\partial\Omega) \times L^2(\partial\Omega)$, of elements $(u, \partial u/\partial n)$, $u \in W^{2,2}(\Omega)$. Then M is dense in $W^{1,2}(\partial\Omega) \times L^2(\partial\Omega)$.*

Proof. Let $(g_0, g_1) \in W^{1,2}(\partial\Omega) \times L^2(\partial\Omega)$, φ_r be the function from the partition of unity in 1.2.4, and let us set $g_{0r} = g_0 \varphi_r$, $g_{1r} = g_1 \varphi_r + g_0 (\partial \varphi_r / \partial n)$. Clearly we have $(g_{0r}, g_{1r}) \in W^{1,2}(\partial\Omega) \times L^2(\partial\Omega)$. Now we use the local charts and we consider (g_{0r}, g_{1r}) as an element of $W^{1,2}(\Delta_r) \times L^2(\Delta_r)$. Let $\varepsilon > 0$; we can find $g_{0r\varepsilon} \in C_0^\infty(\Delta_r)$ such that $|g_{0r\varepsilon} - g_{0r}|_{W^{1,2}(\Delta_r)} < \varepsilon/2$ and $g_{1r\varepsilon} \in C_0^\infty(\Delta_r)$ such that $|g_{1r\varepsilon} - g_{1r}|_{W^{1,2}(\Delta_r)} < \varepsilon/2$. In U_r let us set $u_{rs}(x'_r, x_{rN}) = g_{0r\varepsilon}(x'_r) - h_{rs\varepsilon}(x'{}_r)(x_{rN} - a_{rs}(x'_r))$, denote

$$p_r = \left(1 + \sum_{1=1}^{N} \left(\frac{\partial a_r}{\partial x_{ri}}\right)^2\right)^{1/2}, \quad p_{rs} = \left(1 + \sum_{1=1}^{N} \left(\frac{\partial a_{rs}}{\partial x_{ri}}\right)^2\right)^{1/2},$$

and set:

$$h_{rs\varepsilon} = \frac{g_{1r\varepsilon}}{p_{rs}} - \frac{\sum_{i=1}^{N-1} \frac{\partial g_{0r\varepsilon}}{\partial x_{ri}} \frac{\partial a_{rs}}{\partial x_{ri}}}{p_{rs}^2}.$$

(Cf. the definition of $\mathfrak{M}$.) We have $u_{rs} \in C^\infty(\overline{U_r})$. Let $\tilde{g}_0 = u_{rs}$, $\tilde{g}_1 = \partial u_{rs}/\partial n$ on $\partial\Omega \cap U_r$. We have:

$$\tilde{g}_0 - g_{0r\varepsilon} = g_{1r\varepsilon}(x_r')(a_r - a_{rs}), \qquad \tilde{g}_1 - g_{1r\varepsilon} = \frac{1}{p_r p_{rs}^2}\Big[(p_{rs}^2 \sum_{i=1}^{N-1} \frac{\partial g_{0r\varepsilon}}{\partial x_{ri}} \frac{\partial a_r}{\partial x_{ri}})$$
$$- \left(1 + \sum_{1=1}^{N} \frac{\partial a_r}{\partial x_{ri}} \frac{\partial a_{rs}}{\partial x_{ri}}\right) \sum_{i=1}^{N-1} \frac{\partial g_{r0\varepsilon}}{\partial x_{ri}} \frac{\partial a_{rs}}{\partial x_{ri}}$$
$$+ p_{rs}\left(1 + \sum_{1=1}^{N-1} \frac{\partial a_r}{\partial x_{ri}} \frac{\partial a_{rs}}{\partial x_{ri}}\right) g_{1r\varepsilon} - p_r p_{rs}^2 g_{1r}\Big].$$

Now we use the definition of $\mathfrak{M}$: $\lim_{s\to\infty} a_{rs} = a_r$ in $W^{1,2}(\Delta_r)$, $|a_{rs}|_{C^{0,1}(\overline{\Delta_r})} \le$ const. If s is big enough, we obtain

$$|\tilde{g}_0 - g_{0r\varepsilon}|_{W^{1,2}(\Delta_r)} < \varepsilon/2, \quad |\tilde{g}_1 - g_{1r\varepsilon}|_{W^{1,2}(\Delta_r)} < \varepsilon/2. \tag{5.71}$$

We do not give all details,but only the general idea in the case of $L^2(\Delta_r)$:

$$\lim_{s\to\infty} \frac{1}{p_r} \sum_{i=1}^{N-1} \frac{\partial g_{0r\varepsilon}}{\partial x_{ri}} \frac{\partial a_r}{\partial x_{ri}} - \frac{1}{p_r p_{rs}^2}\left(1 + \sum_{1=1}^{N} \frac{\partial a_r}{\partial x_{ri}} \frac{\partial a_{rs}}{\partial x_{ri}}\right) \sum_{i=1}^{N} \frac{\partial g_{r0\varepsilon}}{\partial x_{ri}} \frac{\partial a_{rs}}{\partial x_{ri}} = 0. \tag{5.72}$$

Indeed: if we put

$$\left(1 + \sum_{1=1}^{N} \frac{\partial a_r}{\partial x_{ri}} \frac{\partial a_{rs}}{\partial x_{ri}}\right) \equiv \lambda_s,$$

then (5.72) is equivalent to

$$\lim_{s\to\infty}\left(p_{rs}^2 \frac{\partial g_{0r\varepsilon}}{\partial x_{ri}} \frac{\partial a_r}{\partial x_{ri}} - \lambda_s \frac{\partial g_{r0\varepsilon}}{\partial x_{ri}} \frac{\partial a_{rs}}{\partial x_{ri}}\right) = 0.$$

But for $i = 1, 2, \ldots, N-1$, $\partial g_{0r\varepsilon}/\partial x_{ri} \in C_0^\infty(\Delta_r)$, we must prove that

$$\lim_{s\to\infty}(p_{rs}^2 \partial a_r/\partial x_{ri} - \lambda_s \partial a_{rs}/\partial x_{ri}) = 0$$

in $L^2(\Delta_r)$. We have $\lim_{s\to\infty}\lambda_s(\partial a_{rs}/\partial x_{ri} - \partial a_r/\partial x_{ri}) = 0$ in $L^2(\Delta_r)$, thus it is sufficient to prove that $\lim_{s\to\infty}(p_{rs}^2 - \lambda_s) = 0$ in $L^2(\Delta_r)$, i.e.

$$\lim_{s\to\infty}\left(\sum_{i=1}^{N-1}\left(\frac{\partial a_{rs}}{\partial x_{ri}}\right)^2 - \sum_{i=1}^{N-1}\frac{\partial a_r}{\partial x_{ri}}\frac{\partial a_{rs}}{\partial x_{ri}}\right) = 0,$$

which is also equivalent to

$$\lim_{s\to\infty}\frac{\partial a_{rs}}{\partial x_{ri}}\left(\frac{\partial a_{rs}}{\partial x_{ri}} - \frac{\partial a_r}{\partial x_{ri}}\right) = 0 \text{ in } L^2(\Delta_r), \quad i = 1,2,\dots,N-1.$$

The last limit is zero, thus we obtain (5.72).

Let now $\psi \in C_0^\infty(U_r)$ be such that $\psi \equiv 1$ in a neighborhood of the supports of g_{0r}, g_{1r} on $\partial\Omega$; let us take $v_{rs} = u_{rs}\psi$. There is $v_{rs} \in C^\infty(\overline{\Omega})$,

$$|v_{rs} - g_{0r}|_{W^{1,2}(\Delta_r)} < \varepsilon, \quad \left|\frac{\partial v_{rs}}{\partial n} - g_{1r}\right|_{L^2(\Delta_r)} < \varepsilon,$$

then for some c_1

$$|v_{rs} - g_{0r}|_{W^{1,2}(\Omega)} < c_1\varepsilon, \quad \left|\frac{\partial v_{rs}}{\partial n} - g_{1r}\right|_{L^2(\partial\Omega)} < c_1\varepsilon.$$

Finally if we set

$$v = \sum_{r=1}^{m} v_{rs} \in C^\infty(\overline{\Omega}),$$

we can construct the element $(v, \partial v/\partial n)$, and we get

$$|v - g_0|_{W^{1,2}(\partial\Omega)} < c_1 m\varepsilon, \quad \left|\frac{\partial v}{\partial n} - g_1\right|_{L^2(\partial\Omega)} < c_1 m\varepsilon.$$

□

5.4.6 Regularity of the Solution

Now we prove:

Theorem 4.1. *Let $\Omega \in \mathfrak{M}$,[6] A be an operator satisfying (5.56)–(5.58), (5.61), (5.66)–(5.68). Let $f \in L^2(\Omega), v$ be the unique solution of $A^*v = f$ in Ω, with*

[6] We can take a more general condition.

$v = \partial v/\partial n = 0$ on $\partial\Omega$. Then there exists a unique linear, bounded mapping $T : L^2(\Omega) \to L^2(\partial\Omega)$, and another mapping $S : L^2(\Omega) \to W^{-1,2}(\partial\Omega)$ such that for all $u \in W^{2,2}(\Omega)$ we have:

$$\int_\Omega uf\,\mathrm{d}x = \langle u, Sf\rangle_{\partial\Omega} - \langle A_{ij}\frac{\partial u}{\partial x_i}n_j, Tf\rangle_{\partial\Omega} + \int_\Omega \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right)C_{kl}\frac{\partial^2 v}{\partial x_k\partial_l}\,\mathrm{d}x$$
$$- \int_\Omega B_{ikl}\frac{\partial u}{\partial x_i}\frac{\partial^2 v}{\partial x_k\partial x_l}\,\mathrm{d}x + \int_\Omega uD_{ij}\frac{\partial^2 v}{\partial x_k\partial x_l}\,\mathrm{d}x + \int_\Omega uE_i\frac{\partial v}{\partial x_i}\,\mathrm{d}x + \int_\Omega Fuv\,\mathrm{d}x. \tag{5.73}$$

Proof. Let v_s be the solution of the problem $A^*v_s = f$ in Ω, $v_s = \partial v_s/\partial n = 0$ on $\partial\Omega$. According to Theorem 4.2.2, $v_s \in W^{4,2}(\Omega)$, and

$$\int_{\Omega_s} u_r f\,\mathrm{d}x = \int_{\partial\Omega_s} u_r A_{ij}\frac{\partial}{\partial x_j}\left(C_{kl}\frac{\partial^2 v_s}{\partial x_k\partial x_l}\right)n_i\,\mathrm{d}S + \int_{\partial\Omega_s} u_r B_{ikl}\frac{\partial^2}{\partial x_k\partial x_l}n_i\,\mathrm{d}S$$
$$- \int_{\partial\Omega_s} A_{ij}\frac{\partial u_r}{\partial x_i}C_{kl}\frac{\partial^2 v_s}{\partial x_k\partial x_l}n_j\,\mathrm{d}S + \int_{\Omega_s}\frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial u_r}{\partial x_i}C_{kl}\frac{\partial^2 v_s}{\partial x_k\partial x_l}\right)n_i\,\mathrm{d}x$$
$$- \int_{\Omega_s} B_{ikl}\frac{\partial u_r}{\partial x_i}\frac{\partial^2 v_s}{\partial x_k\partial x_l}\,\mathrm{d}x + \int_{\Omega_s} u_r D_{ij}\frac{\partial^2 v_s}{\partial x_i\partial x_j}\,\mathrm{d}x + \int_{\Omega_s} u_r E_i\frac{\partial v_s}{\partial x_i}\,\mathrm{d}x + \int_{\Omega_s} Fu_r v_s\,\mathrm{d}x, \tag{5.74}$$

where $u_r = u\varphi_r$. Using Lemmas 4.1, 4.2, and also Lemma 1.4, the following functionals:

$$\int_{\partial\Omega_s} u_r A_{ij}\frac{\partial}{\partial x_j}\left(C_{kl}\frac{\partial^2 v_s}{\partial x_k\partial x_l}\right)n_i\,\mathrm{d}S + \int_{\partial\Omega_s} u_r B_{ikl}\frac{\partial^2 v_s}{\partial x_k\partial x_l}n_i\,\mathrm{d}S$$

can be considered as a sequence of functionals S_{rs} on $W^{1,2}(\Delta_r)$, evaluated at $u_r(x'_r, a_r(x'_r))$. But $\lim_{s\to\infty} u_r(x'_r, a_{rs}(x'_r)) = u_r(x'_r, a_r(x'_r))$ in $W_0^{1,2}(\Delta_r)$; using Lemmas 4.1, 4.2, we have $|S_{rs}|_{W^{-1,2}(\Delta_r)} \le c_1|f|_{L^2(\Omega)}$. We can extract a subsequence S_{rs_t} which is weakly convergent, $\lim_{t\to\infty} S_{rs_t} = S_r$; in the same way we consider the sequence T_{rs} of functionals in $L^2(\Delta_r)$, computed in $u_r(x'_r, a_{rs}(x_{r'}))$:

$$T_{rs} = \int_{\partial\Omega_s} A_{ij}\frac{\partial u_r}{\partial x_i}C_{kl}\frac{\partial^2 v_s}{\partial x_k\partial x_l}n_j\,\mathrm{d}S.$$

As above we can extract a subsequence $T_{rs\tau}$ which is weakly convergent, $\lim_{\tau\to\infty} T_{rs_\tau} = T_r$. Lemma 4.3 gives (5.73) for u_r, S_r, T_r. Let us set

$$S = \sum_{r=1}^m \varphi_r S_r, \quad T = \sum_{r=1}^m \varphi_r S_r;$$

after some computations we get:

$$\sum_{r=1}^{m}\left(A_{ij}\frac{\partial\varphi_r}{\partial x_i}n_j u, T_r f\right)_{\partial\Omega} = 0,$$

then finally (5.73) follows for u by summation of (5.73) calculated for u_r, φ_r, T_r. Lemma 4.4 gives the existence and uniqueness of Sf, Tf for f fixed. $\square$

Remark 4.1. From Theorem 4.1 we can see that $\lim_{s\to\infty} S_{rs} = S_r$, $\lim_{s\to\infty} T_{rs} = T_r$ weakly. Indeed for the first limit, if the result did not hold, there would exist $u\in W_0^{1,2}(\Delta_r)$ and two subsequences $S_{rs_i}, S_{r\sigma_j}$ such that $\lim_{i\to\infty} S_{rs_i}u \neq \lim_{j\to\infty} S_{r\sigma_j}u$. Without lass of generality we can take $\varphi \in C_0^\infty(\Delta_r)$ close to u such that $\lim_{i\to\infty} S_{rs_i}\varphi \neq \lim_{j\to\infty} S_{r\sigma_j}\varphi$. Now by Lemma 4.4 there exist $h_t \in W^{2,2}(\partial\Omega)$, $\lim_{t\to\infty} h_t = \varphi$ in $W^{1,2}(\partial\Omega)$, $\lim_{t\to\infty} \partial h_t/\partial n = 0$ in $L^2(\partial\Omega)$. (We define φ on $\partial\Omega$ by $\varphi(x_r') = \varphi(x_r', a_r(x_r'))$.) Let $\psi\in C_0^\infty(U_r)$, $\psi\equiv 1$ in a neighborhood of supp φ on $\partial\Omega$. We get $\lim_{t\to\infty} h_t\psi = \varphi$ in $W^{1,2}(\partial\Omega)$, $\lim_{t\to\infty}\partial h_t\psi/\partial n = 0$ in $L^2(\partial\Omega)$. If in (5.74) we replace u_r by $h_t\psi$ we obtain $\lim_{i\to\infty} S_{rs_i}h_t\psi \neq \lim_{j\to\infty} S_{r\sigma_j}h_t\psi$, which is a contradiction.

Exercise 4.2. Let $u\in W^{2,2N/(N+1)}(\Omega)$ be a very weak solution of $Au = 0$ in Ω where A is the operator from Theorem 4.1. Let us assume:

$$\lim_{s\to\infty} |u|_{W^{1,2}(\partial\Omega_s)} = 0, \quad \lim_{s\to\infty}\left|\frac{\partial u}{\partial n}\right|_{L^2(\partial\Omega_s)} = 0.$$

Then $u\equiv 0$. Hint: Use Remark 4.1.

5.4.7 Very Weak Solutions

Theorem 4.1 shows that we can use duality to find a *very weak solution* of the Dirichlet problem. First by Theorem 2.4.11, the Dirichlet problem for A can be formulated in the following form: let $(g_0, g_1)\in W^{1,2}(\partial\Omega)\times L^2(\partial\Omega)$ be generated by $u_0\in W^{2,2}(\Omega)$: $u_0 = g_0$ on $\partial\Omega$, $\partial u_0/\partial n = g_1$ on $\partial\Omega$. We are looking for $u\in W^{2,2}(\Omega)$ such that $Au = 0$ in Ω weakly, $u = g_0$, $\frac{\partial u}{\partial n} = g_1$ in the sense of traces. Theorems 3.2.1, 3.3.1 and also Theorem 2.4.12 guarantee the existence of a unique solution. Recall the set M defined in Lemma 4.4; we have

Theorem 4.2. *Let $\Omega\in\mathfrak{M}$, A be an operator satisfying (5.56)–(5.58), (5.61), (5.66)–(5.68). Then the Green operator $G: M\to W^{2,2}(\Omega)$ associated to the problem $Au = 0$ in Ω, $u = g_0$, $\partial u/\partial n = g_1$ on $\partial\Omega$ can be extended to a mapping $G\in[W^{1,2}(\partial\Omega)\times L^2(\partial\Omega)\to L^2(\Omega)]$.*

Proof. We use the local charts (x_r', x_{rN}) with the same notation as before: $p_r = \left(1+\sum_{1=1}^{N}\left(\frac{\partial a_r}{\partial x_{ri}}\right)^2\right)^{1/2}$ as in the proof of Lemma 4.4. Let be given $u_s\in W^{2,2}(\Omega)$ generating $(u_0, \partial u_0/\partial n)$ from M. Let us set $f(x_r') = u_0(x_r', a(x_r'))$, then:

$$\begin{aligned}\frac{\partial u_0}{\partial n} &= \frac{1}{p_r}\left(\sum_{i=1}^{N-1}\frac{\partial u_0}{\partial x_{ri}}\frac{\partial a_r}{\partial x_{ri}} - \frac{\partial u_0}{\partial u_{rN}}\right),\\ \frac{\partial f_r}{\partial x_{ri}} &= \frac{\partial u_0}{\partial x_{ri}} + \frac{\partial u_0}{\partial x_{rN}}\frac{\partial a_r}{\partial x_{ri}}.\end{aligned} \tag{5.75}$$

The determinant of the system (5.75) equals $-p_r \neq 0$, and we can compute $\partial u_s/\partial x_{ir}$ on $\partial\Omega$ using (5.75). This procedure can be applied for $r = 1,2,\ldots,$, and we obtain finally a linear bounded mapping $H \in [M \to L^2(\partial\Omega)]$ such that

$$H\left(u_0, \frac{\partial u_0}{\partial n}\right) = \sum_{i,j=1}^{N} A_{ij}\frac{\partial u_0}{\partial x_j}n_i.$$

Let us consider $u = G(u_0, \partial u_0/\partial n)$, where u is the solution obtained in Theorem 4.1. From (5.73) and from the existence of mappings S,T given in Theorem 4.1 the inequality

$$|u|_{L^2(\Omega)} \le c\left(|u_0|_{W^{1,2}(\partial\Omega)} + \left|\frac{\partial u_0}{\partial n}\right|_{L^2(\partial\Omega)}\right) \tag{5.76}$$

follows and we conclude with Lemma 4.4. □

Clearly, if $u \in C^{0,1}(\partial\Omega)$ on $\partial\Omega$, $\partial u/\partial x_i \in C^0(\partial\Omega)$, then Theorem 4.2 gives the existence of the solution. A natural question is

Problem 4.3. Does the above solution belong to $C^1(\Omega)$?

In the paper of the author [10] for domains with lipschitzian boundary, the space of boundary conditions $W^{1,2}(\partial\Omega) \times L^2(\partial\Omega)$ cannot be extended to the Dirichlet problem. An inequality analogous to (5.76) was proved for $\triangle^2$ and the spaces $C^1(\overline{\Omega}), \Omega$ smooth, $N = 2$, cf. G. Adler [3]:

$$|u|_{C^1(\overline{\Omega})} \le c_1(\partial\Omega)|u|_{C^1(\partial\Omega)} + c_2\left|\frac{\partial u}{\partial n}\right|_{C^0(\partial\Omega)}. \tag{5.77}$$

It is interesting to observe that the constant c_2 is an absolute constant independent of Ω.

It follows from Theorem 4.1.3 that the very weak solution found in Theorem 4.2 belongs to $W^{4,2}(\Omega')$ for every $\Omega' \subset \overline{\Omega}' \subset \Omega$ if the coefficients of the operator A are smooth enough; if the coefficients are in $C^\infty(\Omega)$, then u is in $C^\infty(\Omega)$ too. Concerning the boundary conditions, we can formulate the problem:

Problem 4.4. Let $\Omega \in \mathfrak{M}, u$ be the solution found in Theorem 4.2. Let us denote u_s, $\partial u_s/\partial n$, u, $\partial u/\partial n$, the values of the solutions and their normal derivative on $\partial\Omega_s$, $\partial\Omega$ respectively. Then is it possible to have, in the sense of (5.66): $\lim_{s\to\infty} u_s = u$ in $W^{1,2}(\partial\Omega)$, $\lim_{s\to\infty}\partial u_s/\partial n = \partial u/\partial n$ in $L^2(\partial\Omega)$?

Chapter 6
Boundary Value Problems in Weighted Sobolev Spaces

There exist plenty of references on weighted Sobolev spaces. In this chapter we shall consider only results related to boundary value problems for operators with nondegenerate and nonsingular coefficients. Many examples of weighted Sobolev spaces can be found in the papers of L.D. Kudriavcev [2], J.L. Lions [7, 8], S.V. Uspenskii [1–3], V.P. Illin [2], P.I. Lizorkin [1], J. Nečas [7], E.T. Poulsen [1], A. Kufner [2]; their applications are focused on elliptic equations with degenerate and singular coefficients, cf. for instance L.D. Kudriavcev [2], V.A. Sakharov [1], M.I. Vishik [5], etc. Concerning the boundary value problems studied in Chaps. 1,3, the use of weighted spaces can be found in the papers of M.I. Vishik [4], J. Nečas [3], [7], H. Morel [1], [2], A. Kufner [3], H.O. Cordes [1, 2].

For other questions related with these spaces cf. G. Cimmino [1], I.A. Kiprijanov [1], V.A. Kondratiev [1, 2], N.G. Meyers [1], M.I. Vishik [4].

6.1 A Second Order Equation, Regularity of Solution

6.1.1 The Case of $\partial\Omega$ Regular

Here we shall combine the results obtained in the previous chapters in the setting of weighted spaces.

Let us consider a sufficiently smooth domain Ω, and a second order operator:

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + d, \tag{6.1}$$

assume that

$$a_{ij} = a_{ji}, \quad a_{ij}, b_i \in C^{1,1}(\overline{\Omega}), \quad d \in C^{0,1}(\overline{\Omega})$$

J. Nečas et al., *Direct Methods in the Theory of Elliptic Equations*,
Springer Monographs in Mathematics, DOI 10.1007/978-3-642-10455-8_6,

are real functions; moreover we assume:

$$\xi \in \mathbb{R}^N \Longrightarrow \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq c|\xi|^2.$$

Let a real function $\vartheta \in C^{0,1}(\partial\Omega)$ be given. In this section we assume that all functions are real.

Theorem 1.1. *Let $\Omega \in \mathfrak{N}^{4,1}$, A be the operator (6.1), $h \in W^{1,2}(\partial\Omega)$; let u be the weak solution of the Dirichlet problem $Au = 0$ in Ω, $u = h$ on $\partial\Omega$. Then*

$$\int_\Omega \sum_{|i|\leq 2} \left|D^i u\right|^2 \rho \,\mathrm{d}x \leq c\big(|u|^2_{L^2(\Omega)} + |h|^2_{W^{1,2}(\partial\Omega)}\big) \tag{6.1 bis}$$

holds with $\rho(x) = dist(x, \partial\Omega)$.

Proof. We apply charts (σ, t) in a neighborhood of G_r to define $\partial\Omega$ and a partition of unity associated with these charts; without difficulty we obtain $\sigma \in \mathfrak{N}^{1,1}$ such that $c_1\sigma \leq \rho \leq c_2\sigma$. Let $h_n \in C^{2,1}(\partial\Omega)$, $\lim_{n\to\infty} h_n = h$ in $W^{1,2}(\partial\Omega)$, and let u_n be the solutions of the corresponding Dirichlet problems. We have the imbeddings $C^{2,1}(\partial\Omega) \subset W^{3/2,2}(\partial\Omega)$ algebraically and topologically, hence by Theorem 4.2.2 $u_n \in W^{3,2}(\Omega)$. Let τ be an integer, $1 \leq \tau \leq N$, and let us put $v = (\partial^2 u_n/\partial x_\tau^2)\sigma$; we have $(\partial^2 u_n/\partial^2 x_\tau)\sigma \in W_0^{1,2}(\Omega)$, and then

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^{N} a_{ij} \frac{\partial}{\partial x_i}\left(\frac{\partial^2 u_n}{\partial x_\tau^2}\sigma\right)\frac{\partial u_n}{\partial x_j}\,\mathrm{d}x + \int_\Omega \sum_{i=1}^{N} \frac{\partial^2 u_n}{\partial x_\tau^2}\sigma b_i \frac{\partial u_n}{\partial x_j}\,\mathrm{d}x + \int_\Omega d\sigma \frac{\partial^2 u_n}{\partial x_\tau^2} u_n \,\mathrm{d}x \\
&\qquad = -\int_\Omega \sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u_n}{\partial x_i \partial \tau}\frac{\partial^2 u_n}{\partial x_j \partial x_\tau}\sigma\,\mathrm{d}x - \int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u_n}{\partial x_i \partial x_\tau}\frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_\tau}\,\mathrm{d}x + \\
&\qquad + \int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u_n}{\partial x_\tau^2}\frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_i}\,\mathrm{d}x - \int_\Omega \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial x_\tau}\frac{\partial^2 u_n}{\partial x_i \partial x_\tau}\frac{\partial u_n}{\partial x_j}\sigma\,\mathrm{d}x + \\
&\qquad + \int_\Omega \sum_{i=1}^{N} \frac{\partial^2 u_n}{\partial x_\tau^2}\sigma b_i \frac{\partial u_n}{\partial x_i}\,\mathrm{d}x + \int_\Omega d\sigma \frac{\partial^2 u_n}{\partial x_\tau^2} u_n\,\mathrm{d}x.
\end{aligned} \tag{6.2}$$

Regarding the second integral on the right hand side in (6.2), we have

$$\begin{aligned}
2\int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial^2 u_n}{\partial x_i \partial x_\tau}\frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_\tau}\,\mathrm{d}x &= \int_{\partial\Omega} \sum_{i,j=1}^{N} a_{ij}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_\tau} n_\tau \,\mathrm{d}S \\
-\int_\Omega \sum_{i,j=1}^{N} \frac{\partial a_{ij}}{\partial x_\tau}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_\tau}\,\mathrm{d}x &- \int_\Omega \sum_{i,j=1}^{N} a_{ij}\frac{\partial u_n}{\partial x_\tau}\frac{\partial u_n}{\partial x_j}\frac{\partial^2 \sigma}{\partial x_\tau^2}\,\mathrm{d}x;
\end{aligned} \tag{6.3}$$

for the third integral on the right hand side in (6.2) we get

$$\begin{aligned}&\int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial^2 u_n}{\partial x_\tau^2}\frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_i}\,\mathrm{d}x\\ &=\int_{\partial\Omega}\sum_{i,j=1}^N a_{ij}\frac{\partial u_n}{\partial x_\tau}\frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_i}n_\tau\,\mathrm{d}S-\int_\Omega\sum_{i,j=1}^N\frac{\partial a_{ij}}{\partial x_\tau}\frac{\partial u_n}{\partial x_\tau}\frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_i}\,\mathrm{d}x\\ &-\int_\Omega\sum_{i,j=1}^N a_{ij}\frac{\partial u_n}{\partial x_\tau}\frac{\partial u_n}{\partial x_j}\frac{\partial^2 \sigma}{\partial x_i\partial x_j}\,\mathrm{d}x-\int_\Omega\sum_{i,j=1}^N a_{ij}\frac{\partial u_n}{\partial x_\tau}\frac{\partial^2 u_n}{\partial x_\tau\partial x_j}\frac{\partial \sigma}{\partial x_i}\,\mathrm{d}x;\end{aligned}\tag{6.4}$$

the last integral in (6.4) can be transformed into

$$\begin{aligned}&-\frac{1}{2}\int_\Omega\sum_{i,j=1}^N a_{ij}\frac{\partial}{\partial x_j}\left(\frac{\partial u_n}{\partial x_\tau}\right)^2\frac{\partial \sigma}{\partial x_i}\,\mathrm{d}x\\ &=-\frac{1}{2}\int_{\partial\Omega}\sum_{i,j=1}^N a_{ij}\left(\frac{\partial u_n}{\partial x_\tau}\right)^2\frac{\partial \sigma}{\partial x_i}n_j\,\mathrm{d}S+\frac{1}{2}\int_\Omega\sum_{i,j=1}^N\left(\frac{\partial u_n}{\partial x_\tau}\right)^2\frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial \sigma}{\partial x_i}\right)\mathrm{d}x.\end{aligned}\tag{6.5}$$

For the other terms in (6.2) we have

$$\begin{aligned}&\int_\Omega\sum_{i=1}^N\frac{\partial^2 u_n}{\partial x_\tau^2}\sigma b_i\frac{\partial u_n}{\partial x_i}\,\mathrm{d}x\\ &=-\int_\Omega\sum_{i=1}^N\frac{\partial u_n}{\partial x_\tau}\frac{\partial^2 u_n}{\partial x_i\partial x_\tau}b_i\sigma\,\mathrm{d}x-\int_\Omega\sum_{i=1}^N\frac{\partial u_n}{\partial x_\tau}\frac{\partial u_n}{\partial x_i}\frac{\partial}{\partial x_\tau}(b_i\sigma)\,\mathrm{d}x\\ &=\frac{1}{2}\int_\Omega\sum_{i=1}^N\left(\frac{\partial u_n}{\partial x_\tau}\right)^2\frac{\partial}{\partial x_i}(b_i\sigma)\,\mathrm{d}x-\int_\Omega\sum_{i=1}^N\frac{\partial u_n}{\partial x_\tau}\frac{\partial u_n}{\partial x_i}\frac{\partial}{\partial x_\tau}(b_i\sigma)\,\mathrm{d}x.\end{aligned}\tag{6.6}$$

Finally the last term in (6.2) gives:

$$\int_\Omega d\sigma\frac{\partial^2 u_n}{\partial x_\tau^2}u_n\,\mathrm{d}x=-\int_\Omega\frac{\partial u_n}{\partial x_\tau}\frac{\partial}{\partial x_\tau}(u_n d\sigma)\,\mathrm{d}x.\tag{6.7}$$

Now using the estimate (5.1.15) and (6.2)–(6.7), we have

$$\begin{aligned}&\int_\Omega\sum_{i=1}^N\left(\frac{\partial^2 u_n}{\partial x_i\partial x_\tau}\right)^2\sigma\,\mathrm{d}x\le c_1\Bigg[\left(\int_\Omega\sum_{i=1}^N\left(\frac{\partial^2 u_n}{\partial x_i\partial x_\tau}\right)^2\sigma\,\mathrm{d}x\right)^{1/2}\times\\ &\times\left(\int_\Omega\sum_{i=1}^N\left(\frac{\partial u_n}{\partial x_i}\right)^2\mathrm{d}x\right)^{1/2}+\int_\Omega\sum_{i=1}^N\left(\frac{\partial u_n}{\partial x_i}\right)^2\mathrm{d}x+|h_n|^2_{W^{1,2}(\partial\Omega)}\Bigg].\end{aligned}\tag{6.8}$$

But if λ is sufficiently large, then by Theorem 3.4.1 the sesquilinear form

$$\int_\Omega \left(\sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^N b_i v \frac{\partial u}{\partial x_i} + dvu \right) \mathrm{d}x + \lambda \int_\Omega vu \, \mathrm{d}x$$

is $W_0^{1,2}(\Omega)$-elliptic, hence

$$\int_\Omega \sum_{i=1}^N \left(\frac{\partial u_n}{\partial x_i} \right)^2 \mathrm{d}x \le c_2(|u_n|^2_{L^2(\Omega)} + |h_n|^2_{W^{1,2}(\partial\Omega)}).$$

From this estimate and from (6.8), we get (6.1 bis) for u_n; now letting $n \to \infty$ we have the result for u. □

According to Lemma 5.2.2, Theorem 4.2.2, and the previous theorem, we have:

Theorem 1.2. *Let $\Omega \in \mathfrak{N}^{4,1}$, A be the operator (6.1), $\gamma \in C^{1.1}(\partial\Omega)$, and let $u \in W^{1,2}(\Omega)$ be the solution of the problem $Au = 0$ in Ω, $\partial u/\partial \nu + \gamma u = h$ on $\partial\Omega$, $h \in L^2(\partial\Omega)$. Then*

$$\int_\Omega \sum_{|i|\le 1} \left|D^i u\right|^2 \rho \, \mathrm{d}x \le c(|u|^2_{L^2(\Omega)} + |h|^2_{L^2(\partial\Omega)}).$$

We prove theorems analogous to Theorems 1.1, 1.2 for operators of order $2k$ and for various problems; it is necessary to have inequalities of type (5.2.2). For that, it is possible to use the results of J.L. Lions, E. Magenes [1].

Remark 1.1. We can prove more than what is proved in Theorem 1.1; if the solution is unique for each $h \in W^{1,2}(\partial\Omega)$, let us set:

$$M = \{u \in W^{1,2}(\Omega), \quad \left(\int_\Omega \sum_{|i|\le 1} \left|D^i u\right|^2 \rho \, \mathrm{d}x \right)^{1/2} < \infty, \quad Au = 0 \text{ in } \Omega\}$$

with the natural topology. In this case the mapping $h \to u$ is an isomorphism T of $W^{1,2}(\partial\Omega)$ onto M; in Theorem 1.2: if there exists a unique solution for each $h \in L^2(\partial\Omega)$ then $h \to u$ is an isomorphism T of $L^2(\partial\Omega)$ onto M. Cf. later in this chapter.

6.1.2 The Very Weak Solution

Here we consider the case $\Omega \in \mathfrak{N}^{2,1}$.

Theorem 1.3. *Let us consider $\Omega \in \mathfrak{N}^{2,1}$, and let A be the operator (6.1), not necessarily symmetric, satisfying the hypotheses of Lemma 5.1.4, $h \in L^2(\partial\Omega)$. Let u*

be the very weak solution of the Dirichlet problem $Au = 0$ *in* $\Omega, u = h$ *on* $\partial\Omega$ *in the sense of Chap. 5. Then*

$$\int_\Omega \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2 \rho \,\mathrm{d}x \le c|h|^2_{L^2(\partial\Omega)}. \tag{6.9}$$

Proof. There exists a sequence $h_n \in C^{0,1}(\partial\Omega)$, $\lim_{n\to\infty} h_n = h$ in $L^2(\partial\Omega)$, and denote by u_n the associated solutions. Due to Theorem 4.2.1, we have $u_n \in W^{2,2}(\Omega)$, and according to Theorem 5.1.1:

$$\int_\Omega |u_n|^2 \,\mathrm{d}x \le c_1 \int_{\partial\Omega} |h_n|^2 \,\mathrm{d}S. \tag{6.9 bis}$$

Let us set $v_n = u_n\sigma, \sigma \in \mathfrak{N}^{1,1}$, $c_2\sigma \le \rho \le c_3\sigma$. We have $v_n \in W_0^{1,2}(\Omega)$, and

$$\begin{aligned}
0 &= \int_\Omega \Big(\sum_{i,j=1}^N a_{ij}\frac{\partial v_n}{\partial x_i}\frac{\partial u_n}{\partial x_j} + \sum_{i=1}^N b_i v_n \frac{\partial u_n}{\partial x_i} + d v_n u_n\Big) \,\mathrm{d}x \\
&= \int_\Omega \sum_{i,j=1}^N a_{ij}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}\sigma \,\mathrm{d}x + \int_\Omega \sum_{i,j=1}^N a_{ij} u_n \frac{\partial u_n}{\partial x_j}\frac{\partial \sigma}{\partial x_i}\,\mathrm{d}x \\
&+ \int_\Omega \sum_{i=1}^N b_i u_n \frac{\partial u_n}{\partial x_i}\sigma \,\mathrm{d}x + \int_\Omega u_n^2 d\sigma \,\mathrm{d}x.
\end{aligned} \tag{6.10}$$

The second integral on the right hand side in (6.10) can be written as

$$\begin{aligned}
\int_\Omega \sum_{i,j=1}^N a_{ij} u_n \frac{\partial u_n}{\partial x_i}\frac{\partial \sigma}{\partial x_j}\,\mathrm{d}x &= \frac{1}{2}\int_{\partial\Omega} \sum_{i,j=1}^N a_{ij}\frac{\partial \sigma}{\partial x_i}(u_n)^2 n_j \,\mathrm{d}S \\
&- \frac{1}{2}\int_\Omega \sum_{i,j=1}^N u_n^2 \frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial \sigma}{\partial x_i}\Big)\,\mathrm{d}x,
\end{aligned} \tag{6.11}$$

then due to (6.9 bis), (6.10) we get (6.9) for u_n and letting n tend to infinity the result follows. □

Let us denote

$$M = \{u \in L^2(\Omega), \quad \int_\Omega \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2 \rho \,\mathrm{d}x < \infty, \quad Au = 0 \text{ in } \Omega\}$$

and define on M the natural norm

$$|u|_M = \left(\int_\Omega u^2 dx + \int_\Omega \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2 \rho \,\mathrm{d}x\right)^{1/2}.$$

The previous theorem proves that the mapping $T : L^2(\partial\Omega) \to M$, $T(h) = u$ is linear and continuous. Then we have

Theorem 1.4. *Let us consider $\Omega \in \mathfrak{N}^{2,1}$, and let A be as in the previous theorem. Then $W^{1,2}(\Omega) \cap M$ is dense in M and the mapping $S : W^{1,2}(\Omega) \cap M \to L^2(\partial\Omega)$, $S(u) = h$ can be extended by continuity to $S \in [M \to L^2(\partial\Omega)]$.*

Proof. Let $u \in M$. Using local charts (σ, t) as in 1.2.4, we define $\Omega_s \subset \Omega$, where the boundary $\partial\Omega_s$ is the set of the points x defined by

$$x = y - n/s,\ y \in \partial\Omega, \tag{6.12}$$

with n the exterior normal at the point $y \in \partial\Omega$; without restriction of generality we can assume that $\partial\Omega_s$ is defined for $s = 1, 2, \ldots$.. There is a one-to-one correspondence between $\partial\Omega_s$ and $\partial\Omega$. Moreover, in local charts we have

$$x_s(\sigma) = y(\sigma) - \frac{1}{s} n(\sigma), \quad |i| \le 2, \quad \left| \frac{\partial^{|i|} x_s}{\partial\sigma_1^{i_1} \ldots \partial\sigma_{N-1}^{i_{N-1}}} \right| \le c_1,$$

where c_1 does not depend on s. There exists a function $\sigma \in C^{1,1}(\overline{\Omega})$ equivalent to ρ such that for $x \in \Omega$ and in a neighborhood of $\partial\Omega$, $\sigma(x) = t$ (in local charts, t is the distance between x and $\partial\Omega$ along the normal n). If s is sufficiently large, let us set $\sigma_s(x) = \sigma(x) - 1/s$, $x \in \overline{\Omega}_s$. Now we use (6.11) for u and $v_s = u\sigma_s$ in Ω_s. According to the ellipticity of the sesquilinear form $\sum_{i,j=1}^N a_{ij}\xi_i\xi_j$, we obtain immediately on $\partial\Omega_s$:

$$0 < c_3 \le -\sum_{i,j=1}^N a_{ij} \frac{\partial\sigma_s}{\partial x_i} n_j \le c_4, \tag{6.13}$$

where c_3, c_4 do not depend on s. From (6.10), (6.11), (6.13), it follows

$$\int_{\partial\Omega_s} u^2\, \mathrm{d}S \le c_5 |u|_M^2, \tag{6.14}$$

where c_5 is independent of s. Let us denote by u_s the restriction of u to $\partial\Omega_s$. Using the mapping (6.12) we compute (6.14) on $\partial\Omega$; now we can extract from the sequence u_s a subsequence which is weakly convergent in $L^2(\partial\Omega)$. Let h be the weak limit of this subsequence. Let $f \in L^2(\Omega), v$ the weak solution of $A^*v = f$ in Ω, $v = 0$ on $\partial\Omega$. We have $v \in W^{2,2}(\Omega)$ according to Theorem 4.2.2, and it follows that

$$\int_\Omega uf\, \mathrm{d}x = \int_{\partial\Omega} h\Big(\sum_{i,j=1}^N a_{ji} \frac{\partial v}{\partial x_j} n_i\Big)\, \mathrm{d}S,$$

whereupon theorem 5.1.2 implies that u is a very weak solution of the Dirichlet problem with the boundary condition $u = h$ on $\partial\Omega$. We can find $h_n \in W^{1/2,2}(\partial\Omega)$, such that $\lim_{n\to\infty} h_n = h$ in $L^2(\partial\Omega)$; if u_n solves the problem $Au_n = 0$ in Ω, $u_n = h_n$ on $\partial\Omega$, then $u_n \in W^{1,2}(\Omega)$, and by Theorem 1.3 $\lim_{n\to\infty} u_n = u$ in M; moreover (6.14) implies:

$$|u_n|^2_{L^2(\partial\Omega)} \le c_5 |u_n|^2_M. \tag{6.15}$$

□

Exercise 1.1. The hypotheses are the same as in the previous theorem. For $u \in M$, let $u_s \in L^2(\partial\Omega_s)$ denote the restriction of u to $\partial\Omega_s$. Using (6.12) prove that $u_s \to u$ in $L^2(\partial\Omega)$, where u is the trace of u on $\partial\Omega$; cf. J. Nečas [3].

6.1.3 The Case ∂Ω Non-smooth

Let us consider $\Omega_1, \Omega_2, \dots \Omega_p \in \mathfrak{N}^{2,1}$, $\Omega = \cap_{i=1}^p \Omega_i$. Let us denote by Λ_i the set $\partial\Omega \cap \partial\Omega_i$, and let Λ_i be open in $\partial\Omega$, $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$ and $\operatorname{meas}(\partial\Omega - \cup_{i=1}^p \Lambda_i) = 0$. If $\Omega \in \mathfrak{M}$, cf. Chap. 5, *we shall say that* $\Omega \in \mathfrak{R}$. The domains from $\mathfrak{R}$ have a *"piecewise smooth boundary"*.

We have

Lemma 1.1. *Let us consider $\Omega \in \mathfrak{R}$, $h \in L^2(\partial\Omega)$, $h \neq 0$ on Λ_i. Then there exists a sequence of functions $h_n \in W^{1/2,2}(\partial\Omega)$, $\operatorname{supp} h_n \subset \Lambda_i$, $\lim_{n\to\infty} h_n = h$ in $L^2(\partial\Omega)$.*

Proof. We approximate h by $g \in L^2(\partial\Omega)$, $\operatorname{supp} g \subset \Lambda_i$; we cover $\operatorname{supp} g$ with domains $G_1, G_2, \dots, G_\kappa$ of the type G from 1.2.4 and let $\Psi_j \in C_0^\infty(G_j)$ be such that $x \in \operatorname{supp} g \Longrightarrow \sum_{j=}^{\kappa} \Psi_j(x) = 1$. Let us set $g_j = g\Psi_j$, and approximate g_j by $\tilde{g}_j$ in $W^{1,2}(\partial\Omega)$, $\operatorname{supp} \tilde{g}_j \subset \Lambda_i \cap G_j$, and extend $\tilde{g}_j$ to Ω as in the proof of Lemma 5.1.1. Then $\operatorname{supp} \sum_{j=1}^{\kappa} \tilde{g}_j \subset \Lambda_i$, $\tilde{g}_j \in W^{1,2}(\partial\Omega)$ and $\sum_{j=1}^{\kappa} \tilde{g}_j$ approximates h in $L^2(\partial\Omega)$. □

For each Ω_i defined previously, we can construct a function $\sigma_i \in C^{1,1}(\overline{\Omega}_i)$, $i = 1, 2, \dots, p$, equivalent to $\operatorname{dist}(x, \partial\Omega_i)$. Hereafter we shall use these notations.

Theorem 1.5. *Let us consider $\Omega \in \mathfrak{R}$, $h \in L^2(\partial\Omega)$, $\operatorname{supp} h \subset \Lambda_\alpha$. Let u be a very weak solution of the Dirichlet problem $Au = 0$ in Ω, $u = h$ on $\partial\Omega$, where A is the operator as in Theorem 5.1.2. Then*

$$\int_\Omega \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2 \sigma_\alpha \, dx \le c \int_{\partial\Omega} h^2 \, dS. \tag{6.16}$$

Proof. Let h_n be as in Lemma 1.1, u_n the corresponding solution, $u_n \in W^{1,2}(\Omega)$. According to Theorem 5.1.2 we have:

$$\int_\Omega u_n^2 \, dx \le c_1 \int_{\partial\Omega} h_n^2 \, dS. \tag{6.17}$$

Setting $v_n = u_n\sigma_\alpha$, we have $v_n \in W_0^{1,2}(\Omega)$; it follows from (6.10), (6.11) that inequality (6.16) holds for u_n. When we pass to the limit $\lim_{n\to\infty} u_n = u$ we get (6.16) for u. □

Theorem 1.6. *Let us consider $\Omega \in \mathfrak{N}$, $h \in L^2(\partial\Omega)$, u a very weak solution of the Dirichlet problem $Au = 0$ in Ω, $u = h$ on $\partial\Omega$, where A is the operator as in Theorem 5.1.2. Then*

$$\int_\Omega \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2 \rho \,\mathrm{d}x + \int_\Omega u^2 \,\mathrm{d}x \le c|h|^2_{L^2(\partial\Omega)}. \tag{6.18}$$

Proof. Indeed, we observe that h can be approximated in $L^2(\partial\Omega)$ by $\sum_{\alpha=1}^p h_\alpha$, $\operatorname{supp} h_\alpha \subset \Lambda_\alpha$, and (6.16) implies:

$$\int_\Omega \sum_{i=1}^N \left(\frac{\partial u_\alpha}{\partial x_i}\right)^2 \rho \,\mathrm{d}x \le c_1 \int_\Omega \sum_{i=1}^N \left(\frac{\partial u_\alpha}{\partial x_i}\right)^2 \sigma_\alpha \,\mathrm{d}x \le c_3 \int_{\partial\Omega} h_\alpha^2 \,\mathrm{d}S.$$

□

Let us remark that it is possible to obtain Theorem 1.4 and the convergence as in Exercise 1.1 in the case $\Omega \in \mathfrak{N}$ with other interesting properties (found in applications); cf. the paper of the author [3].

Problem 1.1. Prove Theorem 1.5 if $\Omega \in \mathfrak{N}^{0,1}$.

Remark 1.2. In Neumann, Newton problems, i.e $Au = 0$ in Ω, $\partial u/\partial n + \gamma u = h$ on $\partial\Omega$, $\gamma \in C^{0,1}(\partial\Omega)$, $h \in W^{-1,2}(\partial\Omega)$, cf. Chap. 5, Sect. 5.2, according to Corollary 5.2.1 and (6.18) we have:

$$\int_\Omega \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2 \rho \,\mathrm{d}x \le c|h|^2_{W^{-1,2}(\partial\Omega)}. \tag{6.19}$$

6.2 The Dirichlet Problem and Spaces $W_\sigma^{k,p}$

6.2.1 Density Theorem

In the previous section we have considered the space of $u \in L^2(\Omega)$ such that

$$\int_\Omega \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2 \rho \,\mathrm{d}x < \infty.$$

In a more general case, let Ω be a bounded domain (the case of an unbounded domain also is possible), and σ a function from $C(\Omega)$, $\sigma > 0$ in Ω. Let us define for $p \ge 1$, k an integer:

$$W_\sigma^{k,p}(\Omega) \equiv \{u, \quad \sum_{|i|\le k} \int_\Omega \left|D^i u\right|^p \sigma \,\mathrm{d}x \equiv |u|^p_{W_\sigma^{k,p}(\Omega)} < \infty\}.$$

If $v,u \in W_\sigma^{k,2}(\Omega)$, we put

$$(v,u)_{W_\sigma^{k,2}(\Omega)} = \sum_{|i|\le k} \int_\Omega D^i v D^i \overline{u} \sigma \,\mathrm{d}x.$$

If $\sigma \in C^k(\Omega)$, let us denote:

$$H_\sigma^{k,2}(\Omega) \equiv \{u, \quad \sum_{|i|\le k} \int_\Omega \left|D^i(u\sigma)\right|^2 \sigma^{-1}\,\mathrm{d}x \equiv |u|^2_{H_\sigma^{k,2}(\Omega)} < \infty\},$$

and for $v,u \in H_\sigma^{k,2}(\Omega)$

$$(v,u)_{H_\sigma^{k,2}(\Omega)} = \int_\Omega \sum_{|i|\le k} D^i(v\sigma) D^i(\overline{u}\sigma)\sigma^{-1}\,\mathrm{d}x.$$

Now let us consider a weight $\sigma \in C(\Omega)$, $\sigma > 0$ in Ω and let us assume that $\Omega \in \mathfrak{N}^0$ and that in the local charts (x'_r, x_{rN}), $r = 1,2,\ldots,m$:

$$c_1\sigma(x) \le x_{rN} - a_r(x'_r) + \kappa_r(x'_r) \le c_2\sigma(x), \tag{6.20}$$

where c_1, c_2, are constants, and $0 \le \kappa_r(x'_r) \le c_3$, where c_3 is another constant; $\kappa_r(x'_r)$ is a function defined in Δ_r.

We have:

Theorem 2.1. *Let us consider $\Omega \in \mathfrak{N}^0, \sigma$ a weight satisfying (6.20). Let $\alpha \ge 0$. Then $C^\infty(\overline{\Omega})$ is dense in $W_{\sigma^\alpha}^{k,p}(\Omega)$.*

Proof. Let $u \in W_{\sigma^\alpha}^{k,p}(\Omega)$ and $u_r = u\varphi_r$; for φ_r, U_r, V_r, etc. cf. 1.2.4. We have $u_r \in W_{\sigma^\alpha}^{k,p}(\Omega)$. Let $1 \le r \le m$, and let us suppose in the charts (x'_r, x_{rN}):

$$\lambda > 0, \quad u_{r\lambda}(x'_r, x_{rN}) = u_r(x'_r, x_{rN} + \lambda), \quad x \notin V_r \Longrightarrow u_{r\lambda}(x) = 0.$$

We have in $W_{\sigma^\alpha}^{k,p}(\Omega)$:

$$\lim_{\lambda\to 0} u_{r\lambda} = u_r. \tag{6.21}$$

To prove this, let us consider $D^i u_{r\lambda}, |i| \le k$, and denote $D^i u_r = g$. We have:

$$\begin{aligned}
&\left(\int_{V_r} |g(x'_r,x_{rN}) - g(x'_r,x_{rN}+\lambda)|^p \sigma^\alpha(x)\,\mathrm{d}x\right)^{1/p} \\
\le &c_1\left(\int_{V_r} |g(x'_r,x_{rN})\sigma^{\alpha/p}(x'_r,x_{rN}) - g(x'_r,x_{rN}+\lambda)\sigma^{\alpha/p}(x'_r,x_{rN}+\lambda)|^p\,\mathrm{d}x\right)^{1/p} \\
+&c_1\left(\int_{V_r} |g(x'_r,x_{rN}+\lambda)|^p[\sigma^{\alpha/p}(x'_r,x_{rN}+\lambda) - \sigma^{\alpha/p}(x'_r,x_{rN})]^p\,\mathrm{d}x\right)^{1/p}.
\end{aligned} \tag{6.22}$$

Let $\varepsilon > 0$. According to (6.20), there exists $0 < \delta < \beta/4$, such that for $\lambda \le \delta$, $V_{r\delta} = \{x \in \mathbb{R}^N,\ x = (x'_r, x_{rN}),\ |x_{ri}| < \alpha,\ i = 1,2,\dots,N-1,\ a_r(x'_r) < x_{rN} < a_r(x'_r) + \delta\}$, we have:

$$c_1 \left(\int_{V_{r\delta}} |g(x'_r, x_{rN} + \lambda)|^p [\sigma^{\alpha/p}(x'_r, x_{rN} + \lambda) - \sigma^{\alpha/p}(x'_r, x_{rN})]^p \,\mathrm{d}x \right)^{1/p} \le \varepsilon/3.$$

However,

$$\lim_{\lambda \to 0} \left(1 - \frac{\sigma(x'_r, x_{rN})}{\sigma(x'_r, x_{rN} + \lambda)} \right) = 0$$

uniformly for

$$x'_r \in \Delta_r, \quad a_r(x'_r) + \delta/2 \le x_{rN} \le a_r(x'_r) + \beta/2,\ 0 \le \lambda \le \beta/2$$

and we can find $\lambda_0 > 0$, such that $\lambda \le \lambda_0 \Longrightarrow$

$$c_1 \left(\int_{V_r - V_{r\delta}} |g(x'_r, x_{rN} + \lambda)|^p [\sigma^{\alpha/p}(x'_r, x_{rN} + \lambda) - \sigma^{\alpha/p}(x'_r, x_{rN})]^p \,\mathrm{d}x \right)^{1/p} < \varepsilon/3.$$

Now using Theorem 2.1.1, we find λ_1 such that $\lambda < \lambda_1 \Longrightarrow$

$$c_1 \left(\int_{V_r} |\sigma^{\alpha/p}(x'_r, x_{rN} + \lambda) g(x'_r, x_{rN} + \lambda) - \sigma^{\alpha/p}(x'_r, x_{rN}) g(x'_r, x_{rN})|^p \,\mathrm{d}x \right)^{1/p} \le \varepsilon/3,$$

and then (6.22) and the previous computations imply (6.21). We have $u_{r\lambda} \in W^{k,p}(\Omega_\lambda)$ for $\lambda > 0$ where $\overline{\Omega} \subset \Omega_\lambda$, and using the regularization operator with h sufficiently small, by Theorem 2.3.1 on Ω, we can construct a sequence of functions $u_{r\lambda n} \in C^\infty(\overline{\Omega})$, such that $\lim_{n \to \infty} u_{r\lambda n} = u_{r\lambda}$ in $W^{k,p}(\Omega)$ and *a fortiori* in $W^{k,p}_{\sigma^\alpha}(\Omega)$. We have $u_{m+1} \in W^{k,p}(\Omega)$, $\operatorname{supp} u_{m+1} \subset \Omega$ and hence also $\lim_{\lambda \to 0} u_{m+1,\lambda} = u_{m+1}$ in $W^{k,p}(\Omega)$, and then in $W^{k,p}_{\sigma^\alpha}(\Omega)$. □

Remark 2.1. Without difficulty we prove that for $\Omega \in \mathfrak{N}^{0,1}$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ satisfies (6.20) with $\kappa_r \equiv 0$. If $y \in \partial\Omega$, $\Omega \in \mathfrak{N}^0$, $\sigma = |x - y|$ and if there exists a cone with vertex y in the complement of Ω, with axis parallel to the axis x_{rN}, r well chosen, $y \in U_r$, then σ satisfies the condition (6.20) with $\kappa_r(x'_r) = |y - x^\bullet|$, $x^\bullet = (x'_r, a_r(x'_r))$.

6.2.2 The Trace Problem

We define *traces* as in Chap. 2, Sect. 4.2, using the mapping $T : C^\infty(\overline{\Omega}) \to L^p(\partial\Omega)$: for $u \in C^\infty(\overline{\Omega})$, $Tu = u$ on $\partial\Omega$; here we shall prove only:

Theorem 2.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, σ satisfying (6.20), $0 \le \alpha < p-1$. Then the mapping T can be extended by continuity to $T \in [W_{\sigma^\alpha}^{1,p}(\Omega) \to L^p(\partial\Omega)]$.*

Proof. According to (6.20) and Remark 2.1, we have $W_{\sigma^\alpha}^{1,p}(\Omega) \subset W_{\rho^\alpha}^{1,p}(\Omega)$ algebraically and topologically, where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. It is enough to consider the last spaces.

In local charts (x_r', x_{rN}), we have for $u \in C^\infty(\overline{\Omega})$:

$$u(x_r', a_r(x_r')) = u(x_r', \eta) - \int_{a_r(x_r')}^{\eta} \frac{\partial u}{\partial x_{rN}}(x_r', \xi)\,\mathrm{d}\xi, \quad a_r(x_r') \le \eta \le a_r(x_r') + \beta,$$

thus

$$|u(x_r', a_r(x_r')|^p \le 2^{p-1}\left[|u(x_r', \eta)|^p + \left(\int_{a_r(x_r')}^{a_r(x_r')+\beta} (\xi - a_r(x_r'))^{-\alpha(p-1)}\,\mathrm{d}\xi\right)^{p-1}\right] \times \\ \times \left[\int_{a_r(x_r')}^{a_r(x_r')+\beta} \left|\frac{\partial u}{\partial x_{rN}}(x_r', \xi)\right|^p (\xi - a_r(x_r'))^\alpha\,\mathrm{d}\xi\right]. \tag{6.23}$$

Integrating (6.23) with respect to $\eta \in [a_r(x_r') + \beta/2, a_r(x_r') + \beta]$, we get:

$$|u(x_r', a_r(x_r'))|^p \le c_1\left[\int_{a_r(x_r')+\beta/2}^{a_r(x_r')+\beta} |u(x_r', \xi)|^p\,\mathrm{d}\xi \right. \\ \left. + \int_{a_r(x_r')}^{a_r(x_r')+\beta} \left|\frac{\partial u}{\partial x_{rN}}(x_r', \xi)\right|^p (\xi - a_r(x_r'))^\alpha\,\mathrm{d}\xi\right], \tag{6.24}$$

and by integration with respect to x_r' on Δ_r we get the result. □

Example 2.1. If $\alpha = p-1$, then Theorem 2.2 does not hold in general. Let $\Omega = \{x, 0 < x_i < 1/2\}$, and let us put

$$u(x) = \int_{1/2}^{x_N} \frac{\mathrm{d}t}{t \log t};$$

we have $u \in W_{\rho^{p-1}}^{1,p}$, but

$$-\int_0^{1/2} \frac{\mathrm{d}t}{t \log t} = \infty.$$

Cf. also E.T. Poulsen [1].

Exercise 2.1. Modify Lemma 2.5.2 (it is possible to modify Theorem 2.5.3 and the converse theorems; we can use the methods from Chap. 2, Sect. 5.2; cf. also P.I. Lizorkin [1], J.L. Lions [7]) in the following direction: let

$$\int_{\Delta}\left(|u|^p + \left|\frac{\partial u}{\partial x_1}\right|^p + \left|\frac{\partial u}{\partial x_2}\right|^p\right)(x_1 - x_2)^\alpha \, \mathrm{d}x < \infty, \quad p > 1, -1 < \alpha < p-1.$$

Then we get

$$\int_0^1 \int_0^1 \frac{|u(t,t) - u(\tau,\tau)|^p}{|t-\tau|^{p-\alpha}} \, \mathrm{d}t \, \mathrm{d}\tau \le c \int_{\Delta}\left(|u|^p + \left|\frac{\partial u}{\partial x_1}\right|^p + \left|\frac{\partial u}{\partial x_2}\right|^p\right)(x_1 - x_2)^\alpha \, \mathrm{d}x.$$

Hint: Use the Hardy inequalities given in 2.5.2.

6.2.3 Some Imbedding Theorems

We prove as in Lemma 2.5.1:

Lemma 2.1. *Let $p > 1$, $\alpha \neq p-1$, $\int_0^\infty |u(x)|^p x^\alpha \, \mathrm{d}x < \infty$. Then we have for $\alpha < p-1$*

$$\int_0^\infty \left(\int_0^x |u(\xi)| \, \mathrm{d}\xi\right)^p x^{\alpha-p} \, \mathrm{d}x \le \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty |u(x)|^p x^\alpha \, \mathrm{d}x \tag{6.25}$$

and for $\alpha > p-1$

$$\int_0^\infty \left(\int_x^\infty |u(\xi)| \, \mathrm{d}\xi\right)^p x^{\alpha-p} \, \mathrm{d}x \le \left(\frac{p}{\alpha-p+1}\right)^p \int_0^\infty |u(x)|^p x^\alpha \, \mathrm{d}x. \tag{6.26}$$

We denote by $W^{k,p}_{0,\sigma^\alpha}(\Omega)$ (resp. by $H^{k,2}_{0,\sigma^\alpha}(\Omega)$) the closure of $C_0^\infty(\Omega)$ in $W^{k,p}_{\sigma^\alpha}(\Omega)$ (resp. in $H^{k,2}_{\sigma^\alpha}(\Omega)$).

Theorem 2.3. *Let $\Omega \in \mathfrak{N}^0$, $\alpha > p-1$, $p > 1$, and suppose σ satisfies (6.20). Then $W^{k,p}_{0,\sigma^\alpha}(\Omega) \subset W^{k-l,p}_{0,\sigma^{(\alpha-lp)}}(\Omega)$, $l = 1,2,\ldots k$, algebraically and topologically. Moreover for $u \in W^{k,p}_{0,\sigma^\alpha}(\Omega)$,*

$$c_1 |u|_{W^{k,p}_{\sigma^\alpha}(\Omega)} \le \left(\sum_{|i|=k} \int_\Omega |D^i u|^p \sigma^\alpha \, \mathrm{d}x\right)^{1/p} \le c_2 |u|_{W^{k,p}_{\sigma^\alpha}(\Omega)}.$$

Proof. Let $u \in C_0^\infty(\Omega)$; it is sufficient to consider the case $k=1$. According to (6.25), we have

$$\begin{aligned}
\int_{V_r} |u(x)|^p \sigma^{(\alpha-p)} \, \mathrm{d}x &\le c_3 \int_{\Delta_r} \mathrm{d}x_r' \int_{a_r(x_r')}^{a_r(x_r')+\beta} |u(x_r', x_{rN})|^p (x_{rN} - a_r(x_r') + \kappa_r(x_r'))^{\alpha-p} \, \mathrm{d}x_{rN} \\
&\le c_4 \int_{\Delta_r} \mathrm{d}x_r' \int_{a_r(x_r')}^{a_r(x_r')+\beta} \left|\frac{\partial u(x_r', x_{rN})}{\partial x_{rN}}\right|^p (x_{rN} - a_r(x_r') + \kappa_r(x_r'))^\alpha \, \mathrm{d}x_{rN} \\
&\le c_5 \int_{V_r} \left|\frac{\partial u}{\partial x_{rN}}\right|^p \sigma^\alpha \, \mathrm{d}x.
\end{aligned} \tag{6.27}$$

□

Theorem 2.4. *Let $\Omega \in \mathfrak{N}^0$, $p > 1$, $\alpha > p-1$, and suppose σ satisfies (6.20). Then $W^{1,p}_{\sigma^\alpha}(\Omega) \subset L^p_{\sigma^{(\alpha-p)}}(\Omega)$ algebraically and topologically.*

Proof. Using a partition of unity as in 1.2.4, we consider $u_r = u\varphi_r$, $u \in W^{1,p}_{\sigma^\alpha}$. Obviously,

$$|u_r|_{W^{1,p}_{\sigma^\alpha}(V_r)} \le c_1 |u|_{W^{1,p}_{\sigma^\alpha}(\Omega)}. \tag{6.28}$$

Using (6.26), and computing as in (6.27), it is sufficient to assume $u \in C^\infty(\Omega)$ and use Theorem 2.1. □

Theorem 2.5. *Let $\Omega \in \mathfrak{N}^0$, $p > 1$, $\alpha > p-1$ (resp. $\alpha < p-1$), and suppose σ satisfies (6.20). Let $\alpha_1 > \alpha$. Then the imbedding $W^{1,p}_{\sigma^\alpha}(\Omega) \to L^p_{\sigma^{(\alpha_1-p)}}(\Omega)$ (resp. $W^{1,p}_{0,\sigma^\alpha}(\Omega) \to L^p_{\sigma^{(\alpha_1-p)}}(\Omega)$) is compact.*

Proof. Let us consider first the case $\alpha > p-1$ and the unit ball $|u|_{W^{1,p}_{\sigma^\alpha}(\Omega)} \le 1$. In $L^p_{\sigma^{(\alpha_1-p)}}(\Omega)$, for each $\varepsilon > 0$, there exists an ε-net. Indeed, let us denote:

$$V_\lambda = \{x, x = (x'_r, x_{rN}),\ x'_r \in \Delta_r, \quad a_r(x'_r) < x_{rN} < a_r(x'_r) + \lambda\}, \quad \lambda \le \beta.$$

We can find a sufficiently small $\lambda > 0$ such that

$$|u|_{W^{1,p}_{\sigma^\alpha}(\Omega)} \le 1 \Longrightarrow |u|_{L^p_{\sigma^{(\alpha_1-p)}}(V_\lambda)} \le \varepsilon/3m. \tag{6.29}$$

Indeed we have, for $u \in C^\infty(\overline{\Omega})$, $a_r(x'_r) < x_{rN} < a_r(x'_r) + \lambda$, $a_r(x'_r) + \beta/2 < y < a_r(x'_r) + \beta$,

$$\begin{aligned} |u(x'_r, x_{rN})|^p \le c_1 \Bigg[|u(x'_r, y)|^p + \left(\int_{x_{rN}}^{y} (\xi - a_r(x'_r) + \kappa_r(x'_r))^{-\alpha/(p-1)} \,\mathrm{d}\xi \right)^{p-1} \times \\ \times \int_{a_r(x'_r)}^{a_r(x'_r)+\beta} \left| \frac{\partial u}{\partial x_{rN}}(x'_r, \xi) \right|^p \sigma^\alpha \,\mathrm{d}\xi \Bigg]. \end{aligned} \tag{6.30}$$

Let us integrate with respect to $y \in (a_r(x'_r) + \beta/2 < y < a_r(x'_r) + \beta)$. We get

$$\begin{aligned} |u(x'_r, x_{rN})|^p \le c_2 \int_{a_r(x'_r)+\beta/2}^{a_r(x'_r)+\beta} |u(x'_r, y)|^p \sigma^{\alpha-p} \,\mathrm{d}y \\ + c_2 (x_{rN} - a_r(x'_r) + \kappa_r(x'_r))^{(p-1)} \int_{a_r(x'_r)}^{a_r(x'_r)+\beta} \left| \frac{\partial u}{\partial x_{rN}}(x'_r, \xi) \right|^p \sigma^\alpha \,\mathrm{d}\xi, \end{aligned}$$

and then

$$\begin{aligned}\int_{a_r(x_r')}^{a_r(x_r')+\lambda} &|u(x_r',x_{rN})|^p\sigma^{\alpha_1-p}\,\mathrm{d}x_{rN}\\ &\le c_3[(c_4+\lambda)^{\alpha_1-p+1}-c_4^{\alpha_1-p+1}]\int_{a_r(x_r')+\beta/2}^{a_r(x_r')+\beta}|u(x_r',y)|^p\sigma^{\alpha-p}\,\mathrm{d}y \\ &+c_5[(c_6+\lambda)^{(\alpha_1-\alpha)}-c_6^{(\alpha_1-\alpha)}]\int_{a_r(x_r')}^{a_r(x_r')+\beta}\left|\frac{\partial u}{\partial x_{rN}}(x_r',\xi)\right|^p\sigma^{\alpha}\,\mathrm{d}\xi,\end{aligned} \tag{6.31}$$

where $0\le c_4, 0\le c_6$.

From (6.31) we get the inequality (6.29).

If $\alpha<p-1$, we have for $\varphi\in C_0^\infty(\Omega)$, $a_r(x_r')<x_{rN}<a_r(x_r')+\lambda$:

$$\begin{aligned}|u(x_r',x_{rN})|^p\le c_7\left(\int_{a_r(x_r')}^{x_{rN}}(\xi-a_r(x_r')+\kappa_r(x_r'))^{-\alpha/(p-1)}\,\mathrm{d}\xi\right)^{(p-1)}\times\\ \times\int_{a_r(x_r')}^{a_r(x_r')+\beta}\left|\frac{\partial u}{\partial x_{rN}}(x_r',\xi)\right|^p\sigma^{\alpha}\,\mathrm{d}\xi,\end{aligned}$$

thus

$$\begin{aligned}\int_{a_r(x_r')}^{a_r(x_r')+\lambda}&|u(x_r',x_{rN})|^p\sigma^{(\alpha_1-p)}\,\mathrm{d}x_{rN}\\ &\le c_8((c_9+\lambda)^{(\alpha_1-\alpha)}-c_9^{(\alpha_1-\alpha)})\int_{a_r(x_r')}^{a_r(x_r')+\beta}\left|\frac{\partial u}{\partial x_{rN}}(x_r',\xi)\right|^p\sigma^{\alpha}\,\mathrm{d}\xi\end{aligned}$$

which implies (6.29), where $c_9\ge 0$.

Let $\Omega_1=\Omega-\bigcup_{i=1}^m V_i$. It follows from Theorem 2.6.1 that there exists an $\varepsilon/3$-net for the set of u such that $|u|_{W^{1,p}_{\sigma^\alpha}(\Omega_1)}\le 1$, say $u_1,u_2,\dots,u_h$; it is an ε-net in $L^p_{\sigma^{\alpha_1-p}}(\Omega)$ as well. □

6.2.4 The Dirichlet Problem, Very Weak Solution

Let Ω be a bounded domain, and A be the operator

$$A=\sum_{|i|,|j|\le k}(-1)^{|i|}D^i\left(a_{ij}D^j\right),$$

with the associated sesquilinear form $A(v,u)$. Let $\sigma\in C^{k,1}(\overline{\Omega})$, $\sigma>0$ a weight on Ω satisfying (6.20). We have the fundamental theorem, cf. also M.I. Vishik [4]:

Theorem 2.6. *Let us consider $\Omega\in\mathfrak{N}^0$, $\sigma\in C^{k,1}(\overline{\Omega})$, $\sigma>0$, satisfying (6.20). We assume the sesquilinear form $A(v,u)$ is $W_0^{k,2}(\Omega)$-elliptic with coefficients a_{ij} from $C^{0,1}(\overline{\Omega})$, which are real functions for $|i|=|j|=k$ and for real numbers ξ_i satisfy*

$$\frac{1}{2}\sum_{|i|=|j|=k}(a_{ij}+a_{ji})\zeta_i\zeta_j \geq c\sum_{|i|=k}\zeta_i^2. \tag{6.32}$$

Let us assume $f \in W^{-k,2}(\Omega)$ and u be the solution of the Dirichlet problem $Au = f$ in $\Omega, u \in W_0^{k,2}(\Omega)$. Then

$$|u|_{W_{\sigma}^{k,2}(\Omega)} \leq c(\alpha)|f|_{(W_{0,\sigma^{-\alpha}}^{k,2}(\Omega))'}, \tag{6.33}$$

where α is an arbitrary number, $0 < \alpha < 1$.[1] *The Green operator $G : W^{k,2}(\Omega) \times W^{-k,2}(\Omega) \to W^{k,2}(\Omega)$ corresponding to the problem $Au = f$ in Ω, $u - u_0 \in W_0^{k,2}(\Omega)$ can be extended continuously to a mapping*

$$G \in [W_{\sigma^{\alpha}}^{k,2}(\Omega) \times W_{\sigma^{\alpha}}^{-k,2}(\Omega) \to W_{\sigma^{\alpha}}^{k,2}(\Omega)], \quad W_{\sigma^{\alpha}}^{-k,2}(\Omega) = \left(W_{0,\sigma^{-\alpha}}^{k,2}(\Omega)\right)'.$$

Proof. Let us consider $A_{\lambda}(v,u) = A(v,u) + \lambda(v,u)$, where λ will be fixed later. Let $v = u\sigma; v \in W_0^{k,2}(\Omega)$, and let us write:

$$A_{\lambda}(\sigma u, u) = \int_{\Omega}\sum_{|i|,|j|=k} a_{ij}D^i u D^j \overline{u}\sigma \, \mathrm{d}x + \lambda\int_{\Omega}|u|^2\sigma \, \mathrm{d}x + Z(u),$$

where $Z(u)$ is a sum of integrals of following type:

$$\int_{\Omega} aD^i uD^j\overline{u}\,\mathrm{d}x, \quad |i| = k-1, \quad |j| = k, \quad a \text{ real}, \quad a \in C^{0,1}(\overline{\Omega}), \tag{6.34a}$$

$$\int_{\Omega} aD^i uD^j\overline{u}\,\mathrm{d}x, \quad |i| \leq k-2, \quad |j| = k, \quad a \text{ real}, \quad a \in L^{\infty}(\Omega), \tag{6.34b}$$

$$\int_{\Omega} aD^i uD^j\overline{u}\,\mathrm{d}x, \quad |i| \leq k-1, \quad |j| \leq k-1, \quad a \text{ real}, \quad a \in L^{\infty}(\Omega). \tag{6.34c}$$

Let us consider first (6.34b). We have, according to Theorem 2.3:

$$\begin{aligned}\left|\int_{\Omega} aD^i uD^j\overline{u}\,\mathrm{d}x\right| &\leq c_1|u|_{W_{\sigma}^{k,2}(\Omega)}\left(\int_{\Omega}|D^i u|^2\sigma^{-1}\,\mathrm{d}x\right)^{1/2} \\ &\leq c_2|u|_{W_{\sigma}^{k,2}(\Omega)}\left(\int_{\Omega}|D^i u|^2\sigma^{-2}\,\mathrm{d}x\right)^{1/2} \leq c_3|u|_{W_{\sigma}^{k,2}(\Omega)}|u|_{W^{k-1,2}(\Omega)}.\end{aligned} \tag{6.35}$$

[1] We can use $W_{0,\sigma^{-1}\sigma_2}^{k,2}(\Omega)$ instead of $W_{0,\sigma^{-\alpha}}^{k,2}(\Omega)$, where σ_2 is of logarithmic type. The idea can be found in M.I. Vishik [4]. Cf. an exercise later on in this chapter.

Concerning (6.34c), we get:

$$\left|\int_\Omega aD^iuD^j\overline{u}\,\mathrm{d}x\right| \le c_4|u|^2_{W^{k-1,2}(\Omega)}. \tag{6.36}$$

Let us consider (6.34a); by integration by parts, we obtain:

$$\int_\Omega aD^iuD^j\overline{u}\,\mathrm{d}x = \int_\Omega aD^\alpha uD^\beta\overline{u}\,\mathrm{d}x + R(u),$$

where $R(u)$ is a sum of integrals of the type (6.34c); the multi-indices α,β are such that $\beta_{i_\tau}-\alpha_{i_\tau}=1$, $\tau=1,2,\ldots,h_1$, $\beta_{j_\tau}-\alpha_{j_\tau}=0$, $\tau=1,2,\ldots,h_2$, $\beta_{l_\tau}-\alpha_{l_\tau}=-1$, $i=1,2,\ldots,h_3$. Let $\gamma_i=\min(\alpha_i,\beta_i)$, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_N)$. We have

$$\int_\Omega aD^\alpha uD^\beta\overline{u}\,\mathrm{d}x = \int_\Omega a\frac{\partial^{(|\alpha|-|\gamma|)}}{\partial x_{l_1}\partial x_{l_2}\ldots\partial x_{l_{h_3}}}D^\gamma u\frac{\partial^{(|\beta|-|\gamma|)}}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_{h_1}}}D^\gamma\overline{u}\,\mathrm{d}x.$$

Let us fix the index i_1, and let $\delta=(\delta_1,\delta_2,\ldots,\delta_N), |\delta|=|\alpha|-|\gamma|$,

$$\begin{aligned}&\delta_m = 0 \text{ for } m\neq l_1,\ldots,l_{h_3},i_2,\ldots,i_{h_1},\\&\delta_m = 0 \text{ or } 1 \text{ in the opposite case.}\end{aligned}$$

Let us denote by M the set of these δ and let $\delta'\in M$ be the complementary index. By integration by parts, we get

$$\int_\Omega aD^\alpha uD^\beta\overline{u}\,\mathrm{d}x = \binom{2|\delta|}{|\delta|}^{-1}\sum_{\delta\in M}\int_\Omega aD^\delta\left(D^\gamma u\right)\frac{\partial}{\partial x_{i_1}}D^{\delta'}\left(D^\gamma\overline{u}\right))\,\mathrm{d}x + S(u),$$

where $S(u)$ is a sum of integrals of the type (6.34c).

We have

$$\begin{aligned}&\mathrm{Re}\int_\Omega aD^\alpha uD^\beta\overline{u}\,\mathrm{d}x\\&\quad=\binom{2|\delta|}{|\delta|}^{-1}\frac{1}{2}\sum_{\delta\in M}\int_\Omega a\frac{\partial}{\partial x_{i_1}}\left[D^\delta\left(D^\gamma u\right)D^{\delta'}\left(D^\gamma\overline{u}\right)\right]\mathrm{d}x + \mathrm{Re}\,S(u).\end{aligned} \tag{6.37}$$

By integrations by parts, we can write the first integral on the right hand side of (6.37) as a sum of integrals of the type (6.34c). Concerning the integral (6.34a) we get

$$\left|\mathrm{Re}\int_\Omega aD^iuD^j\overline{u}\,\mathrm{d}x\right| \le c_5|u|^2_{W^{k-1,2}(\Omega)}. \tag{6.38}$$

From (6.32), (6.35), (6.36), (6.38) it follows that

$$\begin{aligned}&\operatorname{Re} A_{\lambda}(\sigma u, u)\\ &\geq c_6 \int_{\Omega} \sum_{|i|=k} |D^i u|^2 \sigma \,\mathrm{d}x + c_6 \int_{\Omega} |u|^2 \sigma \,\mathrm{d}x - c_7 |u|^2_{W^{k-1,2}(\Omega)} + (\lambda - c_6) \int_{\Omega} |u|^2 \sigma \,\mathrm{d}x.\end{aligned} \tag{6.39}$$

We have

$$|u|_{W^{k,2}_{\sigma}(\Omega)} \leq c_8 (\int_{\Omega} \sum_{|i|=k} |D^i u|^2 \sigma \,\mathrm{d}x + \int_{\Omega} |u|^2 \sigma \,\mathrm{d}x). \tag{6.40}$$

To prove this, we use Theorem 2.7.6, so that $\Omega' \subset \overline{\Omega}' \subset \Omega$ implies

$$|u|^2_{W^{k-1,2}(\Omega')} \leq c_9(\Omega') (\int_{\Omega} \sum_{|i|=k} |D^i u|^2 \sigma \,\mathrm{d}x + \int_{\Omega} |u|^2 \sigma \,\mathrm{d}x). \tag{6.41}$$

To estimate $\int_{\Omega} |D^i u|^2 \sigma \,\mathrm{d}x$, $|i| = k-1$, we use (6.49). Then we obtain:

$$\int_{\Omega} \sum_{|i|=k-1} |D^i u|^2 \sigma \,\mathrm{d}x \leq c_{10} (\int_{\Omega} \sum_{|i|=k} |D^i u|^2 \sigma \,\mathrm{d}x + \int_{\Omega} |u|^2 \sigma \,\mathrm{d}x). \tag{6.42}$$

The estimate of integrals $\int_{\Omega} |D^i u|^2 \sigma \,\mathrm{d}x$, $|i| \leq k-2$, follows from (6.42), and applying Theorem 2.7.6, we get (6.40).

If $0 < \varepsilon < 1$, we have

$$|u|_{W^{k,2}_{\sigma^{1+\varepsilon}}(\Omega)} \leq c_{11} |u|_{W^{k,2}_{\sigma}(\Omega)}; \tag{6.43}$$

on the other hand, from (6.43) and Theorem 2.5, it follows that the imbedding $W^{k,2}_{\sigma^{1+\varepsilon}}(\Omega) \subset W^{k-1,2}(\Omega)$ is compact. Then we can apply Lemma 2.6.1 and find, according to (6.39), (6.40), λ sufficiently large such that

$$\operatorname{Re} A_{\lambda}(\sigma u, u) \geq c_{12} |u|^2_{W^{k,2}_{\sigma}(\Omega)}. \tag{6.44}$$

Let us fix this λ and let $\omega \in W^{k,2}_0(\Omega)$ be the solution of the problem $A\omega + \lambda\omega = f$ in Ω. Due to (6.44) we have:

$$c_{12} |\omega|^2_{W^{k,2}_{\sigma}(\Omega)} \leq \operatorname{Re} \langle \omega\sigma, \overline{f} \rangle. \tag{6.45}$$

For $\omega \in W^{k,2}_0(\Omega)$ we have

$$|\omega\sigma|_{W^{k,2}_{\sigma^{-\alpha}}} \leq c_{13} |\omega|_{W^{k,2}_{\sigma}(\Omega)}. \tag{6.46}$$

Indeed,

$$|\omega\sigma|^2_{W^{k,2}_{\sigma^{-\alpha}}} \leq c_{14}\Big(\sum_{|i|=k}\int_\Omega |D^i\omega|^2\sigma^{(2-\alpha)}\,\mathrm{d}x + \sum_{|i|\leq k-1}\int_\Omega |D^i\omega|^2\sigma^{-\alpha}\,\mathrm{d}x\Big)$$
$$\leq c_{15}\sum_{|i|=k}\int_\Omega |D^i\omega|^2\sigma\,\mathrm{d}x + c_{14}\sum_{|i|\leq k-1}\int_\Omega |D^i\omega|^2\sigma^{-\alpha}\,\mathrm{d}x.$$

It is sufficient to apply Theorem 2.4 to obtain (6.46). It follows also that $\omega\sigma \in W^{k,2}_{0,\sigma^{-\alpha}}(\Omega)$, and then (6.46) and (6.45) imply

$$|\omega|_{W^{k,2}_\sigma(\Omega)} \leq c_{16}|f|_{W^{-k,2}_{\sigma^\alpha}(\Omega)}. \tag{6.47}$$

As a particular case of (6.47), according to $|u|_{W^{k-1,2}(\Omega)} \leq c_{19}|u|_{W^{k,2}_\sigma(\Omega)}$ used previously we have

$$|\omega|_{L^2(\Omega)} \leq c_{20}|f|_{W^{-k,2}_{\sigma^\alpha}(\Omega)}. \tag{6.48}$$

Let $w \in W^{k,2}_0(\Omega)$ be the solution of problem $Aw = \lambda\omega$ in Ω. We have

$$|w|_{W^{k,2}(\Omega)} \leq c_{21}|\omega|_{L^2(\Omega)} \leq c_{22}|f|_{W^{-k,2}_{\sigma^\alpha}(\Omega)}. \tag{6.49}$$

But the solution of the problem $Au = f$ in $\Omega, u \in W^{k,2}_0(\Omega)$, is equal to $\omega + w$, hence (6.33) follows. According to Theorem 2.1, the space $W^{k,2}(\Omega)$ is dense in $W^{k,2}_{\sigma^\alpha}(\Omega)$; obviously $W^{-k,2}(\Omega)$ is dense in $W^{-k,2}_{\sigma^\alpha(\Omega)}$. □

Exercise 2.2. Let σ be as in the previous theorem. With the hypotheses given in this theorem prove that $W^{1,p}_\sigma \subset L^p_{\sigma_1}(\Omega)$, where $\sigma_1 = \sigma^{-1}\,|\log(\sigma/2M)|^{-\lambda}$, $\lambda > 2$, $\sigma \leq M$. Cf. also M.I. Vishik [4]. Modify Theorem 2.6 taking into account this result.

Remark 2.2. If $\Omega \in \mathfrak{R}$, cf. 6.1.3, with $\Omega = \cap_{i=1}^p \Omega_i$, $\Omega_i \in \mathfrak{N}^{k+1,1}$, there exists $\sigma_i \in C^{k,1}(\Omega_i)$, equivalent to ρ in Ω_i, where σ_i satisfies the condition (6.20).

Remark 2.3. If $\Omega \in \mathfrak{N}^{0,1}$, by Theorem 2.3 it follows that the generalized solution obtained in Theorem 2.6 has traces on $\partial\Omega$ for u, $\partial u/\partial n, \ldots \partial^{k-1}u/\partial n^{k-1}$ in the sense of this theorem, computed from u_s.

Problem 2.1. With the hypotheses as in Theorem 2.6, is the following inequality

$$|u|_{W^{k,2}_\sigma(\Omega)} \leq c|f|_{W^{-k,2}_{\sigma^\alpha}}$$

true?

If f and the coefficients a_{ij} of the operator are smooth in Ω, the solution u from the previous theorem is also smooth.

Example 2.2. Let Ω be a conical domain, with vertex at the origin and placed in the half space $x_N \geq 0$. Then with obvious conditions, $\sigma(x) = x_N$ satisfies the hypotheses given in the theorem.

Example 2.3. For $\Omega \in \mathfrak{N}^0$, we assume that there exists a ball of fixed radius such that for each y in the complement $\complement\Omega$ of Ω, in particular for $y \in \partial\Omega$, it is possible to put this ball B in $\complement\Omega$ in such a way that $y \in \partial B$. Under obvious conditions $\sigma(x) = \operatorname{dist}(x, \partial B)$ satisfies the hypotheses of the previous theorem.

Remark 2.4. Let Ω be a bounded domain such that $\Omega \subset \Omega_1$, $\Omega_1 \in \mathfrak{N}^0$, $\sigma \in C(\Omega_1)$ satisfies the condition (6.20) for Ω_1. Then for $W^{k,2}_{0,\sigma^\alpha}$, Theorems 2.3, 2.4, 2.5 hold. Moreover Theorem 2.6 is satisfied.

As a consequence of the previous theorem, we obtain a theorem whose significance will be given in the next chapter: in the case $N = 3$ and for domains such that for all $y \in \complement\Omega$ there exists a ball with fixed radius which does not depend on y, $B \subset \complement\Omega$, $y \in \partial B$, we shall prove that the Dirichlet problem for an equation of order $2k$ has a "classical " solution.

Theorem 2.7. *Let $\Omega \in \mathfrak{N}^0$, $\sigma \in C^{k,1}(\overline{\Omega})$,[2] $\sigma > 0$ in $\overline{\Omega}$ satisfying the condition (6.20). Let us assume that the sesquilinear form $A(v,u)$ verifies the hypotheses of Theorem 2.6. Let $f \in W^{-k,2}(\Omega)$, u be the solution of the problem $Au = f$ in $\Omega, u \in W_0^{k,2}(\Omega)$. Then*

$$|u|_{H^{k,2}_{\sigma^{-1}}} \leq c(|f|_{W^{-k,2}(\Omega)} + |f|_{W^{-k,2}_{\sigma^{-1}}(\Omega)}). \tag{6.50}$$

Proof. Let u be the solution of the problem mentioned and λ as in Theorem 2.6. Let us put $u/\sigma = v$. We have $\operatorname{Re} A_\lambda(v, \sigma v) \geq c_1 |v|^2_{W_\sigma^{k,2}(\Omega)}$, and then

$$c_1 |u|^2_{H^{k,2}_{\sigma^{-1}}} \geq \operatorname{Re} A_\lambda(u/\sigma, u). \tag{6.51}$$

We have:

$$\begin{aligned} &|A_\lambda(u/\sigma, u)| \leq \left|\langle u/\sigma, \overline{f}\rangle\right| + \lambda \int_\Omega \left(|u|^2/\sigma\right) \mathrm{d}x \\ &\leq |u|_{H^{-k,2}_{\sigma^{-1}}} |f|_{W^{-k,2}_{\sigma^{-1}}(\Omega)} + c_2 |u|^2_{W^{1,2}(\Omega)} \leq |u|_{H^{-k,2}_{\sigma^{-1}}} |f|_{W^{-k,2}_{\sigma^{-1}}(\Omega)} + c_3 |f|^2_{W^{-k,2}(\Omega)}. \end{aligned} \tag{6.52}$$

But

$$|u|_{H^{-k,2}_{\sigma^{-1}}} |f|_{W^{-k,2}_{\sigma^{-1}}(\Omega)} \leq (c_1/2) |u|^2_{H^{k,2}_{\sigma^{-1}}(\Omega)} + (1/2c_1) |f|^2_{W^{-k,2}_{\sigma^{-1}}(\Omega)};$$

this last inequality and (6.51), (6.52) imply the result. □

[2]Cf. Remark 2.4.

6.3 Sesquilinear Forms on $W^{k,2}_{\sigma^{-1}}(\Omega) \times W^{k,2}_{\sigma}(\Omega)$

6.3.1 The B, H_2-ellipticity

We can generalize in a natural way the notion of V-ellipticity, cf. Chap. 3, for a pair of Hilbert spaces.[3] Let $B(v,u)$ be a bounded sesquilinear form on $H_1 \times H_2$, i.e. linear on H_1 and antilinear on H_2, with complex values and bounded:

$$|B(v,u)| \leq c|v|_{H_1}|u|_{H_2}. \tag{6.53}$$

Moreover let B be a Banach space, such that $H_1 \subset B$ algebraically and topologically. We say that $B(v,u)$ is *H_2-elliptic* if

$$\sup_{|v|_{H_1}\leq 1} |B(v,u)| \geq c|u|_{H_2}, \tag{6.54}$$

and *B-elliptic* if

$$\sup_{|u|_{H_2}\leq 1} |B(v,u)| \geq c|v|_{B}. \tag{6.55}$$

We denote the scalar product on H_i by $(\omega, w)_{H_i}$. We have:

Theorem 3.1. *Let $f \in H_1'$, and let $B(v,u)$ be a sesquilinear form which is H_2-elliptic and B-elliptic. Then there exists a unique $u \in H_2$ such that for all $v \in H_1$ we have $B(v,u) = fv$. Moreover we have*

$$|u|_{H_2} \leq (1/c)|f|_{H_1'}, \tag{6.56}$$

where the constant c is the same as in (6.54).

Proof. By the Riesz theorem, for every $u \in H_2$, there exists a unique element $Zu \in H_1$ such that $v \in H_1 \Longrightarrow B(v,u) = (v, Zu)_{H_1}$. The mapping $Z \in [H_2 \to H_1]$ is open:

$$\sup_{|v|_{H_1}\leq 1} |(v,Zu)_{H_1}| = \sup_{|v|_{H_1}\leq 1} |B(v,u)| \geq c_1|u|_{H_2}. \tag{6.57}$$

The range ZH_2 is closed in H_1; from (6.57) it follows that $Z^{-1}(0) = 0$. If $ZH_2 \neq H_1$, it will be possible to find $v \in H_1$, $v \neq 0$, such that $(v,Zu)_{H_1} = 0$ for all $u \in H_2$. Then (6.55) will imply $v \equiv 0$, which contradicts to the hypothesis; the inequality (6.56) is a consequence of (6.57). □

Example 3.1. Let H_1 be the set of sequences $x = (x_1, x_2, \ldots)$ such that $(x,y)_{H_1} = \sum_{n=1}^{\infty} n^{-\alpha} x_n \bar{y}_n$, α a real number, and H_2 the set of sequences $x' = (x_1', x_2', \ldots)$ such

[3] It is possible to replace H_1, H_2 by two reflexive Banach spaces.

that $(x',y')_{H_2} = \sum_{n=1}^{\infty} n^{\alpha} x'_n \bar{y}'_n$. Denote $B(x,x') = \sum_{n=1}^{\infty} x_n \bar{x}'_n$. The form B is H_i-elliptic, $i = 1,2$; let us prove, e.g., condition (6.54). Let $x' \in H_2$, and $x = (x'_1, x'_2 2^{\alpha}, x'_3 3^{\alpha}, \dots)$. Then $\sum_{n=1}^{\infty} |x_n|^2 n^{-\alpha} = \sum_{n=1}^{\infty} |x'_n|^2 n^{\alpha}$, hence $x \in H_1$. But $B(x,x') = \sum_{n=1}^{\infty} |x'_n|^2 n^{\alpha}$; if $x^* = (x/|x|_{H_1})$, then we get $B(x^*,x') = |x'|^2_{H_2}$.

The investigation of sesquilinear forms of the type $((v,u))$ in Chap. 3 is not sufficiently developed; the hypothesis of the regularity of the boundary $\partial\Omega$ of the domain considered plays an essential role. In boundary value problems the most natural pair H_1, H_2[4] is $W^{k,p}(\Omega)$, $W^{k,q}(\Omega)$, $1/p + 1/q = 1$, or V_p, V_q, where $V_p = \overline{V}$, $V_q = \overline{V}$ with the set V such that $C_0^{\infty}(\Omega) \subset V \subset C^{\infty}(\overline{\Omega})$ and the closures are taken in $W^{k,p}(\Omega)$, $W^{k,q}(\Omega)$ respectively. Such pairs in the case of smooth boundary $\partial\Omega$ are investigated in S. Agmon [2], M. Schechter[10], J.L. Lions, E. Magenes [3], L. Mulkin, K.T. Smith [1]. Another pair is $W^{k-\theta,2}(\Omega)$, $W^{k+\theta,2}(\Omega)$, cf. Chap. 2. In this case the results are due to N. Aronszajn [1], J. Nečas [9]. The first author assumes $\partial\Omega$ smooth, the second one assumes $\Omega \in \mathfrak{N}^{0,1}$. In this section we shall consider for H_1, H_2, B, various weighted spaces. We use the results in J. Nečas [7], A. Kufner [2, 3] and the ideas of M. I. Vishik [4]. In this direction there are many open problems.

6.3.2 Ellipticity of Sesquilinear Forms for $W^{k,2}_{0,\sigma^{-\alpha}}$, $W^{k,2}_{0,\sigma^{\alpha}}$

Let $\sigma \in C^{k-1,1}(\Omega)$ be a weight satisfying (6.20), and such that for $|i| \le k$,

$$\left|D^i \sigma\right| \le c\sigma^{(1-|i|)}. \tag{6.58}$$

Remark 3.1. If $\Omega \in \mathfrak{N}^0$, $y \in \partial\Omega$, then $\sigma(x) = |x-y|$ satisfies (6.58). We have

Lemma 3.1. *Let $\Omega \in \mathfrak{N}^{0,1}$. Then there exists $\sigma \in C^{\infty}(\Omega) \cap C^{0,1}(\overline{\Omega})$, such that $c_1\rho(x) \le \sigma(x) \le c_2\rho(x)$, and which satisfies (6.20) and (6.58); here $\rho(x) = dist\,(x,\partial\Omega)$.*

Proof. Let $\tilde{\alpha} < \alpha, \alpha$ defined as in 1.2.4, $\tilde{\alpha}$ close to α and such the corresponding sets $\tilde{U}_r, r = 1,2,\dots,m$ verify $\partial\Omega \subset \cup_{r=1}^m \tilde{U}_r$, supp $\varphi_r \subset \tilde{U}_r$. For $h < \alpha - \tilde{\alpha}$ let us set, with κ as in 2.1.3,

$$a_r(x'_r,h) = \frac{1}{\kappa h^{N-1}} \int_{|y'_r|<h} \exp \frac{|x'_r - y'_r|^2}{|x'_r - y'_r|^2 - h^2} a_r(y'_r)\,dy'_r.$$

The function $a_r(x'_r,h)$ is infinitely differentiable with respect to (x'_r,h), $|x'_{ri}| < \tilde{\alpha}$, $i = 1,2,\dots,N-1$, $0 < h < \alpha - \tilde{\alpha}$. By a simple computation using that $a_r \in C^{0,1}(\overline{\tilde{\Delta}_r})$, we obtain

[4] Cf. the previous footnote.

$$\left|\frac{\partial^{|i|}a_r}{\partial x_{r_1}^{i_1}\partial x_{r_2}^{i_2}\ldots,\partial x_{r_{N-1}}^{i_{N-1}}\partial h^{i_N}}\right| \le \frac{c(|i|)}{h^{|i|-1}}, \quad |i| \ge 1. \tag{6.59}$$

Let us set $b_r(x'_r,h) = a_r(x'_r,h) + (c(1)+1)h$; we have:

$$(c(1)+1) \le \frac{\partial b_r}{\partial h} \le 2(c(1)+1); \tag{6.60}$$

let $(\alpha-\tilde{\alpha})/2 \le h_0 < (\alpha-\tilde{\alpha})$, such that if $x'_r \in \tilde{\Delta}_r$, we have: $b_r(x'_r,h_0) \le a_r(x'_r)+\beta$, and denote

$$A_r = \{x \in \mathbb{R}^N, \quad x'_r \in \tilde{\Delta}_r, \quad a_r(x'_r) < x_{rN} < b_r(x'_r,h_0)\}. \tag{6.61}$$

The inequalities (6.60) imply that to each point in A_r corresponds one and only one point h in the interval $(0,h_0)$ such that $x_{rN} = b_r(x'_r,h)$; the function $h = h_r(x'_r,x_{rN})$ is in $C^\infty(G_r)\cap C^{0,1}(\overline{G}_r)$ and $h=0$, if $x \in \partial\Omega$. Now from (6.61) we deduce:

$$\left|D^i h_r(x)\right| \le \frac{d(|i|)}{h_r^{(|i|-1)}} \le \frac{e(|i|)}{\rho^{(|i|-1)}}.$$

Let us define

$$\sigma_r(x) = h_r(x) \text{ for } x \in \widetilde{V}_r,\ \sigma_r(x) = 0 \text{ for } x \notin \widetilde{V}_r,$$

$$\sigma(x) = \sum_{r=1}^{m} \sigma_r(x)\varphi_r(x) + \varphi_{m+1}(x).$$

Using (6.60) we have in $\tilde{V}_r$:

$$c_3\rho(x) \le h_r(x) \le c_4\rho(x).$$

Then we get

$$\rho(x) = \sum_{r=1}^{m+1} \varphi_r(x)\rho(x) \le c_5\Big(\sum_{r=1}^{m} \sigma_r(x)\varphi_r(x) + \varphi_{m+1}(x)\Big) = c_5\sigma(x).$$

On the other hand,

$$\sigma(x) = \sum_{r=1}^{m} \sigma_r(x)\varphi_r(x) + \varphi_{m+1}(x) \le c_6(\sum_{r=1}^{m} \rho(x)\varphi_r(x) + \varphi_{m+1}(x)) \le c_6\rho(x).$$

□

Let us consider the operator

$$A = \sum_{|i|,|j|\le k} (-1)^{|i|}D^i(a_{ij}D^j), \quad a_{ij} \in L^\infty(\Omega),$$

and define on $C_0^\infty(\Omega) \times C_0^\infty(\Omega)$

$$A(v,u) = \int_\Omega \sum_{|i|,|j| \le k} \overline{a}_{ij} D^i v D^j \overline{u}\, \mathrm{d}x. \tag{6.62 bis}$$

We have

Theorem 3.2. *Let us consider $\Omega \in \mathfrak{N}^0$, σ satisfying (6.20) and (6.58), $A(v,u)$ defined in (6.62 bis) $W_0^{k,2}(\Omega)$-elliptic. We choose $H_1 = W^{k,2}_{0,\sigma^{-\alpha}}$, $H_2 = W^{k,2}_{0,\sigma^\alpha}$. Then there exist intervals $I_i \subset (-1,1)$, $i = 1,2$, containing neighborhoods of the origin, such that for $\alpha \in I_i$ the sesquilinear form $A(v,u)$ is H_i-elliptic.*

Proof. Let $u \in C_0^\infty(\Omega)$ and set $v = u\sigma^\alpha$. We have

$$A(v,u) = A(u\sigma^{\alpha/2}, u\sigma^{\alpha/2}) + B(u);$$

$B(u)$ is a sum of terms of the type

$$\int_\Omega \overline{a} D^{i'} u D^{i''} \sigma^\alpha D^j \overline{u}\, \mathrm{d}x, \quad i' + i'' = i, \quad |i'| \le k-1, \tag{6.63}$$

and of the type

$$\int_\Omega \overline{a} D^{i'} u D^{i''} \sigma^{\alpha/2} D^{j'} \overline{u} D^{j''} \sigma^{\alpha/2}\, \mathrm{d}x, \quad i' + i'' = i,\ j' + j'' = j,\ |i'| + |j'| \le 2k-1. \tag{6.64}$$

Here $a \in L^\infty(\Omega)$ are given functions that do not depend on α. We have:

$$\begin{cases} |D^{i''} \sigma^\alpha| \le c(i'')|\alpha|\sigma^{\alpha - |i''|}, \\ |D^{j''} \sigma^{\alpha/2}| \le c(j'')|\alpha|\sigma^{\alpha/2 - |j''|}. \end{cases} \tag{6.65}$$

Using Theorem 2.3, we get for (6.63),

$$\begin{aligned} \Big| \int_\Omega \overline{a} D^{i'} u D^{i''} \sigma^\alpha D^j \overline{u}\, \mathrm{d}x \Big| &\le c|\alpha| \Big(\int_\Omega |D^{i'} u|^2 \sigma^{(\alpha - 2|i''|)}\, \mathrm{d}x \Big)^{1/2} \Big| \Big(\int_\Omega |D^j u|^2 \sigma^\alpha\, \mathrm{d}x \Big)^{1/2} \\ &\le c_1 |\alpha| |u|^2_{W^{k,2}_{\sigma^\alpha}(\Omega)}. \end{aligned}$$

By the same computations, we obtain

$$\Big| \int_\Omega \overline{a} D^{i'} u D^{i''} \sigma^{\alpha/2} D^{j'} \overline{u} D^{j''} \sigma^{\alpha/2})\, \mathrm{d}x \Big| \le c_2 |\alpha| |u|^2_{W^{k,2}_{\sigma^\alpha}(\Omega)},$$

and then

$$|B(u)| \le c_3 |\alpha| |u|^2_{W^{k,2}_{\sigma^\alpha}(\Omega)}. \tag{6.66}$$

According to the $W_0^{k,2}$-ellipticity of $A(v,u)$, we have

$$|A(u\sigma^{\alpha/2}, u\sigma^{\alpha/2})| \geq c_4 |u\sigma^{\alpha/2}|^2_{W^{k,2}(\Omega)} \geq c_5 |u|^2_{W^{k,2}_{\sigma^\alpha}(\Omega)};$$

using Theorem 2.3 it follows that

$$|u\sigma^\alpha|^2_{W^{k,2}_{\sigma^{-\alpha}}(\Omega)} \leq c_6 |u|^2_{W^{k,2}_{\sigma^\alpha}(\Omega)}. \tag{6.67}$$

If we set $w = (v/|v|_{W^{k,2}_{\sigma^{-\alpha}}(\Omega)})$, then we have

$$|A(w,u)| \geq (c_5 - c_3|\alpha|)(1/c_6)|u|^2_{W^{k,2}_{\sigma^\alpha}} \tag{6.68}$$

and we have the $W^{k,2}_{0,\sigma^\alpha}$-ellipticity for $|\alpha| < c_5/c_3$. The $W^{k,2}_{0,\sigma^{-\alpha}}$-ellipticity can be derived in the same way. □

Problem 3.1. Determine the precise bounds of the intervals I_1, I_2.
Hint: Consider $A(v,u) + \lambda(v,u)$, λ big enough.

Remark 3.2. If $A(v,u)$ is a hermitian sesquilinear form, we have $I_1 = -I_2$.

Example 3.2. Let us consider $\Omega = (0,1)\times(0,1)$ and the form

$$A(v,u) = \int_\Omega \left(\frac{\partial v}{\partial x_1}\frac{\partial \overline{u}}{\partial x_1} + \frac{\partial v}{\partial x_2}\frac{\partial \overline{u}}{\partial x_2} \right) \mathrm{d}x.$$

Let us set $\sigma = x_2$, and let $u \in W^{1,2}_{0,x_2^\alpha}(\Omega)$, $v = x_2^\alpha u$. We get

$$A(v,u) = \int_\Omega \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial x_2}\right|^2 \right) x_2^\alpha \,\mathrm{d}x + \alpha \int_\Omega x_2^{(\alpha-1)} u \frac{\partial \overline{u}}{\partial x_2}\,\mathrm{d}x$$

and obtain

$$\mathrm{Re}\ \alpha \int_\Omega x_2^{(\alpha-1)} u \frac{\partial \overline{u}}{\partial x_2}\,\mathrm{d}x = \frac{\alpha}{2}(1-\alpha)\int_\Omega x_2^{(\alpha-2)}|u|^2\,\mathrm{d}x.$$

If $1 \geq \alpha \geq 0$, then

$$\frac{\alpha}{2}(1-\alpha)\int_\Omega x_2^{(\alpha-2)}|u|^2\,\mathrm{d}x \geq 0;$$

hence we have the $W^{1,2}_{0,x_2^\alpha}(\Omega)$-ellipticity, if $0 \leq \alpha < 1$. If $\alpha < 0$, then according to (6.25)

$$\frac{|\alpha|}{2}(1-\alpha)\int_\Omega x_2^{(\alpha-2)}|u|^2\,\mathrm{d}x \le \frac{2|\alpha|}{1-\alpha}\int_\Omega\left(\left|\frac{\partial u}{\partial x_1}\right|^2+\left|\frac{\partial u}{\partial x_2}\right|^2\right)x_2^\alpha\,\mathrm{d}x,$$

and this implies the $W^{1,2}_{0,x_2^\alpha}(\Omega)$-ellipticity for $-1<\alpha\le 0$.

Example 3.3. Let us consider

$$\Omega=(0,1)^N,\quad A(v,u)=\int_\Omega\sum_{i,j=1}^N\frac{\partial^2 v}{\partial x_i\partial x_j}\frac{\partial^2\overline{u}}{\partial x_i\partial x_j}\,\mathrm{d}x,$$

the sesquilinear form associated to the operator $\triangle^2$. Let us choose $\sigma=x_N$, and $|\alpha|<1$, $H_1=W^{2,2}_{0,\sigma^{-\alpha}}(\Omega)$, $H_2=W^{2,2}_{0,\sigma^\alpha}(\Omega)$. Then $A(v,u)$ is H_1-elliptic and H_2-elliptic if $|\alpha|<1$. Indeed: Let $u\in W^{2,2}_{0,\sigma^\alpha}(\Omega)$. If we set $v=u\sigma^\alpha$ then according to Theorem 2.3 we have $|v|_{W^{2,2}_{\sigma^{-\alpha},}(\Omega)}\le c|u|_{W^{2,2}_{\sigma^\alpha}(\Omega)}$. If $i\ne N$, we have

$$\begin{aligned}&2\mathrm{Re}\int_\Omega\frac{\partial^2 v}{\partial x_i\partial x_N}\frac{\partial^2\overline{u}}{\partial x_i\partial x_N}\,\mathrm{d}x=\\&2\int_\Omega\left|\frac{\partial^2\overline{u}}{\partial x_i\partial x_N}\right|^2x_N^\alpha\,\mathrm{d}x+\alpha(1-\alpha)\int_\Omega\left|\frac{\partial u}{\partial x_i}\right|^2x_N^{(\alpha-2)}\,\mathrm{d}x.\end{aligned}\tag{6.69}$$

On the other hand we have

$$\begin{aligned}&\mathrm{Re}\int_\Omega\frac{\partial^2 v}{\partial x_N^2}\frac{\partial^2\overline{u}}{\partial x_N^2}\,\mathrm{d}x=\\&2\alpha(1-\alpha)\int_\Omega\left|\frac{\partial u}{\partial x_N}\right|^2x_N^{(\alpha-2)}\,\mathrm{d}x-(\alpha/2)(1-\alpha)(2-\alpha)(3-\alpha)\int_\Omega|u|^2x_N^{(\alpha-4)}\,\mathrm{d}x.\end{aligned}\tag{6.70}$$

Starting from (6.35) we get

$$\int_\Omega|u|^2x_N^{(\alpha-4)}\,\mathrm{d}x\le\frac{4}{(3-\alpha)^2}\int_\Omega\left|\frac{\partial u}{\partial x_N}\right|^2x_N^{(\alpha-2)}\,\mathrm{d}x,\tag{6.71}$$

hence for $1\ge\alpha\ge 0$,

$$\mathrm{Re}\int_\Omega\frac{\partial^2 v}{\partial x_N^2}\frac{\partial^2\overline{u}}{\partial x_N^2}\,\mathrm{d}x\ge\left(\int_\Omega|u|^2x_N^{(\alpha-4)}\,\mathrm{d}x\right)\frac{\alpha}{2}(1-\alpha)(3-\alpha).$$

We estimate (6.69) as in the previous example; the result follows for $0\le\alpha<1$. Using Theorem 2.3, we obtain easily that for $|\alpha|<1$, $H^{k,2}_{0,\sigma^\alpha,}(\Omega)=W^{k,2}_{0,\sigma^\alpha}(\Omega)$ algebraically and topologically. Hence for $-1<\alpha\le 0$ we have

$$|A(u\sigma^{\alpha},u)| = |A(v,v\sigma^{-\alpha})| \geq c_1|v|^2_{W^{2,2}_{\sigma^{-\alpha}}(\Omega)} \geq c_1|u|^2_{H^{2,2}_{\sigma^{\alpha}}(\Omega)} \geq c_2|u|^2_{W^{2,2}_{\sigma^{\alpha}}(\Omega)}. \quad (6.72)$$

Problem 3.2. Determine whether for $\sigma \in C^k(\overline{\Omega})$ satisfying (6.20), the intervals I_1, I_2 from Theorem 3.2 are equal to $(-1,1)$. We can consider $A(v,u)+\lambda(v,u)$, λ big enough, and a_{ij} sufficiently smooth.

Example 3.4. Let Ω be bounded in $\mathbb{R}^N$, $N \geq 3$,

$$y \in \partial\Omega, \quad r = |x-y|, \quad A(v,u) = \int_{\Omega} \sum_{i=1}^{N} \frac{\partial v}{\partial x_i}\frac{\partial \overline{u}}{\partial x_i}\,\mathrm{d}x.$$

If $|\alpha| < N-2$, then $A(v,u)$ is H_1 and H_2 elliptic if $H_1 = W^{1,2}_{0,r^{-\alpha}}(\Omega)$, $H_2 = W^{1,2}_{0,r^{\alpha}}(\Omega)$. Indeed: if $-N+2 < \alpha \leq 0$, then

$$\operatorname{Re} A(ur^{\alpha},u) = \int_{\Omega} \sum_{i=1}^{N} \left|\frac{\partial u}{\partial x_i}\right|^2 r^{\alpha}\,\mathrm{d}x + \frac{\alpha}{2}(2-\alpha-N)\int_{\Omega} |u|^2 r^{(\alpha-2)}\,\mathrm{d}x, \quad (6.72\,\text{bis})$$

and according to (6.25),

$$|ur^{\alpha}|_{W^{1,2}_{r^{-\alpha}}(\Omega)} \leq c|u|_{W^{1,2}_{r^{-\alpha}}(\Omega)} \quad (6.73)$$

(we use polar coordinates); inequality (6.73) holds for $-N+2 \leq \alpha$. Now let us consider $|\alpha| < N-2$; we have algebraically and topologically

$$H^{1,2}_{0,r^{\alpha}}(\Omega) = W^{1,2}_{0,r^{\alpha}}(\Omega). \quad (6.74)$$

Using the same argument as in (6.72), we have the conclusion. If for instance $\Omega \in \mathfrak{N}^{0,1}$, (6.73), (6.74) are true for all α and our result holds for $|\alpha| \leq N-2$.

Remark 3.3. Let Ω be a bounded or unbounded domain, $y \in \partial\Omega$ be a fixed point. Let us assume that there exists a cone with vertex y contained in $\complement\Omega$. Let $\sigma = |x-y|$. Then we have algebraically and topologically for every α and $0 \leq l \leq k$: $W^{k,p}_{0,\sigma^{\alpha}}(\Omega) \subset W^{k-l,p}_{0,\sigma^{\alpha-pl}}(\Omega)$.

Example 3.5. Let Ω be a bounded domain, $N=2$ and $y \in \partial\Omega$ fixed. Let us assume that there exists an infinite cone with vertex y and angle ω, $0 \leq \omega < 2\pi$, contained in $\complement\Omega$; if $\omega = 0$, the cone degenerates into a half line. We consider

$$A(v,u) = \int_{\Omega} \left(\frac{\partial v}{\partial x_1}\frac{\partial \overline{u}}{\partial x_1} + \frac{\partial v}{\partial x_2}\frac{\partial \overline{u}}{\partial x_2}\right)\mathrm{d}x.$$

Using the Fourier series, we obtain

$$\int_0^l |h(\varphi)|^2 d\varphi \le \frac{l^2}{\pi^2}\int_0^l |h'(\varphi)|^2 d\varphi \tag{6.75}$$

for the function of the angular variable φ, $h(\varphi)$, $0 < l \le 2\pi$, $h(\varphi)\in W_0^{1,2}(0,l)$. Let us choose $\sigma = |x-y|$. We have

$$u \in W^{1,2}_{0,\sigma^\alpha}(\Omega) \Longrightarrow |u\sigma^\alpha|_{W^{1,2}_{\sigma^{-\alpha}}(\Omega)} \le c_1 |u|_{W^{1,2}_{\sigma^\alpha}(\Omega)}$$

for all α. [The case $\omega = 0$ is a special case, namely $\alpha = 0$, but then there is nothing to prove.] It follows from (6.25) and (6.26) using polar coordinates that

$$\int_\Omega |u|^2 r^{(\alpha-2)}\,\mathrm{d}x \le \frac{4}{\alpha^2}\int_\Omega \left|\frac{\partial u}{\partial r}\right|^2 r^\alpha\,\mathrm{d}x \tag{6.76}$$

and taking into account (6.75) we get

$$\int_\Omega |u|^2 r^{(\alpha-2)}\,\mathrm{d}x \le \frac{l^2}{\pi^2}\int_\Omega \left|\frac{\partial u}{\partial \varphi}\right|^2 r^{(\alpha-2)}\,\mathrm{d}x, \qquad l = 2\pi-\omega.$$

From this inequality and from (6.76) it follows

$$\int_\Omega |u|^2 r^{(\alpha-2)}\,\mathrm{d}x \le \frac{1}{\alpha^2/4+\pi^2/l^2}\int_\Omega \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial x_2}\right|^2\right) r^\alpha\,\mathrm{d}x.$$

Now (6.72 bis) implies that $A(v,u)$ is $W^{1,2}_{0,\sigma^{-\alpha}}$-elliptic and $W^{1,2}_{0,\sigma^\alpha}$-elliptic for $|\alpha| \le 2\pi/l$.

Example 3.6. Under the same hypotheses as in the previous example, we consider

$$A(v,u) = \int_\Omega \left(\frac{\partial^2 v}{\partial x_1^2}\frac{\partial^2 \overline{u}}{\partial x_1^2} + 2\frac{\partial^2 v}{\partial x_1\partial x_2}\frac{\partial^2 \overline{u}}{\partial x_1\partial x_2} + \frac{\partial^2 v}{\partial x_2^2}\frac{\partial^2 \overline{u}}{\partial x_2^2}\right)\mathrm{d}x, \quad \omega > 0.$$

For $v = ur^\alpha$, we obtain

$$\begin{aligned}\mathrm{Re}\,A(ur^\alpha,u) = \int_\Omega \left(\left|\frac{\partial^2 u}{\partial x_1^2}\right|^2 + 2\left|\frac{\partial^2 u}{\partial x_1\partial x_2}\right|^2 + \left|\frac{\partial^2 u}{\partial x_2^2}\right|^2\right) r^\alpha\,\mathrm{d}x \\ -\alpha(\alpha+1)\int_\Omega \left(\left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial x_2}\right|^2\right) r^{(\alpha-2)}\,\mathrm{d}x - \alpha(\alpha-2)\int_\Omega \left|\frac{\partial u}{\partial r}\right|^2 r^{(\alpha-2)}\,\mathrm{d}x \\ + \frac{\alpha^2(\alpha-2)^2}{2}\int_\Omega |u|^2 r^{(\alpha-4)}\,\mathrm{d}x.\end{aligned}$$

As in the previous example we obtain

$$\int_\Omega \left(\left| \frac{\partial u}{\partial x_1} \right|^2 + \left| \frac{\partial u}{\partial x_2} \right|^2 \right) r^{(\alpha-2)} \, \mathrm{d}x$$
$$\leq \frac{1}{\alpha^2/4 + \pi^2/l^2} \int_\Omega \left(\left| \frac{\partial^2 u}{\partial x_1^2} \right|^2 + 2 \left| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right|^2 + \left| \frac{\partial^2 u}{\partial x_2^2} \right|^2 \right) r^\alpha \, \mathrm{d}x,$$

where for $0 \leq \alpha$ the ellipticity holds if

$$\alpha < 2/3 \left(\sqrt{(1 + 3\pi^2/l^2)} - 1 \right),$$

and by (6.72) if

$$|\alpha| < 2/3 \left(\sqrt{(1 + 3\pi^2/l^2)} - 1 \right).$$

Remark 3.4. Let us remark that the coefficients in Theorem 3.2 belong to $L^\infty(\Omega)$.

Exercise 3.1. Let Ω be as in Example 3.3, decomposed into cubes Ω_i with faces parallel to the faces of Ω, $\Omega = \sum_{i=1}^{K} \Omega_i \cup M$, meas $M = 0$. Let us consider a function constant in each Ω_i, $a(x) = a_i > 0$, $x \in \Omega_i$, and set

$$A(v,u) = \int_\Omega a \sum_{i,j=1}^{N} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 \overline{u}}{\partial x_i \partial x_j} \, \mathrm{d}x + \lambda (v, \overline{u}).$$

For λ sufficiently large, prove for $|\alpha| < 1$ the $W^{2,2}_{0,x_N^{-\alpha}}(\Omega)$-ellipticity and the $W^{2,2}_{0,x_N^{\alpha}}(\Omega)$-ellipticity.

Exercise 3.2. Cf. A. Kufner [1]; the hypotheses are the same as in Example 3.5, $\omega > 0$. We decompose Ω into two parts Ω_1, Ω_2, by an half-line, with origin at $y \in \partial\Omega$, see Fig. 6.1.

Let $a = a_i$ in Ω_i, $a_i > 0$ be constants, $i = 1,2$, and let us consider the sesquilinear form

$$A(v,u) = \int_\Omega a \left(\frac{\partial v}{\partial x_1} \frac{\partial \overline{u}}{\partial x_1} + \frac{\partial v}{\partial x_2} \frac{\partial \overline{u}}{\partial x_2} \right) \mathrm{d}x.$$

Let ν be the angle as in Fig. 6.1, and $\kappa = \max(\nu, 2\pi - \omega - \nu)$. Prove the $W^{1,2}_{0,r^{-\alpha}}(\Omega)$-ellipticity and the $W^{1,2}_{0,r^{\alpha}}(\Omega)$-ellipticity for $|\alpha| < \pi/\kappa$.

Exercise 3.3. Prove Theorem 3.1 for systems.

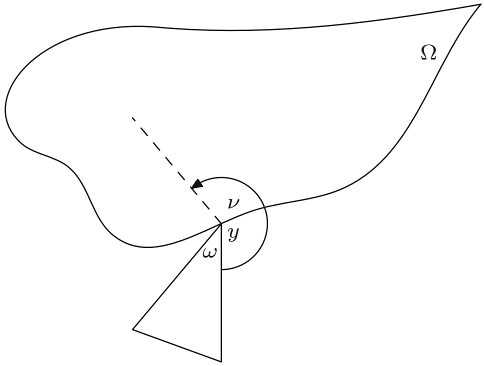

Fig. 6.1

6.3.3 The Dirichlet Problem

Consider $f \in W^{-k,2}_{\sigma^\alpha}(\Omega)$, $u_0 \in W^{k,2}_{\sigma^\alpha}(\Omega)$, $A(v,u)$ the sesquilinear form (6.63); assume that σ satisfies (6.20) and (6.58). The function $u \in W^{k,2}_{\sigma^\alpha}(\Omega)$ is a *weak solution* or *very weak solution* of the problem $Au = f$ in Ω, $\partial^i u/\partial n^i = \partial^i u_0/\partial n^i$, $i = 1,2,\ldots,$ $k-1$ on $\partial\Omega$, if for all $v \in W^{k,2}_{0,\sigma^{-\alpha}}(\Omega)$

$$A(v,u) = \langle v, \overline{f}\rangle, \tag{6.77}$$

$$u - u_0 \in W^{k,2}_{0,\sigma^{-\alpha}}(\Omega). \tag{6.78}$$

Remark 3.5. Let $\Omega \in \mathfrak{N}^{0,1}$, $\sigma = \rho$ with $\rho(x) = \operatorname{dist}(x,\partial\Omega)$, $\alpha > -1$, $g \in L^2_{\sigma^{(\alpha+2(k-|i|))}}(\Omega)$, $|i| \le k$. Then $D^i g \in W^{-k,2}_{\sigma^\alpha}(\Omega)$; this is a direct consequence of Theorem 2.3.

An immediate consequence of Theorems 3.2, 3.1 is

Theorem 3.3. *Assume $\Omega \in \mathfrak{N}^{0,1}$, $\sigma \in C^{k-1,1}(\Omega)$, $\sigma > 0$ a weight satisfying (6.20) and (6.58). Let $A(v,u)$ be the sesquilinear form (6.63), $W^{-k,2}_0(\Omega)$-elliptic, and $H_1 = W^{k,2}_{0,\sigma^{-\alpha}}(\Omega)$, $H_2 = W^{k,2}_{0,\sigma^\alpha}(\Omega)$. Consider $\alpha \in I_1 \cap I_2$ (cf. Theorem 3.2), $f \in W^{-k,2}_{\sigma^\alpha}(\Omega)$, $u_0 \in W^{k,2}_{\sigma^\alpha}(\Omega)$. Then there exists a unique solution u of the Dirichlet problem and it satisfies the estimate*

$$|u|_{W^{k,2}_{\sigma^\alpha}(\Omega)} \le c(|f|_{W^{-k,2}_{\sigma^\alpha}(\Omega)} + |u_0|_{W^{k,2}_{\sigma^\alpha}(\Omega)}). \tag{6.79}$$

Theorem 3.4. *Let us consider $\Omega \in \mathfrak{N}^{0,1}$, $\sigma \in C^{k-1,1}(\Omega), \sigma > 0$ a weight satisfying (6.20) and (6.58). Let $A(v,u)$ be the sesquilinear form (6.63), $W^{k,2}_0(\Omega)$-elliptic,* *and* $H_1 = W^{k,2}_{0,\sigma^{-\alpha}}(\Omega)$, $H_2 = W^{k,2}_{0,\sigma^{\alpha}}(\Omega)$. *Let us assume* $W^{k,2}_{0,\sigma^{-\alpha}}(\Omega) \subset W^{k,2}_0(\Omega) \subset W^{k,2}_{0,\sigma^{\alpha}}(\Omega)$ *algebraically and topologically, which holds for* $\alpha \geq 0$. *The conclusion of the previous theorem and (6.79) holds for* $\alpha \in I_2$, $f \in W^{-k,2}_{\sigma^{\alpha}}(\Omega)$, $u_0 \in W^{k,2}_{\sigma^{\alpha}}(\Omega)$.

In this case it is sufficient to choose $B = W^{k,2}_0(\Omega)$ to apply Theorem 3.1.

Remark 3.6. Consider $\Omega \in \mathfrak{N}^{0,1}$, $\sigma(x) = \operatorname{dist}(x, \partial\Omega)$, $|\alpha| < 1$; condition (6.78) gives a condition for traces $u, \partial u/\partial n, \ldots \partial^{k-1}u/\partial n^{k-1}$, cf. Theorem 2.3.

If a_{ij}, f are smooth enough, then according to Theorem 4.1.2, the solution of the Dirichlet problem is also smooth.

Example 3.7. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $y \in \partial\Omega$. Let us assume that there exists a cone, contained in $\complement\Omega$ with vertex y and angle ω. Consider $r = |x - y|$, A the operator given in (6.63), $u_0 = r^{\gamma}\lambda(\varphi)$, $\lambda(\varphi) \in C^k([0,l])$, $l = 2\pi - \omega$. We apply Theorem 3.4 with $0 < \alpha$, $\alpha \in I_2$. We obtain $u_0 \in W^{k,2}_{r^{\alpha}}$ if $\gamma > k - \alpha/2 - 1$; in particular for $\gamma = k - 1$. We can also consider boundary conditions as in Example 5.4.1.

6.3.4 The Neumann Problem and Other Problems

Consider a boundary value problem as in Chap. 3. Let a V-elliptic sesquilinear form on $W^{k,2}(\Omega) \times W^{k,2}(\Omega)$ be given. Hereafter we shall assume that there exists a space $\mathscr{V}$, $C^{\infty}_0(\Omega) \subset \mathscr{V} \subset C^{\infty}(\overline{\Omega})$, such that $\overline{\mathscr{V}} = V$ in $W^{k,2}(\Omega)$. Let us assume $\sigma \in C(\Omega)$, $\sigma > 0$ in Ω, $V^{k,2}_{\sigma^{\alpha}}(\Omega) \equiv \overline{\mathscr{V}}$ in $W^{k,2}_{\sigma^{\alpha}}(\Omega)$. Let B be a normal Banach space, $V^{k,2}_{\sigma^{-\alpha}} \subset B$ algebraically and topologically, Q a normal Banach space, $Q = \overline{C^{\infty}_0(\Omega)}$, such that $W^{k,2}_{\sigma^{-\alpha}} \subset Q$ algebraically and topologically. Let $f \in Q'$, $g \in (\overline{V^{k,2}_{\sigma^{-\alpha}}(\Omega)})'$, $u_0 \in W^{k,2}_{\sigma^{\alpha}}$ such that for all $\varphi \in C^{\infty}_0(\Omega)$, $g\varphi = 0$. Finally let us assume that the sesquilinear form $((v,u))$ is B and $V^{k,2}_{\sigma^{\alpha}}$ elliptic. A function $u \in W^{k,2}_{\sigma^{\alpha}}$ is a solution of the problem if

$$u - u_0 \in V^{k,2}_{\sigma^{\alpha}} \tag{6.80a}$$

$$v \in V^{k,2}_{\sigma^{-\alpha}} \Longrightarrow ((v,u)) = \langle v, \overline{f} \rangle + \overline{g}v. \tag{6.80b}$$

We deduce from Theorem 3.1 the existence of a unique solution and the following inequality:

$$|u|_{W^{k,2}_{\sigma}(\Omega)} \leq c(|f|_{Q'} + |g| + |u_0|_{W^{k,2}_{\sigma}}). \tag{6.81}$$

The only problem is to prove the B and $W^{k,2}_{\sigma^{\alpha}}$ ellipticity.

We restrict ourselves to considering problems for which $V = W^{k,2}(\Omega)$, $\sigma = |x - y|$, $y \in \partial\Omega$, or $\sigma = \log(2M/|x - y|)$, $M = \operatorname{diam}(\Omega)$. The modifications for mixed problems and for the case

$$V=\{v\in W^{k,2}(\Omega),\quad \frac{\partial^l u}{\partial n^l}=\frac{\partial^{l+1}u}{\partial n^{l+1}}=\dots\frac{\partial^{k-1}u}{\partial n^{k-1}}=0 \text{ on } \partial\Omega\}$$

and others combinations are clear. For the weight σ mentioned and $\Omega\in\mathfrak{N}^{0,1}$, the traces on $\partial\Omega$ are well defined; we shall use another definition for $V^{k,2}_{\sigma^{-\alpha}}(\Omega)$, $V^{k,2}_{\sigma^{\alpha}}(\Omega)$ without the hypothesis of the existence of a $\mathscr{V}$, i.e. we define $V^{k,2}_{\sigma^{\alpha}}=\{v\in W^{k,2}_{\sigma^{\alpha}}$, $B_{is}v=0$ on $\partial\Omega$, $i=1,2,\dots,\kappa$, $s=1,2,\dots,\mu_i\}$; cf. 3.2.3.

6.3.5 An Imbedding Theorem for $W^{k,p}_{r^{\alpha}}(\Omega)$

Theorem 3.5. *Let $\Omega\in\mathfrak{N}^{0,1}$, $\alpha\neq p-N$, $y\in\partial\Omega$,* [5] *$\sigma(x)=|x-y|$. Assuming $\alpha>p-N$, the imbedding $W^{1,p}_{\sigma^{\alpha}}(\Omega)\subset L^p_{\sigma(\alpha-p)}(\Omega)$ is algebraic and topological. If $\alpha<p-N$, $u\in C(\overline{\Omega})\cap W^{1,p}_{\sigma^{\alpha}}(\Omega)$, $u(y)=0$, then $|u|_{L^p_{\sigma(\alpha-p)}(\Omega)}\le c|u|_{W^{1,p}_{\sigma^{\alpha}}(\Omega)}$.*

Proof. Without restriction of generality we can assume $y=(0,0)$ in the system (x_1',x_{1N}). We can consider $W^{1,p}_{\sigma^{\alpha}}(V_1)$, cf. 1.2.4, using the partition of unity from 1.2.4; we take functions in $W^{1,p}_{\sigma^{\alpha}}(V_1)$ with support in $V_1\cup\Lambda_1$, cf. 1.2.4. Let us apply the mapping (2.47). Since the mapping $x=T(y)$ and its inverse are both lipschitzian, we obtain by Lemma 2.3.2: If $u\in W^{1,p}_{\sigma^{\alpha}}(V_1)$, $v(y)=u(T(y))$, then

$$c_1|u|_{W^{1,p}_{\sigma^{\alpha}}(V_1)}\le|v|_{W^{1,p}_{r^{\alpha}}(K_+)}\le c_2|u|_{W^{1,p}_{\sigma^{\alpha}}(V_1)},\ r(z)=|z|. \tag{6.82}$$

Let us consider $v\in W^{1,p}_{r^{\alpha}}(K_+)$, with support in $K_+\cup\Delta$.

First, assume $\alpha<p-N$.

It follows from (6.26) that

$$\begin{aligned}\int_{K_+}|v|^p r^{\alpha-p+N-1}\,\mathrm{d}r\,\mathrm{d}S&\le\frac{p^p}{|\alpha+N-p|^p}\int_{K_+}\left|\frac{\partial v}{\partial r}\right|^p r^{\alpha+N-1}\,\mathrm{d}r\,\mathrm{d}S\\&=\frac{p^p}{|\alpha+N-p|^p}\int_{K_+}\left|\frac{\partial v}{\partial r}\right|^p r^{\alpha}\,\mathrm{d}x,\end{aligned}\tag{6.83}$$

and with (6.82) we will get our assertion.

If $\alpha<p-N$ and $u(y)=0$ we use inequality (6.20) with v and obtain (6.83), and the result holds in the general case. □

We have also

[5] It is sufficient to assume $\partial\Omega$ is lipschitzian in a neighborhood of y.

Theorem 3.6. *Let us assume the hypotheses as in the previous theorem. Then for $\alpha \neq p-N$ (for $\alpha < p-N$, $u(y)=0$)*

$$\left(\int_{\partial\Omega} |u|^p r^{(\alpha-p+1)}\,\mathrm{d}S\right)^{1/p} \leq c|u|_{W^{1,p}_{r^\alpha}(\Omega)}.$$

Proof. Let us set $\rho(x) = |x-y|$, $x \in \partial\Omega$. As in (6.24) we have:

$$\begin{aligned} |u(x'_r, a_r(x'_r))|^p &\leq 2^{(p-1)}\left(|u(x'_r,\eta)|^p + \left(\int_{a_r(x'_r)}^{a_r(x'_r)+\rho} \left|\frac{\partial u}{\partial x_{rN}}(x'_r,\xi)\right| \mathrm{d}\xi\right)^p\right) \\ &\leq 2^{(p-1)}\left(|u(x'_r,\eta)|^p + \rho^{p-1}\int_{a_r(x'_r)}^{a_r(x'_r)+\rho} \left|\frac{\partial u}{\partial x_{rN}}(x'_r,\xi)\right|^p \mathrm{d}\xi\right). \end{aligned} \tag{6.84}$$

Now by integration with respect to η in $(a_r(x'_r), a_r(x'_r)+\rho)$ we get

$$\begin{aligned} &|u(x'_r, a_r(x'_r))|^p \rho^{(\alpha-p+1)} \leq \\ &\leq c_1\Big(\int_{a_r(x'_r)}^{a_r(x'_r)+\rho} |u(x'_r,\eta)|^p r^{(\alpha-p)} d\eta + \int_{a_r(x'_r)}^{a_r(x'_r)+\rho} r^\alpha \left|\frac{\partial u}{\partial x_{rN}}(x'_r,\xi)\right|^p \mathrm{d}\xi\Big) \leq \\ &\leq c_1 \int_{a_r(x'_r)}^{a_r(x'_r)+\beta} |u(x'_r,\eta)|^p r^{(\alpha-p)} d\eta + c_2 \int_{a_r(x'_r)}^{a_r(x'_r)+\beta} r^\alpha \left|\frac{\partial u}{\partial x_{rN}}(x'_r,\xi)\right|^p \mathrm{d}\xi. \end{aligned} \tag{6.85}$$

If we integrate (6.85) with respect to x'_r in Δ_r, the result follows. Clearly if $|x-y| \geq$const, we apply $|u|_{L^p(\partial\Omega)} \leq c_3|u|_{W^{1,p}_{r^\alpha}(\Omega)}$, the inequality obtained in Chap. 2; without using Theorem 2.4.2 we can prove it as in (6.85) using a partition of unity and the domains $V_r, r = 1,2,\dots,m$, cf. 1.2.4. □

Now we prove

Theorem 3.7. *Let $\Omega \in \mathfrak{N}^{0,1}$, $a_{ij} \in L^\infty(\Omega)$, $b_{ij} \in L^\infty(\partial\Omega)$, and suppose the sesquilinear form*

$$((v,u)) = \int_\Omega \sum_{|i|,|j|\leq k} \overline{a}_{ij} D^i v D^j \overline{u}\,\mathrm{d}x + \int_{\partial\Omega} \sum_{i=0}^{k-1} \sum_{|j|\leq k-1} \overline{b}_{ij} \frac{\partial^i v}{\partial n^i} D^j \overline{u}\,\mathrm{d}S, \tag{6.85 bis}$$

is $W^{k,2}(\Omega)$-elliptic. Let $y \in \partial\Omega$, $r = |x-y|$. For N odd, if $(N-1)/2 \geq k$, let $\alpha > 2k-N$; if $(N-1)/2 < k$, let $|\alpha| < 1$. If N is even, let $k \leq (N-2)/2$, $\alpha > 2k-N$. Then there exist intervals I_1, I_2, containing neighborhoods of zero such that for $\alpha \in I_1$, resp. for $\alpha \in I_2$[6] *the sesquilinear form $((v,u))$is $W^{k,2}_{r^{-\alpha}}(\Omega)$-elliptic, resp. $W^{k,2}_{r^\alpha}(\Omega)$-elliptic. We assume for N odd, if $(N-1)/2 \geq k$, $I_2 \subset (2k-N,\infty)$,*

[6] It is possible to find I_1, I_2 uniformly for $y \in \partial\Omega$.

$I_1\subset(-\infty,N-2k)$, if $(N-1)/2<k$, $I_1,I_2\subset(-1,1)$. For N even, we assume $I_2\subset(2k-N,\infty)$, $I_1\subset(-\infty,N-2k)$.

Proof. Let N be even; for $u\in W^{k,2}_{r^\alpha}(\Omega)$, let us set $v=ur^\alpha$. If $\varepsilon>0$, we have for $\alpha\in(2k-N+\varepsilon,\infty)$,

$$|v|_{W^{k,2}_{r^{-\alpha}}(\Omega)}\le c(\varepsilon)|u|_{W^{k,2}_{r^\alpha}(\Omega)}. \tag{6.86}$$

It is sufficient to apply Theorem 3.5 for $p=2$, $\alpha,\alpha-2,\dots,\alpha-2k+2$.

We have:

$$((v,u))=A(ur^{\alpha/2},ur^{\alpha/2})+a(ur^{\alpha/2},ur^{\alpha/2})+B(v,u)+b(v,u), \tag{6.87}$$

where $B(v,u)$ is a sum of terms as in (6.63), (6.64), with $\sigma=r$. We have $|D^i r^\beta|\le c_1|\beta|r^{(\beta-|i|)}$, then

$$\left|\int_\Omega \overline{a}D^{i'}uD^{i''}r^\alpha D^j\overline{u}\,\mathrm{d}x\right|\le c_2|\alpha|\left(\int_\Omega |D^{i'}u|^2 r^{(\alpha-2|i''|)}\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |D^j u|^2 r^\alpha\,\mathrm{d}x\right)^{1/2},$$

and according to Theorem 3.5, we get

$$\left|\int_\Omega \overline{a}D^{i'}uD^{i''}r^\alpha D^j\overline{u}\,\mathrm{d}x\right|\le c_3|\alpha||u|^2_{W^{k,2}_{r^\alpha}(\Omega)}.$$

The term (6.64) can be estimated in the same way, and we get

$$|B(v,u)|\le c_4(\varepsilon)|\alpha||u|^2_{W^{k,2}_{r^\alpha}(\Omega)}. \tag{6.88}$$

Now $b(v,u)$ is a sum of terms of the following type:

$$\begin{gathered}\int_{\partial\Omega}\overline{b}\frac{\partial^{i'}v}{\partial n^{i'}}\frac{\partial^{i''}r^\alpha}{\partial n^{i''}}D^j\overline{u}\,\mathrm{d}S,\\ |i'+i''|\le k-1,\ |j|\le k-1,\ |i''|\ge 1,\end{gathered} \tag{6.89}$$

and of the type

$$\begin{gathered}\int_{\partial\Omega}\overline{b}\frac{\partial^{i'}v}{\partial n^{i'}}\frac{\partial^{i''}r^{\alpha/2}}{\partial n^{i''}}D^{j'}\overline{u}D^{j''}r^{\alpha/2}\,\mathrm{d}S,\\ |i'+i''|\le k-1,\ |j'+j''|\le k-1,\ |i'|+|j'|\le 2k-3,\end{gathered} \tag{6.90}$$

where $b\in L^\infty(\partial\Omega)$. Using Theorem 3.6, we obtain

$$|b(v,u)|\le c_5(\varepsilon)|\alpha||u|^2_{W^{k,2}_{r^\alpha}(\Omega)}. \tag{6.91}$$

We have

$$|((ur^{\alpha/2}, ur^{\alpha/2}))| \geq c_6 |ur^{\alpha/2}|^2_{W^{k,2}(\Omega)} \geq c_7(\varepsilon)|u|^2_{W^{k,2}_{r^\alpha}(\Omega)},$$

hence

$$|((v,u))| \geq [c_7(\varepsilon) - c_4(\varepsilon)|\alpha| - c_5(\varepsilon)|\alpha|]|u|^2_{W^{k,2}_{r^\alpha}(\Omega)}, \tag{6.91 bis}$$

which gives the $W^{k,2}_{r^\alpha}(\Omega)$-ellipticity for $|\alpha| < c_7(\varepsilon)/(c_4(\varepsilon)+c_5(\varepsilon))$; the $W^{k,2}_{r^{-\alpha}}(\Omega)$-ellipticity can be obtained in the same way.

Let N be odd, $(N-1)/2 \geq k$. We proceed as above.

Let N be odd, $(N-1)/2 < k$, $\alpha \in (1-\varepsilon,\, 1+\varepsilon)$. Let $m = (N+1)/2$; we have $W^{k,2}_{r^\alpha}(\Omega) \subset C^{k-m}(\overline{\Omega})$ algebraically and topologically, with

$$|u|_{C^{k-m}(\overline{\Omega})} \leq c_8(\varepsilon)|u|_{W^{k,2}_{r^\alpha}(\Omega)}. \tag{6.92}$$

Indeed we obtain, without difficulty, if $0 \leq \alpha \leq 1-\varepsilon,\ p < 2N/(N+\alpha)$,

$$|u|_{W^{k,p}(\Omega)} \leq c_9(\varepsilon)|u|_{W^{k,2}_{r^\alpha}(\Omega)}; \tag{6.93}$$

for $\alpha < 0$ we have obviously

$$|u|_{W^{k,2}(\Omega)} \leq |u|_{W^{k,2}_{r^\alpha}(\Omega)}. \tag{6.94}$$

If $p = 2N/(N+\alpha+1/2\varepsilon)$, it follows from Theorem 2.3.8:

$$|u|_{C^{k-m}(\overline{\Omega})} \leq c_{10}(\varepsilon)|u|_{W^{k,p}(\Omega)},$$

then using (6.93) we get (6.92); if $\alpha < 0$, (6.92) is a consequence of (6.94) and Theorem 2.3.8. Let

$$\begin{aligned} M_1 &= \{u \in W^{k,2}_{r^\alpha}(\Omega),\ \sum_{|i|\leq k-m} |D^i u(y)| = 0\},\\ N_1 &= \{v \in W^{k,2}_{r^{-\alpha}}(\Omega),\ \sum_{|i|\leq k-m} |D^i v(y)| = 0\}. \end{aligned} \tag{6.95}$$

We have N_1 and M_1 ellipticity if α is sufficiently small. Indeed: If $u \in M_1, v = ur^\alpha$, then according to Theorem 3.5, for $|i| \leq k$ we have

$$\int_\Omega |D^i u|^2 r^{(\alpha-2(k-|i|))}\,\mathrm{d}x \leq c_{11}(\varepsilon)|u|^2_{W^{k,2}_{r^\alpha}(\Omega)}; \tag{6.96}$$

furthermore, for $|i| \leq k$,

$$\int_\Omega |D^i v|^2 r^{(\alpha-2(k-|i|))}\,\mathrm{d}x \leq c_{12}(\varepsilon)|u|^2_{W^{k,2}_{r^\alpha}(\Omega)}, \tag{6.97}$$

and then (6.86) follows. Moreover for $|i|\le k-m$, $\alpha-2(k-|i|)+N-1<-1$, we have (6.97), (6.92) for v, and so $v\in N_1$. Now we can consider (6.87), we obtain as above $I\subset(-1,1)$ such that for $\alpha\in I$ the sesquilinear form $((v,u))$ is N_1 and M_1-elliptic. Let us denote $N_2=\{v\in W^{k,2}_{r^{-\alpha}}(\Omega),\ ((v,u))=0 \text{ for } u\in N_1\}$, $M_2=\{u\in W^{k,2}_{r^{\alpha}}(\Omega),\ ((v,u))=0 \text{ for } v\in N_2\}$, and denote by n the dimension of the space of polynomials $\sum_{|i|\le k-m}a_i(x-y)^i$. We have $\dim M_1=\dim M_2=n$. Indeed: let $p_1,p_2,\dots,p_n$ be a basis of these spaces of polynomials. For every p_i, $i=1,2,\dots,n$ we can find a unique $v_i\in N_1, u_i\in M_1$ such that $v_i+p_i\in N_2, u_i+p_i\in M_2$; this follows from $u\in M_1\Longrightarrow((v_i+p_i,u))=((u,v_i+p_i))^*=0$, whence $((u,v_i))^*=-((u,p_i))^*$. Applying Theorem 3.1 for the form $((u,v))^*$, which is N_1 and M_1 elliptic for $\alpha\in I$, we obtain the existence and uniqueness of v_i. Concerning u_i we have $v\in N_1\Longrightarrow((v,u_i+p_i))=0$, and then we get $((v,u_i))=-((v,p_i))$ and the existence and uniqueness of u_i. In case $\alpha\ge 0$, we moreover have

$$u_i\in W^{k,2}(\Omega). \tag{6.97 bis}$$

Indeed if we denote $W=\{v\in W^{k,2}(\Omega),\sum_{|i|\le k-m}|D^iv(y)|=0\}$, then the sesquilinear form $((v,u))$ is W-elliptic, and the result follows. Hence we have $N_1\dot{+}N_2=W^{k,2}_{r^{-\alpha}}(\Omega)$, $M_1\dot{+}M_2=W^{k,2}_{r^{\alpha}}(\Omega)$; if $u\in W^{k,2}_{r^{\alpha}}(\Omega)$ we have

$$u=u_1+\sum_{|i|\le k-m}\frac{1}{i!}D^iu(y)(x-y)^i,\ u_1\in M_1. \tag{6.98}$$

Clearly, the decomposition $u=u_{M_1}+u_{M_2}$ is unique; indeed if we assume that there exists another decomposition, the difference of these two decompositions, say $\omega_{M_1}+\omega_{M_2}$ will be equal to zero with $\omega_1\ne 0$, $\omega_2\ne 0$, which is not possible since there exists $w\in N_1$, $|w|_{W^{k,2}_{r^{-\alpha}}(\Omega)}=1$, such that $0=((w,\omega_{M_1}+\omega_{M_2}))=((w,\omega_{M_1}))\ge c_{13}(\varepsilon)|\omega_{M_1}|_{W^{k,2}_{r^{\alpha}}(\Omega)}$. The same result holds for $W^{k,2}_{r^{-\alpha}}(\Omega)$. Now it follows from (6.92) for $W^{k,2}_{r^{\alpha}}(\Omega)$, $W^{k,2}_{r^{-\alpha}}(\Omega)$ and from (6.98) that the mappings $P_i,Q_i,i=1,2$ defined for $u\in W^{k,2}_{r^{\alpha}}(\Omega)$, $v\in W^{k,2}_{r^{-\alpha}}(\Omega)$ by $P_iu=u_{M_i}$, $Q_iv=v_{N_i}$ are linear and bounded, $P_i: W^{k,2}_{r^{\alpha}}(\Omega)\to W^{k,2}_{r^{\alpha}}(\Omega)$, $Q_i: W^{k,2}_{r^{-\alpha}}(\Omega)\to W^{k,2}_{r^{-\alpha}}(\Omega)$.

Let us consider now $((v,u))$ on $N_2\times M_2$. We have still the N_2 and M_2 ellipticity. Let, for instance, $u\in M_2$; there always exists $v\in W^{k,2}_{r^{-\alpha}}(\Omega)$, $|v|_{W^{k,2}_{r^{-\alpha}}(\Omega)}=1$, such that $|((v,u))|>0$: if $\alpha\ge 0$, then (6.97 bis) implies $u\in W^{k,2}(\Omega)$. There exists $v\in C^\infty(\overline{\Omega})$ such that $|((v,u))|>0$. If $\alpha\le 0$, $u\in W^{k,2}(\Omega)$ automatically and the result again holds. Let now $v=v_{N_1}+v_{N_2}$; clearly $|((v_{N_2},u))|>0$, and it follows, for $u\in M_2$ that

$$f(u)=\sup_{v\in N_2,|v|_{W^{k,2}_{r^{-\alpha}}(\Omega)}=1,}|((v,u))|>0. \tag{6.99}$$

But the function $f(u)$ is continuous on M_2, $\dim M_2=n<\infty$, and so

$$\inf_{u \in M_2, |u|_{W^{k,2}_{r\alpha}(\Omega)}=1} f(u) > 0,$$

and this implies the M_2-ellipticity. The N_2-ellipticity can be obtained in the same way.

Now let us prove the $W^{k,2}_{r^{-\alpha}}(\Omega)$ and $W^{k,2}_{r^{\alpha}}(\Omega)$ ellipticity. Let $|u|_{W^{k,2}_{r\alpha}(\Omega)} = 1$; we have $u = u_{M_1} + u_{M_2}$. There exists $v_i \in N_i$ such that $((v_i, u_{M_i})) \geq c_{14}(\varepsilon)|u_{M_i}|_{W^{k,2}_{r\alpha}(\Omega)}$, and then

$$((v_1 + v_2, u)) \geq c_{14}(\varepsilon)(|u_{M_1}|_{W^{k,2}_{r\alpha}(\Omega)} + |u_{M_2}|_{W^{k,2}_{r\alpha}(\Omega)}) \geq c_{14}(\varepsilon)|u|_{W^{k,2}_{r\alpha}(\Omega)}.$$

With the same argument we prove the $W^{k,2}_{r^{-\alpha}}(\Omega)$-ellipticity. □

6.3.6 Concluding Remarks

There is a problem in Theorem 3.7 concerning the condition $k \leq (N-2)/2, N$ even; for instance the case $N = 2$ cannot be taken into account, the case $N = 4$ works only for $k = 1$. To eliminate these restrictions, for $N = 2$, we can use the weight $\sigma(x) = \log(2M/|x-y|), M =$ diam (Ω). This is left as an exercise:

Exercise 3.4. Prove for $\alpha < p-1, p \leq 2$ that

$$\begin{aligned}&\int_0^{1/2} \left| \int_r^{1/2} u(\rho)\mathrm{d}\rho \right|^p (-\log r)^{(\alpha-p)} r^{(-p+1)}\, \mathrm{d}r \\ &\leq \left(\frac{p}{p-1-\alpha} \right)^p \int_0^{1/2} |u(r)|^p (-\log r)^{\alpha} r\, \mathrm{d}r,\end{aligned} \tag{6.100}$$

and for $\alpha > 2,\ p-\beta-1 > 0,\ 2(2-\alpha) \leq -\log b(p-\beta-1)$ that

$$\begin{aligned}&\int_0^{b} \left| \int_0^{r} u(\rho)d\rho \right|^p (-\log r)^{\alpha-2} r^{(\beta-p)}\, \mathrm{d}r \\ &\leq \left(\frac{2p}{p-1-\beta} \right)^p \int_0^{b} |u(r)|^p (-\log r)^{\alpha-2} r^{\beta}\, \mathrm{d}r.\end{aligned} \tag{6.101}$$

Exercise 3.5. Using (6.100), (6.101) prove the existence of $J \subset (-1,1)$ including a neighborhood of zero, such that the sesquilinear form (6.85 bis) is $W^{k,2}_{\log^{-\alpha}(2M/r)}(\Omega)$-elliptic and $W^{k,2}_{\log^{\alpha}(2M/r)}(\Omega)$-elliptic.

Hint: Use the same method as in Theorem 3.7.

Example 3.8. Let $N \geq 3$, Ω a sector in the unit ball with center at the origin, $V^{1,2}_{r\alpha} = \{v \in W^{1,2}_{r\alpha},\ r = |x|,\ v(x) = 0 \text{ if } |x| = 1\}$. For the sesquilinear form

$$A(v,u) = \int_\Omega \sum_{i=1}^N \frac{\partial v}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i} \, \mathrm{d}x,$$

we have the $V^{1,2}_{r^\alpha}(\Omega)$-ellipticity if $|\alpha| < N-2$.
Indeed, we obtain for $v = ur^\alpha, |\alpha| < N-2, u \in V^{1,2}_{r^\alpha}(\Omega)$ that $|v|_{W^{k,2}_{r^{-\alpha}}(\Omega)} \leq c(\alpha)|u|_{W^{k,2}_{r^\alpha}(\Omega)}$. Using integration by parts, we get:

$$\operatorname{Re} A(ur^\alpha, u) = \frac{\alpha}{2} \sum_{i=1}^N \int_{\partial\Omega} r^{(\alpha-2)} x_i n_i |u|^2 \, \mathrm{d}S +$$

$$\frac{\alpha}{2}(2-\alpha-N) \int_\Omega r^{(\alpha-2)} |u|^2 \, \mathrm{d}x + \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 r^\alpha \, \mathrm{d}x.$$

$$\sum_{i=1}^N \int_{\partial\Omega} r^{(\alpha-2)} x_i n_i |u|^2 \, \mathrm{d}S = 0;$$

the $V^{1,2}_{r^\alpha}(\Omega)$-ellipticity of A follows if $2-N < \alpha \leq 0$. If $|\alpha| < N-2$, then on $V^{1,2}_{r^\alpha}$ the norms induced by $W^{1,2}_{r^\alpha}(\Omega)$ and $H^{1,2}_{r^\alpha}(\Omega)$ are equivalent. As in Example 3.3, the result follows for $|\alpha| < N-2$.

Example 3.9. Let $\Omega \subset \mathbb{R}^2$ be the sector $\Omega = \{x, 0 < r < 1/2, |\varphi| < \pi - \omega/2\}$, Ω_i, $i = 1,2$, the subdomains

$$\Omega_1 = \{x, 0 < r < 1/2, 0 < \varphi < \pi - \omega/2\},$$

$$\Omega_2 = \{x, 0 < r < 1/2, -\pi + \omega/2 < \varphi < 0\},$$

and

$$A(v,u) = a_1 \int_{\Omega_1} \left(\frac{\partial v}{\partial r} \frac{\partial \overline{u}}{\partial r} + \frac{1}{r^2} \frac{\partial v}{\partial \varphi} \frac{\partial \overline{u}}{\partial \varphi} \right) \mathrm{d}x + a_2 \int_{\Omega_2} \left(\frac{\partial v}{\partial r} \frac{\partial \overline{u}}{\partial r} + \frac{1}{r^2} \frac{\partial v}{\partial \varphi} \frac{\partial \overline{u}}{\partial \varphi} \right) \mathrm{d}x$$

the sesquilinear form defined for $a_1 > 0, a_2 > 0$.

Using inequality (6.100), and the spaces $H^{1,2}_{(-\log r)^\alpha}(\Omega)$, we obtain the $V^{1,2}_{(-\log r)^\alpha}(\Omega)$-ellipticity for $|\alpha| < 1$; here we take $V^{1,2}_{(-\log r)^\alpha}(\Omega) = \{v \in W^{1,2}_{(-\log r)^\alpha}(\Omega), v(x) = 0 \text{ for } |x| = 1/2\}$.

If $V^{1,2}_{(-\log r)^\alpha}(\Omega) = W^{1,2}_{(-\log r)^\alpha}(\Omega)$, the result holds for $A(v,u) + \lambda(v,u), \lambda$ sufficiently large.

Problem 3.3. Using the weight $\sigma(x) = |x-y|$ with $y \in \Omega$ fixed, does the method of Theorem 3.7 hold? For more precise results concerning the values of α, it seems useful to prove *a priori* estimates, cf. H.O. Cordes [1], [2].

Chapter 7
Regularity of the Solution for Non-smooth Coefficients and Non-regular Domains

The problem of solving a boundary value problem has been already investigated, but it remains to deal with the regularity of this solution in the interior of the domain and at the neighborhood of the boundary in the cases where the coefficients and the boundary are non-smooth. There are many open questions; a general theory for operators of order $2k$ in the following form:

$$\sum_{|i|,|j|\le k} (-1)^{|i|} D^i(a_{ij}D^j), \quad a_{ij} \in L^\infty(\Omega),$$

concerning for instance interior estimates in Ω^*, $\overline{\Omega}^* \subset \Omega$, of the following type:

$$|D^\alpha u|_{L^\infty(\Omega^*)} \le c(\Omega^*)|f|, \quad |\alpha| \le k-1, \tag{$*$}$$

where $|f|$ is a suitable norm and u the solution of a problem with homogeneous boundary conditions. It is also reasonable to study the regularity in a neighborhood of the boundary.

We consider second order operators of the following form,

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + d, \quad a_{ij}, b_i, d \in L^\infty(\Omega);$$

there exists a very general theory based on techniques and fundamental results of E. de Giorgi [1], and extended by G. Stampacchia [2], [3, 5–9, 12], C.B. Morrey Jr. [2], W.G. Littman, G. Stampacchia, and H. Weinberger [1]. The fundamental result states that the weak solution is hölderian in Ω. For the Dirichlet problem this result holds in $\overline{\Omega}$ with some conditions on $\partial\Omega$ ($\Omega \in \mathfrak{N}^{0,1}$ is sufficient). For other problems, cf. G. Stampacchia [3]. Another method related to Harnack's inequality can be found in J. Moser [1], [2]; this method is used by S.N. Kruzhkov [1]. G. Fichera [8] used another idea to prove that $u \in L^\infty(\Omega)$. The Brelot axiomatics and the maximum principle are proved in R.M. Hervé [1].

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Operators of the type

$$A = -\sum_{i,j=1}^{N} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + d$$

were considered by C. Pucci [1, 3], H.O. Cordes [1, 2], C. Miranda [1]. The maximum principle for these operators can be found in many papers by A.D. Aleksandrov [1–6].

For operators of order > 2, there exist some particular results. A result of the type (*) is not known to the author. If the coefficients a_{ij}, $|i|$, $|j| \leq k$, are smooth enough without regularity on $\partial\Omega$, the regularity of a weak solution in a neighborhood of the boundary is considered in the papers of the author [2, 5, 7, 9, 10, cf. also Chaps. 5, 6, this chapter and V.A. Kondratiev [1, 2].

For literature on connected topics cf. also G. Adler [1–3], S.N. Kruzhkov, L.P. Kupcov [1], W. Littman[1].

7.1 Second Order Operators

7.1.1 Auxiliary Lemmas

Let us consider a second order operator, with real coefficients:

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} + d, \quad a_{ij}, b_i, d \in L^\infty(\Omega). \tag{7.1}$$

Let us assume that for $\xi = (\xi_1, \xi_2, \ldots \xi_N)$,

$$\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j \geq c|\xi|^2, \quad c > 0. \tag{7.2}$$

Let us consider a bounded domain Ω in $\mathfrak{N}^{0,1}$. We remark that in this chapter we can deal with domains for which the imbedding theorems hold.

Let V be a closed subspace of $W^{1,2}(\Omega)$; in this section we consider only real functions; V satisfies the condition $W_0^{1,2}(\Omega) \subset V \subset W^{1,2}(\Omega)$, and we assume that

$$v \in V \Longrightarrow |v| \in V. \tag{7.3}$$

If $v \in V$, we denote by v^n, $n = 1, 2, \ldots$, the *cut-off function* defined as

$$v^n(x) = v(x) \text{ for } |v(x)| \leq n;$$

$$v^n(x) = n \operatorname{sgn} v(x) \text{ for } |v(x)| > n.$$

Let us assume that for $n \geq n_0$,

$$v \in V \Longrightarrow v^n \in V. \tag{7.4}$$

Let us denote $v_+ = \frac{1}{2}(|v|+v)$, $v_- = \frac{1}{2}(|v|-v)$. We shall assume that for $\lambda \geq 0$, $n \geq n_0$,

$$v \in V \Longrightarrow (v_+^n)^{1+\lambda} \in V, \quad (v_-^n)^{1+\lambda} \in V. \tag{7.5}$$

Using these definitions we shall prove a theorem due to G. Stampacchia (and in some sense a generalization of this result), cf. G. Stampacchia [2, 6] using in author's opinion a more simple method.

Lemma 1.1. *Let Ω be a bounded domain. Then $W^{1,2}(\Omega)$ satisfies (7.3)–(7.5).*

Proof. According to Theorem 2.2.3, if we modify the function $v \in W^{1,2}(\Omega)$ on a set of zero measure, this function is absolutely continuous on almost all lines parallel to the coordinate axes $x_1, x_2, \ldots, x_N$ and the usual derivatives coincide almost everywhere in Ω with the distributional derivatives. For instance, let us consider v_+ on a line parallel to x_1.

Let x be a point where $\partial v/\partial x_1$ exists in the classical sense; if $v(x) > 0$, we have $(\partial v/\partial x_1)(x) = (\partial v_+/\partial x_1)(x)$; if $v(x) = 0 = (\partial v/\partial x_1)(x)$, then $(\partial v_+/\partial x_1)(x) = 0$. The set of points x such that $v(x) = 0$, $(\partial v/\partial x_1)(x) \neq 0$ has zero measure on the considered line. For $v(x) < 0$, $(\partial v_+/\partial x_1)(x) = 0$. Now if we take into account Theorem 2.2.3 the result follows. □

The following lemmas will be useful:

Lemma 1.2. *Let Ω be a bounded domain. If $\lim_{n\to\infty} u_n = u$ in $W^{1,2}(\Omega)$, then $\lim_{n\to\infty} |u_n| = |u|$ in $W^{1,2}(\Omega)$.*

Proof. Clearly we get $\lim_{n\to\infty} |u_n| = |u|$ in $L^2(\Omega)$, then similarly as in the proof in Proposition 2.2.4, $\lim_{n\to\infty} |u_n| = |u|$ weakly in $W^{1,2}(\Omega)$; but $||u_n||_{W^{1,2}(\Omega)} = |u_n|_{W^{1,2}(\Omega)}$. □

Lemma 1.3. *Let Ω be a bounded domain. Then $W_0^{1,2}(\Omega)$ satisfies conditions (7.3) – (7.5).*

Proof. We proceed as in the proof of Lemma 1.1; using the previous lemma it follows that $|u|$, u_+, u_+^n etc. are continuous mappings from $W_0^{1,2}(\Omega)$ to $W_0^{1,2}(\Omega)$. □

Lemma 1.4. *Let us consider $\Omega \in \mathfrak{N}^{0,1}$, $\Lambda \subset \partial\Omega$, Λ open in $\partial\Omega$, $V = \{v \in W^{1,2}(\Omega), v = 0$ on $\Lambda\}$. Then V satisfies conditions (7.3)–(7.5).*

This result is a direct consequence of Theorems 2.2.3, 2.4.3.

7.1.2 Fundamental Lemmas

Lemma 1.5. *Let us consider the operator given by (7.1), (7.2):*

$$A = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + 1,$$

let V be the space as in (7.3)–(7.5), $1/2 \le 1/q < 1 - 1/N$. Let $V_q = \overline{V}$ in $W^{1,q}(\Omega)$, $F \in (V_q)'$, $u \in V$ such that

$$v \in V \Longrightarrow A(v,u) \equiv \int_\Omega \left(\sum_{i,j=1}^{N} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_j} + vu\right) \mathrm{d}x = Fv.$$

Then $u \in L^r(\Omega)$ with $1/r = 1/p - 1/N$, where $1/p + 1/q = 1$, and

$$|u|_{L^r(\Omega)} + |(u_+)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} + |(u_-)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} \le cr|F|_{(V_q)'}, \tag{7.6}$$

where $(1+\lambda/2)[2N/(N-2)] = r$.

Proof. According to Theorem 2.3.4 it follows:

$$\left(\int_\Omega (u_+)^{(1+\lambda/2)(2N/(N-2))}\,\mathrm{d}x\right)^{(N-2)/2N} \le c_1|(u_+)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)} \qquad (7.6\,\mathrm{bis}),$$

then $|u_+|_{L^r(\Omega)} \le c_1^{1/(1+\lambda/2)}|(u_+)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)}$; hence it is sufficient to prove the inequalities:

$$|(u_+)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} \le c_2\, r|F|_{(V_q)'}, \tag{7.7}$$

$$|(u_-)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} \le c_3\, r|F|_{(V_q)'}. \tag{7.8}$$

We will restrict ourselves to (7.7). Let $v = (u_+^n)^{(1+\lambda)}$. We have

$$A(v,u) = \int_\Omega \left(\sum_{i,j=1}^{N} (1+\lambda)(u_+^n)^\lambda a_{ij}\frac{\partial u_+^n}{\partial x_i}\frac{\partial u}{\partial x_j} + (u_+^n)^{1+\lambda}u\right) \mathrm{d}x = F(u_+^n)^{1+\lambda}.$$

Using Theorem 2.2.3, we get:

$$\begin{aligned}
A(v,u) &= \int_\Omega \left(\sum_{i,j=1}^N (1+\lambda)(u_+^n)^\lambda a_{ij} \frac{\partial u_+^n}{\partial x_i} \frac{\partial u_+^n}{\partial x_j} + (u_+^n)^{1+\lambda} u \right) \mathrm{d}x \\
&\geq \int_\Omega \left((1+\lambda) \sum_{i,j=1}^N (u_+^n)^\lambda a_{ij} \frac{\partial u_+^n}{\partial x_i} \frac{\partial u_+^n}{\partial x_j} + (u_+^n)^{(2+\lambda)} \right) \mathrm{d}x \\
&= \int_\Omega \left(\frac{1+\lambda}{(1+\lambda/2)^2} \sum_{i,j=1}^N a_{ij} \frac{\partial}{\partial x_i} (u_+^n)^{1+\lambda/2} \frac{\partial}{\partial x_j} (u_+^n)^{1+\lambda/2} + (u_+^n)^{(2+\lambda)} \right) \mathrm{d}x \\
&\geq \frac{c_2}{1+\lambda/2} |(u_+^n)^{1+\lambda/2}|^2_{W^{1,2}(\Omega)},
\end{aligned}$$

then

$$|(u_+^n)^{1+\lambda/2}|^2_{W^{1,2}(\Omega)} \leq c_3(1+\lambda/2)|F|_{(V_q)'}|(u_+^n)^{1+\lambda}|_{W^{1,q}(\Omega)}. \tag{7.9}$$

As $\lambda = [(2-q)N]/(qN-q-N)$, we get, according to (7.6 bis):

$$\begin{aligned}
&|(u_+^n)^{1+\lambda}|_{W^{1,q}(\Omega)} \leq \mu(\Omega)^{1/N} \left(\int_\Omega (u_+^n)^r \,\mathrm{d}x \right)^{1/q-1/N} \\
&+ \sum_{i=1}^N (1+\lambda) \left(\int_\Omega (u_+^n)^r \,\mathrm{d}x \right)^{(2-q)/2q} \left(\int_\Omega (u_+^n)^\lambda \left(\frac{\partial u_+^n}{\partial x_i} \right)^2 \mathrm{d}x \right)^{1/2} \\
&\leq c_4 |(u_+^n)^{1+\lambda/2}|^{1+\lambda/(2+\lambda)}_{W^{1,2}(\Omega)}.
\end{aligned} \tag{7.10}$$

Then (7.9), (7.10) imply the inequality:

$$|(u_+^n)^{1+\lambda/2}|^{1/(1+\lambda/2)}_{W^{1,2}(\Omega)} \leq c_2\, r |F|_{(V_q)'}. \tag{7.11}$$

According to Theorems 2.6.1 or 1.1.4, we can extract from the sequence $(u_+^n)^{1+\lambda/2}$ a subsequence which converges in $L^2(\Omega)$ and almost everywhere to $(u_+)^{1+\lambda/2}$; then, using Proposition 2.2.4, we obtain (7.7). □

Remark 1.1. If

$$A = -\sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial}{\partial x_j} \right) + d$$

with $d \geq 0$ and $d \neq 0$ if $V \neq W_0^{1,2}(\Omega)$, then we obtain Lemma 1.5 for this operator.

Let also $\sigma \in L^\infty(\partial\Omega)$, and let us define for the operator A satisfying (7.1), (7.2) the sesquilinear form

$$((v,u)) = A(v,u) + \int_{\partial\Omega} \sigma v u \,\mathrm{d}S. \tag{7.12}$$

We have

Lemma 1.6. *Let $((v,u))$ be given as in (7.12), V be the space verifying (7.3)–(7.5), F as in the previous lemma, $u \in V$ such that for all $v \in V$ we have $((v, u)) = Fv$. Then*

$$|(u_+)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} + |(u_-)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} \leq c(\lambda)(|F|_{(V_q)'} + |u|_{W^{1,2}(\Omega)}). \tag{7.13}$$

Proof. The function u satisfies

$$\begin{aligned}&\int_\Omega \Big(\sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j} + vu \Big) \mathrm{d}x \\ &= Fv - \int_\Omega \left(-vu + dvu + \sum_{i=1}^N b_i v \frac{\partial u}{\partial x_i} \right) \mathrm{d}x - \int_{\partial\Omega} \sigma v u \,\mathrm{d}S \equiv Fv - F_1 v.\end{aligned} \tag{7.14}$$

According to Theorems 2.3.4, 2.4.2 it follows: $|F_1|_{(V_{q_1})'} \leq c_1 |u|_{W^{1,2}(\Omega)}$, with $1/q_1 = 1/2 + 1/N$. If $q_1 \leq q$, (7.13) is a consequence of (7.7), which is true for $N \leq 4$. Let $N > 4$, and $q_1 > q$. Let us set $r_1 = 2N/(N-4)$, $r_{m+1} = [(N-1)/(N-2)]r_m$, $m = 1, 2, \ldots, \lambda_m = ((N-2)/N)r_m - 2$. Let $r_m \leq \tilde{r} \leq r_{m+1}$, $\tilde{r} \leq r$. We get as in (7.7):

$$\begin{aligned}&|(u_+^u)^{(1+\tilde\lambda/2)}|_{W^{1,2}(\Omega)}^2 \leq c_2(1+\lambda_{m+1}) \left(|(u_+)^{(1+\tilde\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} |F|_{(V_q)'} \right) \\ &+ c_2(1+\lambda_{m+1}) \Big(\int_\Omega (u_+^n)^{(1+\tilde\lambda-\lambda_m/2)} (u_+^n)^{\lambda_m/2} \left(u_+ + \sum_{i=1}^N \left| \frac{\partial u_+}{\partial x_i} \right| \right) \mathrm{d}x \\ &+ \int_{\partial\Omega} (u_+^n)^{(1+\tilde\lambda)} u_+ \,\mathrm{d}S \Big) \leq c_3(1+\lambda_{m+1}) \Big(|(u_+^n)^{(1+\tilde\lambda/2)}|_{W^{1,2}(\Omega)}^{1+\tilde\lambda/(2+\tilde\lambda)} |F|_{(V_q)'} \\ &+ \left(\int_\Omega (u_+)^{(2+2\tilde\lambda-\lambda_m)} \mathrm{d}x \right)^{1/2} |(u_+)^{(1+\lambda_m/2)}|_{W^{1,2}(\Omega)} + \int_{\partial\Omega} (u_+)^{2+\tilde\lambda} \,\mathrm{d}S \Big).\end{aligned} \tag{7.15}$$

We have $2 + 2\tilde\lambda - \lambda_m \leq 2 + 2\lambda_{m+1} - \lambda_m \leq r_m$,

$$2 + \tilde\lambda \leq 2(N-1)/(N-2)(1+\lambda_m/2);$$

this, (7.15), and Theorem 2.4.2, leads finally by an induction over $\lambda_1 \to \lambda_2$, $\lambda_2 \to \lambda_3,\dots,\lambda_m \to \lambda$, to

$$\begin{aligned}&|(u_+^n)^{(1+\tilde\lambda/2)}|^2_{W^{1,2}(\Omega)}\\&\le c_4(\lambda)\left(|(u_+^n)^{(1+\tilde\lambda/(2)}|^{(1+\overline{\lambda}/(2+\tilde\lambda)}_{W^{1,2}(\Omega)}|F|_{(V_q)'}+(|F|_{(V_q)'}+|u|_{W^{1,2}(\Omega)})^{(2+\tilde\lambda)}\right),\end{aligned} \tag{7.16}$$

thus

$$\frac{|(u_+^n)^{(1+\tilde\lambda/2}|^{(1+\tilde\lambda/2)}_{W^{1,2}(\Omega)}}{(|F|_{(V_q)'}+|u|_{W^{1,2}(\Omega)})^{(2+\tilde\lambda)}}\le c_5(\lambda),$$

and by the same procedure as in the proof of Lemma 1.5 we have

$$|(u_+)^{(1+\tilde\lambda/2)}|^{1/(1+\tilde\lambda/2)}_{W^{1,2}(\Omega)}\le c_6(\lambda)(|F|_{(V_q)'}+|u|_{W^{1,2}(\Omega)}). \tag{7.17}$$

Finally, by induction (7.17) holds for λ; the same computations work for u_-. □

We don't consider the possible generalizations concerning the coefficients b_i, d. These questions are considered in G. Stampacchia's paper [12].

Using Theorem 3.6.1, it is easy to prove

Exercise 1.1. Let A be the operator (7.1), (7.2) with $b_i \equiv 0$, $d \in L^{N/2}(\Omega)$, $N \ge 3$, $d \ge 0$, $d \ne 0$. Then (7.6) holds.

7.1.3 Regularity of the Solution of the Boundary Value Problem

From the previous lemmas, we obtain immediately interesting consequences for the solutions of boundary value problems.

The Dirichlet problem:

Theorem 1.1. *Let us consider $\Omega \in \mathfrak{N}^{0,1}$, $V = W_0^{1,2}(\Omega)$, A the operator given in (7.1), (7.2), $1/N < 1/p \le 1/2$, $f_i \in L^p(\Omega)$, $i = 1,2,\dots,N$, $u_0 \in W^{1,p}(\Omega)$. Let $u \in W^{1,2}(\Omega)$ be the solution of the Dirichlet problem $Au = \sum_{i=1}^N \partial f_i/\partial x_i$ in Ω,*[1] *$u - u_0 \in W_0^{1,2}(\Omega)$. Then if $1/r = 1/p - 1/N$, we have*

$$|u|_{L^r(\Omega)}+|u|_{L^{r(N-1)/N}(\partial\Omega)}\le c(p)\Big(\sum_{i=1}^N|f_i|_{L^p(\Omega)}+|u_0|_{W^{1,p}(\Omega)}+|u|_{W^{1,2}(\Omega)}\Big). \tag{7.18}$$

[1]In other words, $Au = f$ with $f \in W^{-1,p}(\Omega)$.

The proof follows from Lemma 1.6 and from Theorems 2.3.4, 2.4.2. The same situation appears in

Theorem 1.2. *We keep the same hypotheses as in the previous theorem but we replace* $\sum_{i=1}^{N} \partial f_i/\partial x_i$ *by* $f \in L^s(\Omega)$, $1/s = 1/p + 1/N$. *Then for the solution* u *of the problem* $Au = f$ *in* $\Omega, u - u_0 \in W_0^{1,2}(\Omega)$ *we have*

$$|u|_{L^r(\Omega)} + |u|_{L^{r(N-1)/N}(\partial\Omega)} \leq c(p)(|f|_{L^s(\Omega)} + |u_0|_{W^{1,p}(\Omega)} + |u|_{W^{1,2}(\Omega)}). \quad (7.19)$$

The Neumann-Newton problem and the mixed problem:

Let us consider $\partial\Omega = \Lambda_1 \cup \Lambda_2 \cup M$, where Λ_1, Λ_2 are two open sets in $\partial\Omega$, meas $_{\partial\Omega} M = 0$, $\Lambda_1 \neq \emptyset$, $\sigma \in L^\infty(\Lambda_1)$; we denote

$$V = \{v \in W^{1,2}(\Omega), v = 0 \text{ on } \Lambda_2\}. \quad (7.20)$$

We have

Theorem 1.3. *Let us consider* $\Omega \in \mathfrak{N}^{0,1}, V$ *the space (7.20), A the operator given in (7.1), (7.2),* $1/N < 1/p \leq 1/2$, $1/s = 1/p + 1/N$, $f \in L^s(\Omega)$, $u_0 \in W^{1,p}(\Omega)$, $g \in L^t(\Lambda_1)$, $1/t = N/p(N-1)$, $u \in W^{1,2}(\Omega)$ *the solution of the problem* $Au = f$ *in* Ω, $\partial u/\partial \nu + \sigma u = g$ *on* $\Lambda_1, u = u_0$ *on* Λ_2. *Then if* $1/r = 1/s - 2/N$, *we have*

$$|u|_{L^r(\Omega)} + |u|_{L^{r(N-1)/N}(\partial\Omega)} \leq c(p)(|f|_{L^s(\Omega)} + |u_0|_{W^{1,p}(\Omega)} + |g|_{L^t(\Lambda_1)} + |u|_{W^{1,2}(\Omega)}).$$

The proof follows from Lemma 1.6 and from Theorems 2.3.4, 2.4.2.

Remark 1.2. The existence of solutions mentioned in Theorems 1.1–1.3 or in the negative case the Fredholm alternative, follows from Theorem 3.2.1 or Theorem 3.3.1.

Remark 1.3. We have $1/r = 1/s - 2/N$, as if $u \in W^{s,2}(\Omega)$, which in general does not hold under our hypotheses. We still have $s < N/2$.

7.1.4 The Case $1/N > 1/p$

Now we shall consider the case $1/N > 1/p$. We shall not consider the particular case $1/N = 1/p$, for this case see G. Stampacchia [7]. We obtain $|u|_{L^\infty(\Omega)} < \infty$.

Lemma 1.7. *Let A be the operator defined by (7.1), (7.2), V the space defined in (7.3), (7.4),* $\sigma \in L^\infty(\partial\Omega)$, $F \in (V_q)'$ *with* $1 - 1/N < 1/q$. *Let* $u \in V$ *be a function such that we have for all* $v \in V$,

$$\int_\Omega \left(\sum_{i,j=1}^{N} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^{N} b_i v \frac{\partial u}{\partial x_i} + dvu \right) \mathrm{d}x + \int_{\partial\Omega} \sigma v u \, \mathrm{d}S = Fv.$$

Then

$$|u|_{L^\infty(\Omega)} \le c(|F|_{(V_q)'} + |u|_{W^{1,2}(\Omega)}). \tag{7.20bis}$$

Proof. According to Lemma 1.6, for all $\lambda \ge 0$ we have

$$|(u_+)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} + |(u_-)^{(1+\lambda/2)}|_{W^{1,2}(\Omega)}^{1/(1+\lambda/2)} \le c_1(\lambda)(|F|_{(V_q)'} + |u|_{W^{1,2}(\Omega)}), \tag{7.21}$$

then, if $N \ge 3$,

$$|u|_{L^{(1+\lambda/2)(2N/(N-2))}(\Omega)} + |u|_{L^{(1+\lambda/2)(2(N-1)/(N-2))}(\Omega)} \le c_2(\lambda)(|F|_{(V_q)'} + |u|_{W^{1,2}(\Omega)}). \tag{7.22}$$

Now, it follows that

$$F_2 v = \int_\Omega (1-d) v u \,\mathrm{d}x + \int_{\partial\Omega} \sigma v u \,\mathrm{d}S$$

is a functional on $(V_q)'$, and

$$|F_2|_{(V_q)'} \le c_3(|F|_{(V_q)'} + |u|_{W^{1,2}(\Omega)}). \tag{7.23}$$

Let us set $F_3 = F + F_2$; without loss of generality we can assume $1/q < 1$. Let us consider $\lambda_{n+1} = [(2-q)/q]\nu(1+\lambda_n/2)$, $n = 1,2,\ldots,\nu > 2q/(2-q)$, if $N=2$ and $\nu = 2N/(N-2)$, if $N \ge 3$; λ_1 must be sufficiently large such that $\nu(2-q)/q + 2/(1+\lambda_1/2) \le \nu$. As in the proofs of Lemmas 1.5, 1.6, we obtain for (u_+) (and also for (u_-)):

$$\begin{aligned} &|(u_+)^{1+\lambda_{n+1}/2}|_{W^{1,2}(\Omega)}^2 \le c_4\,(1+\lambda_{n+1}/2)\,|F_3|_{(V_q)'}\,|(u_+)^{1+\lambda_{n+1}}|_{W^{1,q}(\Omega)} \\ &+ c_5 |(u_+)^{1+\lambda_{n+1}/2}|_{L^2(\partial\Omega)} |(u_+)^{1+\lambda_{n+1}/2}|_{W^{1,2}(\Omega)}. \end{aligned} \tag{7.24}$$

We have

$$|(u_+)^{1+\lambda_{n+1}}|_{W^{1,q}(\Omega)} \le c_6 |(u_+)^{1+\lambda_n/2}|_{W^{1,2}(\Omega)}^{[(2-q)/2q]\nu} |(u_+)^{1+\lambda_{n+1}/2}|_{W^{1,2}(\Omega)}, \tag{7.25}$$

$$|(u_+)^{1+\lambda_{n+1}/2}|_{L^2(\Omega)} \le c_7 |(u_+)^{1+\lambda_n/2}|_{W^{1,2}(\Omega)}^{[(2-q)/2q]\nu + 1/(1+\lambda_n/2)}. \tag{7.26}$$

Now, it follows from (7.24)–(7.26), if we set:

$$z_n = \frac{|(u_+)^{1+\lambda_n/2}|_{W^{1,2}(\Omega)}^{1/(1+\lambda_n/2)}}{|F_3|_{(V_q)'}},$$

that

$$z_{n+1}^{(1+\lambda_n/2)} \le c_8\,(1+\lambda_{n+1}/2)\, z_n^{\lambda_{n+1}/2} + c_9 z_n^{(1+\lambda_{n+1}/2)}. \tag{7.27}$$

Let us set $m_1 = 1 + z_1, c_{10} = \max(1, c_8, c_9)$ and define:

$$m_{n+1} = c_{10}^{1/(1+\lambda_{n+1}/2)} (1+\lambda_{n+1}/2)^{1/(1+\lambda_{n+1}/2)} m_n, \quad n = 1,2\dots$$

Obviously $z_n \leq m_n$, so we now have

$$\begin{aligned}\ln m_{n+1} &= \ln m_1 + \ln c_{10} \sum_{i=1}^{n} \frac{1}{1+\lambda_{i+1}/2} + \sum_{i=1}^{n} \frac{\ln(1+\lambda_{i+1}/2)}{1+\lambda_{i+1}/2} \\ &\leq \ln m_1 + \ln c_{10} \sum_{i=1}^{n} \frac{1}{(((2-q)/2q)\nu)^i} + \sum_{i=1}^{n} \frac{\ln(1+\lambda_1/2) + i\ln((2-q)/2)\nu}{(((2-q)/2q)\nu)^i} \\ &\leq \ln m_1 + \ln c_{11};\end{aligned}$$

this holds due to $\nu > 2q/(2-q)$, then $m_{n+1} \leq c_{11} m_1$, which with (7.23) implies:

$$|(u_+)^{1+\lambda_n/2}|_{W^{1,2}(\Omega)}^{1/(1+\lambda_n/2)} \leq c_{12}(|F|_{(V_q)'} + |u|_{W^{1,2}(\Omega)}). \tag{7.28}$$

Finally we have:

$$\left(\int_\Omega (u_+)^{(2+\lambda_n)}\,\mathrm{d}x\right)^{1/(2+\lambda_n)} \leq c_{13}^{1/(1+\lambda_n/2)} |(u_+)^{1+\lambda_n/2}|_{W^{1,2}(\Omega)}^{1/(1+\lambda_n/2)},$$

this last result and the same estimate for (u_-) give the result. □

7.1.5 Regularity of the Solution

Let us formulate some consequences of Lemma 1.7 for boundary value problems:

Theorem 1.4. *Let us consider $\Omega \in \mathfrak{N}^{0,1}$, $V = W_0^{1,2}(\Omega)$, A the operator (7.1), (7.2), $1/p < 1/N, f_i \in L^p(\Omega)$, $i = 1,2,\dots,N$, $u_0 \in W^{1,p}(\Omega)$. Let u be the weak solution of the Dirichlet problem $Au = \sum_{i=1}^N \partial f_i/\partial x_i$ in $\Omega, u = u_0$ on $\partial\Omega$ with $u \in W^{1,2}(\Omega)$. Then*

$$|u|_{L^\infty(\Omega)} \leq c(\sum_{i=1}^{N} |f_i|_{L^p(\Omega)} + |u_0|_{W^{1,p}(\Omega)} + |u|_{W^{1,2}(\Omega)}). \tag{7.29}$$

Theorem 1.5. *Let be $\Omega \in \mathfrak{N}^{0,1}, V = W^{1,2}(\Omega)$, A be the operator (7.1), (7.2), $1/s < 2/N$, $f \in L^s(\Omega)$, $u_0 \in W^{1,p}(\Omega)$, $1/p = 1/s - 1/N$; let $u \in W^{1,2}(\Omega)$ be the weak solution of the Dirichlet problem $Au = f$ in Ω, $u = u_0$ on $\partial\Omega$. Then*

$$|u|_{L^\infty(\Omega)} \leq c(|f|_{L^s(\Omega)} + |u_0|_{W^{1,p}(\Omega)} + |u|_{W^{1,2}(\Omega)}). \tag{7.30}$$

Theorem 1.6. *Let be $\Omega \in \mathfrak{N}^{0,1}$, V the space defined in (7.20), A the operator (7.1), (7.2), $1/s < 2/N$, $f \in L^s(\Omega)$, $u_0 \in W^{1,p}(\Omega)$, $1/p = 1/s - 1/N$, $g \in L^t(\Lambda_1)$, $1/t = (1/p)[N/(N-1)]$, $\sigma \in L^\infty(\Lambda_1)$. Let $u \in W^{1,2}(\Omega)$ be the weak solution of the problem $Au = f$ in Ω, $\partial u/\partial \nu + \sigma u = g$ on Λ_1, $u = u_0$ on Λ_2. Then*

$$|u|_{L^\infty(\Omega)} \le c(|f|_{L^s(\Omega)} + |u_0|_{W^{1,p}(\Omega)} + |g|_{L^t(\Lambda_1)} + |u|_{W^{1,2}(\Omega)}). \tag{7.31}$$

Let us remark that as in the previous part, the data are such that using the imbedding theorems, according to Theorems 1.4, 1.6, the existence of the solution or at least the Fredholm alternative takes place.

We can also obtain estimates of type

$$|u|_{W^{s,2}(\Omega)} \le c_1 |f|_{L^s(\Omega)}, \quad s > N/2, \tag{$*$}$$

which implies that $|u|_{L^\infty(\Omega)} \le c_2 |f|_{L^s(\Omega)}$; but in general the estimate $(*)$ is not true with our hypotheses.

As we have seen previously, for the Dirichlet problem with the hypotheses of our Theorems 1.4, 1.5, we obtain that $u \in C^{0,\nu}(\overline{\Omega})$.

7.1.6 The Dual Method

Theorems 1.1.–1.6. concern the regularity of the solution. Then, using the duality, we obtain a class of *very weak solutions*. For this let us consider the operator A^* adjoint of A, i.e.

$$A^* = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ji}\frac{\partial}{\partial x_j}\right) - \sum_{i=1}^N \frac{\partial}{\partial x_i}(b_i) + d. \tag{7.32}$$

Let us consider $\sigma \in L^\infty(\partial\Omega)$, and

$$((v,u)) = \int_\Omega \left(\sum_{i,j=1}^N a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_j} + \sum_{i=1}^N b_i v \frac{\partial u}{\partial x_j} + dvu\right) \mathrm{d}x + \int_{\partial\Omega} \sigma v u \,\mathrm{d}S.$$

To simplify let us assume the V-ellipticity of the form $((v,u))$, i.e.

$$v \in V \Longrightarrow |((v,v))| \ge c|v|^2_{W^{1,2}(\Omega)} \quad c > 0. \tag{7.33}$$

First, let us consider the Dirichlet problem.

Theorem 1.7. *Let $\Omega \in \mathfrak{N}^{0,1}$, $V = W_0^{1,2}(\Omega)$, A the operator (7.1), (7.2), with (7.33), $G^* \in [L^2(\Omega) \to W_0^{1,2}(\Omega)]$ the Green operator corresponding to the problem $A^*u = f$ in $\Omega, u = 0$ on $\partial\Omega$. If $1/2 \le 1/q < 1 - 1/N$, let $1/s = 1/q + 1/N$; if $1 - 1/N < 1/q$, let $s = 1$. Then G^* can be extended by continuity to $G^* \in [L^s(\Omega) \to W_0^{1,q}(\Omega)]$.*

Proof. Let us consider $f \in L^2(\Omega)$, $u = G^* f$, $F = -\sum_{i=1}^N \partial f_i/\partial x_i$, $f_i \in L^p(\Omega)$, $1/p + 1/q = 1$ (without loss of generality we assume $1/q < 1$). According to Theorems 1.1, 1.4, if v is the solution of the problem $Av = -\sum_{i=1}^N \partial f_i/\partial x_i$ in Ω, $v = 0$ on $\partial\Omega$, then v satisfies the following estimate:

$$|v|_{L^r(\Omega)} + |v|_{W^{1,2}(\Omega)} \leq c_1 (\sum_{i=1}^N |f_i|^p_{L^p(\Omega)})^{1/p}, \quad 1/r + 1/p = 1. \tag{7.34}$$

On the other hand,

$$((v,u)) = \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} f_i \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x,$$

then using (7.34), we get:

$$\sup_{(\sum_{i=1}^N |f_i|^p_{L^p(\Omega)})^{1/p} \leq 1} \left| \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i} f_i \,\mathrm{d}x \right| = \left(\sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^q_{L^q(\Omega)} \right)^{1/q} \leq c_2 |f|_{L^s(\Omega)}. \tag{7.35}$$

Now it is sufficient to apply Theorem 2.4.10. □

Remark 1.4. As for $N \geq 3$, $q < N$, $W^{1,q}(\Omega) \subset L^{Nq/(N-q)}(\Omega)$, algebraically and topologically, G^* obtained from Theorem 1.7 is such that $G^* \in [L^s(\Omega) \to L^q(\Omega)]$, $1/q = 1/s - 2/N$ if $1/2 + 1/N \leq 1/s < 1$, and $1/q > 1/s - 2/N$ if $s = 1$. If $b_i \in W^{1,\infty}(\Omega)$, we can apply Theorems 1.2, 1.5 for the operator A^* and it follows that $G^* \in [L^s(\Omega) \to L^q(\Omega)]$, with $1/q = 1/s - 2/N$, if $2/N < 1/s \leq 1/2 + 1/N$, and $q = \infty$, if $1/s < N/2$. This remark holds also for other problems in this section.

Problem 1.1. Let G^* be the Green operator defined in Theorem 1.7. Determine the values of q such that $G^* \in [W^{-1,q}(\Omega) \to W_0^{1,q}(\Omega)]$.

Theorem 1.8. *Let us consider $\Omega \in \mathfrak{N}^{0,1}$, V the space (7.20), A^* the operator defined in the previous theorem, let G^* the Green operator such that $G^* \in [L^2(\Omega) \times L^2(\Lambda_1) \to W^{1,2}(\Omega)]$ and corresponding to the problem $A^* u = f$ in Ω, $\partial u/\partial \nu^* + \sum_{i=1}^N b_i n_i u + \sigma u = g$ on Λ_1, $u = 0$ on Λ_2. Let s be as in the previous theorem, $1/t = (N-s)/[s(N-1)]$, if $1/2 + 1/N \leq 1/s \leq 1$; $1/q = 1/s - 2/N$, if $1/2 + 1/N \leq 1/s < 1$; $1/q > 1/s - 2/N$, if $s = 1$. Then G^* can be extended by continuity to $G^* \in [L^s(\Omega) \times L^t(\Lambda_1) \to L^q(\Omega)]$.*

Proof. Let us consider $f \in L^2(\Omega)$, $g \in L^2(\Lambda_1)$, let u be the solution of the problem $A^* u = f$ in Ω, $\partial u/\partial \nu^* + \sum_{i=1}^N b_i n_i u + \sigma u = g$ on Λ_1, $u = 0$ on Λ_2, $F \in L^\tau(\Omega)$, with $1/\tau + 1/\rho = 1$, and v the solution of the problem $Av = F$ in Ω, $\partial v/\partial \nu + \sigma v = 0$ on $\Lambda_1, u = 0$ on Λ_2. According to Theorems 1.3, 1.6, for v we have

$$|v|_{L^r(\Omega)} + |v|_{L^{r[(N-1)/N]}(\partial\Omega)} \leq c_1 |F|_{L^\tau(\Omega)}, \tag{7.36}$$

if $1/2+1/N \leq 1/s < 1$, and

$$|u|_{L^\infty(\Omega)} + |u|_{L^\infty(\partial\Omega)} \leq c_2 |F|_{L^2(\Omega)}, \tag{7.37}$$

if $s = 1$. We have

$$((v,u)) = \int_\Omega v f \,\mathrm{d}x + \int_{\Lambda_1} v g \,\mathrm{d}S = \int_\Omega u F \,\mathrm{d}x,$$

then (7.36), (7.37) imply:

$$|u|_{L^\rho(\Omega)} \leq c_3 (|f|_{L^s(\Omega)} + |g|_{L^t(\Lambda_1)}).$$

□

7.2 The Maximum Principle

This section is devoted to a generalization of the maximum principle. For simplicity we consider only the case of the operator

$$A = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^N b_i \frac{\partial}{\partial x_i} + d\,,$$

where the form $A(v,u)$ is $W_0^{1,2}(\Omega)$-elliptic and $d \geq 0$. A more general case, without the restriction $d \geq 0$ is treated in a paper by G. Stampacchia [9]. For the Dirichlet problem we shall prove the following: if $u_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, and $u \in W^{1,2}(\Omega)$ is the solution of the problem $Au = 0$ in Ω, $u - u_0 \in W_0^{1,2}(\Omega)$, then $\sup_{x\in\Omega} |u(x)| \leq \sup_{x\in\Omega} |u_0(x)|$. We shall deduce the existence of a solution for the Dirichlet problem corresponding to the operator A with continuous data on the boundary. We have observed that a such solution is continuous in Ω; in the paper by W. Littman, G. Stampacchia, A.F. Weinberger [1] the behaviour of the solution in a neighborhood of the boundary is investigated; the result is that $y \in \partial\Omega$ is a regular point if and only if this property is true for the Laplace operator; we assume $b_i = d = 0$.

7.2.1 The Maximum Principle

Let Ω be any bounded domain, A the operator

$$A = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^N b_i \frac{\partial}{\partial x_i} + d \tag{7.38}$$

with real coefficients, $a_{ij} \in L^\infty(\Omega)$, $b_i \in L^N(\Omega)$, $d \in L^{N/2}(\Omega)$, if $N \geq 3$ and $b_i \in L^{2+\varepsilon}(\Omega)$, $d \in L^{1+\varepsilon}(\Omega)$, if $N = 2$, $\varepsilon > 0$. We assume $d \geq 0$, and the form

$$A(v,u) = \int_\Omega \left(\sum_{i,j=1}^{N} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^{N} b_i v \frac{\partial u}{\partial x_i} + dvu \right) \mathrm{d}x$$

$W_0^{1,2}(\Omega)$ strongly coercive, i.e.

$$v \in W_0^{1,2}(\Omega) \Longrightarrow A(v,v) \geq c|v|^2_{W^{1,2}(\Omega)}, \quad c > 0. \tag{7.39}$$

We can prove immediately

Theorem 2.1. *Let A be given as in (7.38), (7.39), Ω be a bounded domain, $u_0 \in W^{1,2}(\Omega)$, and almost everywhere $u_0(x) \leq M < \infty$, $M \geq 0$. Let $u \in W^{1,2}(\Omega)$ be the weak solution of the Dirichlet problem $Au = 0$ in Ω, $u - u_0 \in W_0^{1,2}(\Omega)$. Then*

$$\operatorname*{ess\,sup}_{x \in \Omega} u(x) \leq M. \tag{7.40}$$

Proof. [2] Let us consider $\varepsilon > 0$, and $u_1 = \max(M+\varepsilon, u) - (M+\varepsilon)$ almost everywhere. According to Theorem 2.2.3, $u_1 \in W^{1,2}(\Omega)$, moreover $u_1 \in W_0^{1,2}(\Omega)$. Indeed: $u = u_0 + h$, $h \in W_0^{1,2}(\Omega)$, let us consider a sequence $h_n \in C_0^\infty(\Omega)$, $\lim_{n\to\infty} h_n = h$ in $W^{1,2}(\Omega)$. Lemma 1.2 implies $\lim_{n\to\infty}[\max(M+\varepsilon,\ u_0 + h_n) - (M+\varepsilon)] = u_1$ in $W_0^{1,2}(\Omega)$; clearly $\max(M+\varepsilon, u_0+h_n) - (M+\varepsilon) \in W_0^{1,2}(\Omega)$. Then

$$0 = A(u_1,u) = \int_\Omega \left(\sum_{i,j=1}^{N} \frac{\partial u_1}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^{N} b_i u_1 \frac{\partial u}{\partial x_i} + du_1 u \right) \mathrm{d}x \geq A(u_1,u_1),$$

this implies $u_1 \equiv 0$, and hence $\operatorname*{ess\,sup}_{x\in\Omega} u(x) \leq M + \varepsilon$. □

Exercise 2.1. Let us consider a sequence of domains, $\Omega_n \subset \overline{\Omega_n} \subset \Omega$, such that for every compact $K \subset \Omega, \Omega_n \supset K$, $n \geq n_0$, and let be $M \geq \inf_n \sup_{x \in \Omega - \overline{\Omega}_n} u_0(x)$, $M \geq 0$. With the hypotheses given in Theorem 2.1, prove that $\operatorname*{ess\,sup}_{x\in\Omega} u(x) \leq M$.

Corollary 2.1. *We assume the hypotheses of Theorem 2.1; then we have:*

$$\begin{aligned} \min(0, \operatorname*{\mathit{ess\,inf}}_{x\in\Omega} u_0(x)) &\leq \operatorname*{\mathit{ess\,inf}}_{x\in\Omega} u(x) \\ &\leq \operatorname*{ess\,sup}_{x\in\Omega} u(x) \leq \max(0, \operatorname*{ess\,sup}_{x\in\Omega} u_0(x)), \end{aligned} \tag{7.41}$$

$$|u|_{L^\infty(\Omega)} \leq |u_0|_{L^\infty(\Omega)}. \tag{7.42}$$

[2]The basic idea of this proof is due to J. Kadlec.

Theorem 2.2. *Considering the hypotheses as in Theorem 2.1 and assuming $\Omega \in \mathfrak{N}^{0,1}$, we have*

$$\min(0, \operatorname*{ess\,inf}_{x\in\Omega} u_0(x)) \le \operatorname*{ess\,inf}_{x\in\partial\Omega} u(x) \le \operatorname*{ess\,sup}_{x\in\Omega} u(x) \le \max(0, \operatorname*{ess\,sup}_{x\in\Omega} u_0(x)), \tag{7.43}$$

Proof. Let $M = \max_{x\in\Omega}(0, \text{ess sup} u_0(x))$, $u_1 := \max(M,u) - M$. It follows from Theorems 2.2.3, 2.4.10 that $u_1 \in W_0^{1,2}(\Omega)$; we finish as the previous proof. □

We prove as above

Theorem 2.3. *Let Ω be a bounded domain, $u_0 \in C(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$. Then*

$$\min(0, \min_{x\in\Omega} u_0(x)) \le \operatorname*{ess\,inf}_{x\in\Omega} u(x) \le \operatorname*{ess\,sup}_{x\in\Omega} u(x) \le \max(0, \max_{x\in\Omega} u_0(x)), \tag{7.44}$$

$$|u|_{L^\infty(\Omega)} \le \max_{x\in\Omega} |u_0(x)|. \tag{7.45}$$

Exercise 2.2. With the hypotheses of Theorem 2.3 prove that if $d \equiv 0$, we get:

$$\min_{x\in\partial\Omega} u_0(x) \le \operatorname*{ess\,inf}_{x\in\Omega} u(x) \le \operatorname*{ess\,sup}_{x\in\Omega} u(x) \le \max_{x\in\partial\Omega} u_0(x).$$

Let $g \in C(\partial\Omega)$. Let us extend g on $\mathbb{R}^N$ to a function continuous on $\mathbb{R}^N$ using the Urysohn theorem. By a process of regularization, we prove that the space of restriction of functions from $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ on $\partial\Omega$ is a dense subset of $C(\partial\Omega)$. Let us denote by M the set of restrictions of functions from $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ to $\partial\Omega$. The Green operator G defined by $Gg = u$ is, according to (7.45), such that $G \in [M \to L^\infty(\Omega)]$; as M is dense in $C(\partial\Omega)$, we have:

Theorem 2.4. *Let Ω be a bounded domain, A the operator as in (7.38), (7.39), G be defined above. Then G can be extended continuously to a mapping from $[C(\partial\Omega) \to L^\infty(\Omega)]$.*

Remark 2.1. If the coefficients of the operator are sufficiently smooth in Ω, then using Theorem 4.1.3 we prove that the solution belongs to $C^2(\Omega)$; for the Laplace operator, we study the solution in a neighborhood of the boundary by the "barrier functions" method, cf. for instance I.G. Petrovskii [1]. If $\Omega \in \mathfrak{N}^{0,1}$ we prove using Theorem 3.4 below that $u \in C(\overline{\Omega})$.

Moreover, the following theorem holds:

Theorem 2.5. *Let Ω be bounded, A be the operator as in (7.38), (7.39), $g \in C(\partial\Omega)$, $Gg = u$ be defined in Theorem 2.4, $u^* \in C(\overline{\Omega}) \cap W_{\text{loc}}^{1,2}(\Omega)$ be a weak solution of $Au^* = 0$ in Ω, $u^* = g$ on $\partial\Omega$. Then $u^* = u$ almost everywhere.*

Proof. Let $\Omega_n \in \mathfrak{N}^\infty$ be a sequence of subdomains $\Omega_n \subset \overline{\Omega}_n \subset \Omega$ such that for every compact set $K \subset \Omega$ there exists n_0 such that $n_0 > n \Longrightarrow K \subset \Omega_n$. Let $\varepsilon > 0$, $g_\varepsilon \in M$, $|g - g_\varepsilon|_{C^0(\partial\Omega)} < \varepsilon$. We denote by u_ε (resp. $u_{\varepsilon n}$) the solution corresponding to g_ε in Ω (resp. in Ω_n). Then using (7.45) we get

$$|u_\varepsilon - u^*|_{L^\infty(\Omega_n)} \le |g_\varepsilon - u^*|_{C(\partial\Omega_n)} < \varepsilon.$$

On the other hand, according to Theorem 3.6.7, the functions u_n extended by g_ε on Ω converge to u_ε in $W^{1,2}(\Omega)$, and then

$$|u_\varepsilon - u^*|_{L^\infty(\Omega)} \le |g_\varepsilon - g|_{C(\partial\Omega)} < \varepsilon,$$

and

$$|u_\varepsilon - u|_{L^\infty(\Omega)} \le |g_\varepsilon - g|_{C(\partial\Omega)} < \varepsilon.$$

□

Starting from the inequality (7.45) we can extend the Green operator G obtained in Theorem 2.4 on $C(\partial\Omega)$ to larger spaces.

7.3 Higher Order Equations

We are interested in the behaviour of the solution in a neighborhood of the boundary for equations of order ≥ 2 with sufficiently smooth coefficients, and sufficiently general hypotheses about Ω: for instance the "cone property", $\mathfrak{N}^{0,1}$, convexity, etc. We restrict ourselves to the Dirichlet problem. The basic idea can be found in R. Courant, D. Hilbert [1] and was developed by the author [5]; here we assume $N = 2$. Using the results obtained in the previous chapter concerning weighted spaces, we shall also prove some results for $N = 3$.

7.3.1 "The Mean Inequality"

Let us denote $K_r = K(0,r)$ the ball with center at the origin and radius r; we consider an operator of order $2k$,

$$A = \sum_{|i|,|j|\le k} (-1)^{|i|} D^i (a_{ij} D^j), \tag{7.46}$$

$$a_{ij} \in C^{\kappa_i,1}(\overline{\Omega}), \quad \kappa_i = \max(0, |i| - k - 1 + [N/2]).$$

We have the following

Lemma 3.1. *Let $r \leq 1, A$ be the operator (7.46) in K_r, with $|a_{ij}|_{C^{\kappa_i,1}(\overline{K_r})} \leq c_1$; we assume the form*

$$A(v,u) = \int_{K_r} \sum_{|i|,|j| \leq k} \overline{a}_{ij} D^i v D^j \overline{u} \, \mathrm{d}x$$

to be $W_0^{k,2}(K_r)$-elliptic with $|A(v,v)| \geq c_2 |v|^2_{W^{k,2}(K_r)}$.

Let us consider $f \in W^{-k+[N/2],2}(K_r)$, $u \in W^{k,2}(K_r)$ the weak solution of $Au = f$ in K_r. Then $u \in C^{k-1}(K_r)$, and

$$\sum_{|i| \leq k-1} \left| D^i u(0) \right| \leq c(c_1,c_2) \Big(\sum_{|i| \leq k} r^{(-N/2-k+1+|i|)} \left| D^i u \right|_{L^2(K_r)} + r^{(1+[N/2]-N/2)} |f|_{W^{-k+[N/2],2}(K_r)} \Big). \tag{7.47}$$

Proof. Set $y = x/r, v(y) = u(ry)$ and denote D^i_y the derivative with respect to the coordinate variable y. Further denote $b_{ij}(y) = r^{(2k-(|i|+|j|))} a_{ij}(ry)$. Let us prove the $W_0^{k,2}(K_1)$-ellipticity with constant c_2 of the bilinear form

$$\int_{K_1} \sum_{|i|,|j| \leq k} \overline{b}_{ij}(y) D^i_y w D^j_y \overline{w} \, dy.$$

If $w \in W_0^{k,2}(K_1)$, let $m \in W_0^{k,2}(K_r)$ be defined by $m(x) = w(x/r)$. It is convenient to take the norms in $W_0^{k,2}(\Omega)$ in the following form

$$\left(\int_\Omega \sum_{|i|=k} |D^i u|^2 \, \mathrm{d}x \right)^{1/2},$$

cf. Theorem 1.1.1. We get

$$\begin{aligned}
&\left| \int_{K_1} \sum_{|i|,|j| \leq k} \overline{b_{ij}(y)} D^i_y w D^j_y \overline{w} \, dy \right| = \\
&= \left| r^{-N} \int_{K_r} \sum_{|i|,|j| \leq k} \overline{a_{ij}(x)} r^{2k} D^i m D^j \overline{m} \, \mathrm{d}x \right| \geq r^{(2k-N)} c_2 \sum_{|i|=k} \int_{K_r} |D^i m|^2 \, \mathrm{d}x \qquad (7.48) \\
&= c_2 \sum_{|i|=k} \int_{K_1} |D^i_y w|^2 \, \mathrm{d}y = c_2 |w|^2_{W_0^{k,2}(K_1)}.
\end{aligned}$$

Now, we have

$$|b_{ij}|_{C^{\kappa_i,1}(\overline{K_1})} \leq |a_{ij}|_{C^{\kappa_i,1}(\overline{K_r})}. \tag{7.49}$$

If $[N/2] < k$, let us define $g \in W^{-k+[N/2],2}(K_1)$ by

$$< \psi, g >= r^{(-N+2k)} < \varphi, f >, \quad \varphi(x) = \psi(x/r). \tag{7.50a}$$

If $[N/2] \geq k$, let

$$g(y) = r^{2k} f(ry). \tag{7.50b}$$

For $\psi \in C_0^\infty(K_1)$ we get

$$\int_{K_1} \sum_{|i|,|j| \leq k} \overline{b_{ij}(y)} D_y^i \psi D_y^j \overline{v} \, \mathrm{d}y = \int_{K_r} \sum_{|i|,|j| \leq k} r^{(2k-N)} \overline{a_{ij}(x)} D^i \varphi D^j \overline{u} \, \mathrm{d}x =< \psi, g > .$$

According to Theorem 4.1.2 and Proposition 4.1.1, we have $v \in W^{k+[N/2],2}(K_\rho)$, $\rho < 1$, then by Theorem 2.3.8, $v \in C^{k-1}(K_1)$.

If $[N/2] > k$, it follows according to Theorem 4.1.2 and Proposition 4.1.1 that

$$|v|_{W^{k+[N/2],2}(K_{1/2})} \leq c_3 \big(|v|_{W^{k,2}(K_1)} + \sum_{|i|=[N/2]-k} |D^i g|_{L^2(K_1)} \big), \tag{7.51}$$

where c_3 depend only on c_1, c_2. Indeed: using Theorem 2.7.6 we get

$$\begin{aligned} |v|_{W^{k+[N/2],2}(K_{1/2})} &\leq c_4 \big(|v|_{W^{k,2}(K_1)} + \sum_{|i| \leq [N/2]-k} |D^i g|_{L^2(K_1)} \big) \\ &\leq c_5 \big(|v|_{W^{k,2}(K_1)} + \sum_{|i|=[N/2]-k} |D^i g|_{L^2(K_1)} + \sum_{|i|<[N/2]-k} |D^i g|_{L^2(K_{1/2})} \big), \\ \sum_{|i|<[N/2]-k} |D^i g|_{L^2(K_{1/2})} &\leq c_6 |v|_{W^{k+[N/2]-1,2}(K_{1/2})} \\ &\leq \frac{1}{2c_5} |v|_{W^{k+[N/2],2}(K_{1/2})} + c_7 |v|_{W^{k,2}(K_{1/2})}, \end{aligned} \tag{7.52}$$

and (7.51) follows from (7.52).

Now let us consider the case $k < [N/2]$. We have

$$\sum_{|i| \leq k-1} |D_y^i v(0)| \leq c_8 \big(|v|_{W^{k,2}(K_1)} + |g|_{W^{-k+[N/2],2}(K_1)} \big),$$

where the constant c_8 depends only on c_1, c_2. We get

$$\sum_{|i| \leq k-1} r^{|i|} |D^i u(0)| \leq c_8 \big(\sum_{|i| \leq k} r^{|i|-N/2} |D^i u|_{L^2(K_r)} + r^{(k-N/2-[N/2])} |f|_{W^{-k+[N/2],2}(K_r)} \big),$$

and hence (7.47).

If $k \geq [N/2]$, we obtain from (7.51),

$$\sum_{|i| \leq k-1} |D_y^i v(0)| \leq c_9 \big(|v|_{W^{k,2}(K_1)} + \sum_{|i|=[N/2]-k} |D^i g|_{L^2(K_1)} \big),$$

where c_9 depends only on c_1, c_2. Now we get

$$\begin{aligned} &\sum_{|i| \leq k-1} r^{|i|} |D^i v(0)| \\ &\qquad \leq c_9 \big(\sum_{|i| \leq k} r^{(|i|-N/2)} |D^i u|_{L^2(K_r)} + \sum_{|i|=[N/2]-k} r^{(k+[N/2]-N/2)} |D^i f|_{L^2(K_r)} \big) \\ &\qquad \leq c_9 \big(\sum_{|i| \leq k} r^{(|i|-N/2)} |D^i u|_{L^2(K_r)} + r^{(k+[N/2]-N/2)} |f|_{W^{([N/2]-k),2}(K_r)} \big), \end{aligned}$$

and we get (7.47) also in this case. □

7.3.2 *Existence of a Classical Solution in the Case* $N = 2$

Let us recall that a bounded domain satisfies the *exterior cone* property if there exists a fixed open cone such for every $y \in \partial\Omega$ it is possible to locate this cone in $\complement\Omega$ in such a way that y is the vertex of the cone. In this case we shall write $\Omega \in C_{ext}$, cf. Fig. 7.1. A bounded domain verifies the *exterior ball* property if there exists a ball B such that for every $y \in \partial\Omega$ this ball can be located in $\complement\Omega$ in such a way that $y \in \partial B$. In this case we shall write $\Omega \in B_{ext}$, cf. Fig. 7.2.

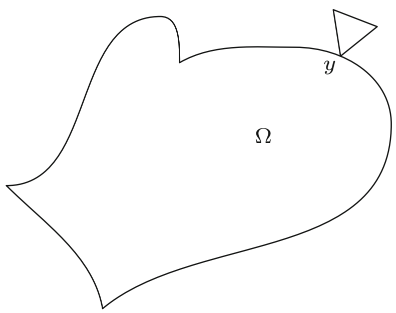

Fig. 7.1

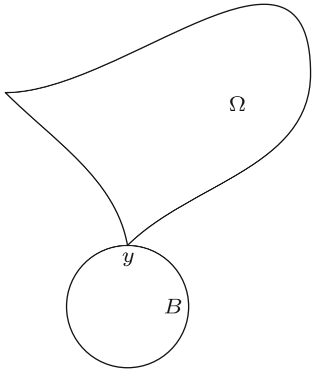

Fig. 7.2

We have the following theorem:

Theorem 3.1. *Let us consider $\Omega \in C_{ext}$ and A is the operator (7.46), $A(v,u)$ is $W_0^{k,2}(\Omega)$-elliptic, $f \in W^{-k+1,2}(\Omega)$, the function u is a weak solution of the problem $Au = f$ in Ω, $u \in W_0^{k,2}(\Omega)$ and $N = 2$. Then $u \in C^{k-1}(\overline{\Omega})$ for $x \in \partial\Omega, |i| \le k-1$, $D^i u(x) = 0$, and*

$$|u|_{C^{k-1}(\overline{\Omega})} \le c|f|_{W^{-k+1,2}(\Omega)}. \tag{7.53}$$

Proof. According to Theorem 4.2.1, $u \in C^{k-1}(\Omega)$. Let us consider $x_n \in \Omega$, such that $\lim_{n\to\infty} x_n = x \in \partial\Omega$, $d_n = \operatorname{dist}(x_n, \partial\Omega)$, $\xi_n \in \partial\Omega$ such that $d_n = |x_n - \xi_n|$. Let C be the cone defined previously, $C \subset \complement\Omega$, with vertex ξ_n, and $\sigma(x) = |x - \xi_n|$. Let us consider the ball $K(x_n, d_n/2)$. Let us introduce the local coordinates (y', y_N) with origin at ξ_n for which the axis y_N coincides with the axis of the cone C and consider lines parallel to the axis y_N which cut the ball $K(x_n, d_{n/2})$. Without loss of generality we assume d_n sufficiently small that all parallel lines considered cut C. Let us denote by M_n the union of open segments on the parallel lines with one end on ∂C and the other end in $K(x_n, d_n/2)$. As in (6.27) we obtain

$$\sum_{|i|\le k} |D^i u|_{L^2_{r^{-2(k-|i|)}}(M_n)} \le c_1 |u|_{W^{k,2}(M_n)},$$

which gives

$$\sum_{|i|\le k} |D^i u|_{L^2(K(x_n, d_n/2))} d_n^{-(k-|i|)} \le c_2 |u|_{W^{k,2}(M_n)}.$$

Now it suffices to use (7.47) to prove that $\lim_{n\to\infty} D^i u(x_n) = 0$, $|i| \le k-1$; then (7.47) implies (7.53). □

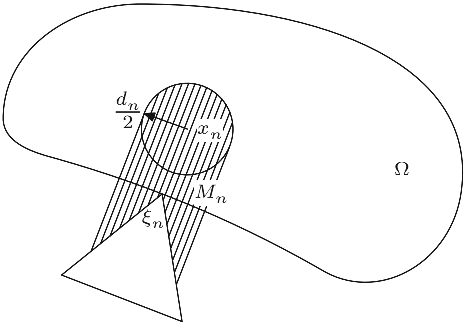

Fig. 7.3

For the Dirichlet problem with nonhomogeneous boundary conditions which was not considered in Theorem 3.1, we have

Theorem 3.2. *Let us consider $\Omega \in C_{ext}, N = 2, A$ as in the previous theorem,*

$$f \in W^{-k+1,2}(\Omega),\ u_0 \in W^{k+1,2}(\Omega) \cap C^{k-1}(\overline{\Omega}).$$

Then the weak solution u of the Dirichlet problem $Au = f$ in Ω, $u - u_0 \in W_0^{k,2}(\Omega)$ satisfies $u \in C^{k-1}(\overline{\Omega})$ and for $x \in \partial\Omega$, $|i| \le k-1$, $D^i u(x) = D^i u_0(x)$. Moreover we have the inequality

$$|u|_{C^{k-1}(\overline{\Omega})} \le c(|f|_{W^{-k+1,2}(\Omega)} + |u_0|_{W^{k+1,2}(\Omega)} + |u_0|_{C^{k-1}(\overline{\Omega})}).$$

If $\Omega \in \mathfrak{N}^{0,1}$ then $W^{k+1,2}(\Omega) \subset C^{(k-1)}(\overline{\Omega})$ algebraically and topologically.

Example 3.1. Let $\Omega = \{x \in \mathbb{R}^2; 0 < x_1 < 1, |x_2| < x_1^\alpha, \alpha \ge 0\}$, $u_0 \in W^{3,2}(\mathbb{R}^2)$, $f \in L^2(\Omega)$. Then the solution of the problem $\triangle^2 u = f$ in $\Omega, u - u_0 \in W_0^{2,2}(\Omega)$ belongs to $C^1(\overline{\Omega})$.

Remark 3.1. If $\Omega \in \mathfrak{N}^{0,1}$, and if the operator A is strongly elliptic, certain results obtained by the author in [9] imply that the solution obtained in Theorem 3.1 is $\kappa-$hölderian in $\overline{\Omega}$; if the coefficients a_{ij}, $|i| = |j| = k$, are real, we can choose $\kappa < 1/2$.

7.3.3 The Case $N = 3$

Theorem 3.3. *Let us consider $\Omega \in B_{ext}$ and let A be the operator (7.46) verifying the hypotheses given in Theorem 6.2.6. Let $N = 3, f \in W^{-k+1,2}(\Omega)$ and let $u \in W_0^{k,2}(\Omega)$ be the solution of $Au = f$ in Ω. Then $u \in W^{k-1,\infty}(\Omega) \cap C^{k-1}(\Omega)$, and*

$$|u|_{W^{k-1,\infty}(\Omega)} \le c|f|_{W^{-k+1,2}(\Omega)}. \tag{7.54}$$

Proof. Suppose $x \in \Omega$, $d = \mathrm{dist}(x, \partial\Omega)$, $\xi \in \partial\Omega$, $d = |x - \xi|$. Let B the ball as in Fig. 7.2, $B \subset \complement\Omega$ such that $\xi \in \partial B$; let ρ be the radius of B and y its center. Without loss of generality we assume $d < \rho/2$. Let B_1 be the ball of radius $\rho/2$ with center $y_1; y_1$ is a point on the segment (y, ξ) such that $\mathrm{dist}(\xi, \partial B_1) = d$. Let us set $\sigma_x(\eta) = \mathrm{dist}\,(\eta, \partial B_1)$. According to Theorem 6.2.7 and Remark 6.2.4, we get

$$|u|_{H^{k,2}_{\sigma_x^{-1}}(\Omega)} \le c_1(|f|_{W^{-k+1,2}(\Omega)} + |f|_{W^{-k,2}_{\sigma_x^{-1}}(\Omega)}), \tag{7.55}$$

where c_1 does not depend on x. By Theorems 6.2.3, 6.2.4, for all $\varphi \in C_0^\infty(\Omega)$,

$$|\varphi|_{W^{k-1,2}(\Omega)} \le c_2 |\varphi|_{W^{k,2}_{\sigma_x}(\Omega)},$$

where c_2 does not depend on x, and hence

$$|f|_{W^{-k,2}_{\sigma_x^{-1}}(\Omega)} \le c_2 |f|_{W^{-k+1,2}(\Omega)}. \tag{7.56}$$

Now let M_x be the union of segments as in Fig. 7.3, where one end of each segment is on ∂B and the other one is in $K(x, d/2)$. From (7.55) and (7.56) we get

$$\left|\frac{u}{\sigma_x}\right|_{W^{k,2}(M_x)} \le c_3 d^{-1/2} |f|_{W^{-k+1,2}(\Omega)},$$

where c_3 does not depend on x; the function $u/\sigma_x = v \in W_0^{k,2}(\Omega)$; using the inequality (6.25) on M_x, $|i| \le k$, we get:

$$|D^i v|_{L^2_{\sigma_x^{-2(k-|i|)}}(M_x)} \le c_4 d^{-1/2} |f|_{W^{-k+1,2}(\Omega)},$$

then for $|i| \le k$:

$$|D^i u|_{L^2_{\sigma_x^{-2(k-|i|)-2}}(M_x)} \le c_5 d^{-1/2} |f|_{W^{-k+1,2}(\Omega)},$$

and

$$|D^i u|_{L^2(K(x,d/2))} d^{(-(k-|i|)-1)} \le c_6 d^{-1/2} |f|_{W^{-k+1,2}(\Omega)},$$

and finally

$$|D^i u|_{L^2(K(x,d/2))} d^{(-(k-|i|)-1/2)} \le c_6 |f|_{W^{(-k+1),2}(\Omega)}. \tag{7.57}$$

It follows from (7.57) and (7.47) that

$$\sum_{|i| \le k-1} |D^i u(x)| \le c_7 |f|_{W^{-k+1,2}(\Omega)}. \tag{7.58}$$

□

Problem 3.1. Prove that with the hypotheses of Theorem 3.3, we have $u \in C^{(k-1)}(\overline{\Omega})$.

Moreover we get

Corollary 3.1. *Let us assume the hypotheses given in Theorem 3.3. Assume*

$$f \in W^{-k+1,2}(\Omega), \quad u_0 \in W^{k+1,2}(\Omega) \cap W^{k-1,\infty}(\Omega),$$

and let u be the solution of the Dirichlet problem $Au = f$ in Ω, $u - u_0 \in W_0^{k,2}(\Omega)$. Then

$$|u|_{W^{k-1,\infty}(\Omega)} \le c(|f|_{W^{-k+1,2}(\Omega)} + |u_0|_{W^{k+1,2}(\Omega)} + |u_0|_{W^{k-1,\infty}(\Omega)}).$$

Remark 3.2. If $\Omega \in \mathfrak{N}^{0,1} \cap B_{ext}$, $N = 3$, then $W^{k+1,2}(\Omega) \subset C^{k-1}(\overline{\Omega})$ algebraically and topologically and it suffices in the previous Corollary to assume only $u_0 \in W^{k+1,2}(\Omega)$.

7.3.4 Continuity of the Solution for the Laplace Operator on $\overline{\Omega}$

The weighted space method and Lemma 3.1 give again the well known results for the second order Dirichlet problems. Here we consider only the Laplace operator, but clearly the results can be extended to more general cases.

Theorem 3.4. *Let be $\Omega \in C_{ext}$, $f \in W^{-1+[N/2],2}(\Omega) \cap L^q(\Omega)$, with $1/q = 1/2 - ([N/2] - 1/N)$,*[3] *u the solution of the Dirichlet problem $-\triangle u = f$ in $\Omega, u \in W_0^{k,2}(\Omega)$. Then $u \in C(\overline{\Omega})$, and*

$$|u|_{C(\overline{\Omega})} \le c(|f|_{W^{[N/2]-1,2}(\Omega)} + |f|_{L^q(\Omega)}). \tag{7.59}$$

Proof. Fix $y \in \partial\Omega$, $r_y = |x-y|$, $H_1 = W^{1,2}_{0,r_y^{-\alpha}}(\Omega)$, $H_2 = W^{1,2}_{0,r_y^{\alpha}}(\Omega)$. We assert that the sesquilinear form

$$A(v,u) = \int_\Omega \sum_{i=1}^N \frac{\partial v}{\partial x_i}\frac{\partial \overline{u}}{\partial x_i}\,\mathrm{d}x$$

is H_2-elliptic for $\alpha = -N + 2 - \varepsilon$, $0 < \varepsilon < 1$. Indeed, according to Example 3.3, we have:

$$\mathrm{Re}\,A(ur_y^\alpha, u) = \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2 r_y^\alpha\,\mathrm{d}x + \frac{\alpha}{2}(2 - N - \alpha)\int_\Omega |u|^2 r_y^{(\alpha-2)}\,\mathrm{d}x.$$

[3] If $\Omega \in \mathfrak{N}^{0,1}$, then according to Theorem 2.3.4, $W^{[N/2]-1,2}(\Omega) \subset L^q(\Omega)$ algebraically and topologically.

We have

$$\frac{\alpha}{2}(2-N-\alpha)\int_{\Omega}|u|^2 r_y^{(\alpha-2)}\,\mathrm{d}x = \frac{2-N-\varepsilon}{2}\varepsilon\int_{\Omega}|u|^2 r_y^{(-N-\varepsilon)}\,\mathrm{d}x.$$

Let C be the cone in $\complement\Omega$, cf. Fig. 7.1, with vertex y. Let us introduce the local charts (x', x'_N) with the origin at y and with the axis x'_N parallel to the axis of the cone C. If a is sufficient small, let us consider the prism $P=\{x=(x',x'_N),|x'_i|<a,|x'_N|<c_1 a\}$. If c_1 is big enough we obtain that the face located in $x'_N=-b$ is in $\complement\Omega$. Denote by $P_i^{+,-}$, $i=1,2,\dots N-1$, the faces of P located in the hyperplanes $x'_i=\pm a$ and by $P_N^{+,-}$ the faces in the hyperplanes $x'_N=\pm b$. By elementary computations as in Sect. 1.1, Chap. 1, we obtain for $i=1,2,\dots,N-1$:

$$\int_{P_i^{+,-}}|u|^2 dP_i^{+,-} \le c_2 a^2\int_{P_i^{+,-}}\Big(\sum_{j=1}^{N}\left|\frac{\partial u}{\partial x_j}\right|^2-\left|\frac{\partial u}{\partial x_i}\right|^2\Big)dP_i^{+,-}, \tag{7.60}$$

and then

$$\underbrace{\int_{-a}^{a}\cdots\int_{-a}^{a}}_{(N-2)\text{times}}|u(a,x_2,\dots,x_{N-1},b)|^2\,\mathrm{d}x_2\dots\mathrm{d}x_{N-1} \le c_3 a\int_{P_1^+}\Big(\sum_{j=1}^{N}\left|\frac{\partial u}{\partial x_j}\right|^2-\left|\frac{\partial u}{\partial x_1}\right|^2\Big)dP_1^+. \tag{7.61}$$

From (7.61) it follows that

$$\int_{P_N^+}|u|^2 dP_N^+ \le c_4 a^2\Big(\int_{P_N^+}\Big(\sum_{j=1}^{N}\left|\frac{\partial u}{\partial x_j}\right|^2-\left|\frac{\partial u}{\partial x_N}\right|^2\Big)dP_N^+ + \int_{P_1^+}\Big(\sum_{j=1}^{N}\left|\frac{\partial u}{\partial x_j}\right|^2-\left|\frac{\partial u}{\partial x_1}\right|^2\Big)dP_1^+\Big). \tag{7.62}$$

On the faces $P_i^{+,-}$, we have $c_5 a\le r_y\le c_6 a$, and then using (7.60), (7.62) it follows that

$$\int_{\Omega}|u^2|r^{(\alpha-2)}\,\mathrm{d}x \le c_7\int_{\Omega}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^2 r^{\alpha}\,\mathrm{d}x.$$

Now choosing $\varepsilon<\max(1,2/[c_7(N-1)])$ we have the $W^{1,2}_{0,r_y^{\alpha}}(\Omega)$-ellipticity. It is easy to see that ε does not depend on $y\in\partial\Omega$. Now we have

$$\sup_{y\in\partial\Omega}|u|_{W^{1,2}_{r_y^{(-N+2-\varepsilon)}}(\Omega)} \le c_8|f|_{L^q(\Omega)}. \tag{7.62 bis}$$

Indeed, it is sufficient to prove

$$\sup_{|v|_{W^{1,2}_{0,r_y^{(-N+2-\varepsilon)}}(\Omega)}\leq 1} \int_\Omega |vf|\,\mathrm{d}x \leq c_9|f|_{L^q(\Omega)}; \tag{7.63}$$

using this last result we apply Theorem 6.3.1. According to Theorem 6.2.3 we get

$$\int_\Omega |v|^2 r_y^{(N-4+\varepsilon)}\,\mathrm{d}x \leq c_{10},$$

and then (7.63) follows. Theorem 4.2.2 gives $u \in C(\Omega)$. If we now have a sequence $x_n \in \Omega$ such that $\lim_{n\to\infty} x_n = x \in \partial\Omega$, $d_n =$ dist $(x_n, \partial\Omega)$, $\xi_n \in \partial\Omega$, $d_n = |\xi_n - x_n|$, then it follows from Theorem 6.2.3 that

$$\left(\int_\Omega |u|^2 r_y^{(-N-\varepsilon)}\,\mathrm{d}x\right)^{1/2} \leq c_{11}|f|_{L^q(\Omega)},$$

taking into account (7.62 bis) and Lemma 3.1, and we obtain $\lim_{n\to\infty} u(x_n) = 0$. □

Remark 3.3. If we assume Ω only bounded, we prove without difficulty

$$\sup_{y\in\partial\Omega} |u|_{W^{1,2}_{r_y^{(-N+2)}}(\Omega)} \leq c|f|_{L^q(\Omega)},$$

(cf. notations in the previous theorem and in the proof), and then

$$|u|_{L^\infty(\Omega)} \leq c(|f|_{L^q(\Omega)} + |f|_{W^{[N/2]-1,2}(\Omega)}).$$

Remark 3.4. If

$$f \in W^{[N/2]-1,2}(\Omega) \cap L^q(\Omega), \quad u_0 \in W^{(N/2]+1,2}(\Omega) \cap C(\overline{\Omega}),$$

then according to Theorem 3.4, we have for the weak solution of the problem $-\triangle u = f$ in $\Omega, u - u_0 \in W_0^{1,2}(\Omega)$,

$$|u|_{C(\overline{\Omega})} \leq c(|f|_{W^{[N/2]-1,2}(\Omega)} + |f|_{L^q(\Omega)} + |u_0|_{W^{[N/2]+1,2}(\Omega)} + |u_0|_{C(\overline{\Omega})}).$$

Remark 3.5. If A is a second-order elliptic operator with sufficiently smooth coefficients and $a_{ij} = a_{ji}$, we modify Theorem 3.4 replacing r_y by

$$\rho_y(x) = (\sum_{i,j=1}^N A_{ij}(y)(x_i - y_i)(x_j - y_j))^{1/2},$$

where $A_{ij}(y)$ are the elements of the matrix inverse to the matrix of the coefficients $a_{ij}(y)$.

7.3.5 The Case of the Right Hand Side Equal to Zero

We return now to the hypothesis $N = 2$. For the solution of the problem $Au = 0$ in Ω, $u - u_0 \in W_0^{k,2}(\Omega)$, $u_0 \in W^{k,2}(\Omega) \cap C^{k-1}(\overline{\Omega})$, we shall prove that $u \in W^{k,2}(\Omega) \cap C^{k-1}(\overline{\Omega})$. This result was derived by the author in his paper [5] for polyharmonic operators. We shall use the results on Poisson kernels given in Chap. 4; for simplicity we assume that

$$a_{ij} \in C^\infty(\overline{\Omega}). \tag{7.64}$$

Now we prove

Lemma 3.2. *Let us consider $r < 1$, the operator A defined in (7.46) and satisfying (7.64) in K_r, and suppose the sesquilinear form $A(v,u)$ is $W_0^{k,2}(K_r)$-elliptic with ellipticity constant c^*. Let $u \in W^{k,2}(K_r)$ be the weak solution of $Au = 0$ in K_r. Then $u \in C^{k-1}(\bar{K}_r)$, and for $|\mu| = k-1$, μ fixed, we have*

$$D^\mu u(0) = \frac{1}{r^{(k+N-1)}} \int_{K_r} N(x,r)u(x)\,\mathrm{d}x, \tag{7.65}$$

where

$$N(x,r) \in C(\overline{K_r}),\ |N|_{C(\overline{K_r})} \le c(a_{ij}, c^*).$$

Proof. According to Theorem 4.1.2, we have $u \in C^\infty(K_r)$. With the notation used in Lemma 3.1 we set

$$y = x/r, \quad v(y) = u(ry), \quad b_{ij}(y) = r^{(2k-|i|-|j|)} a_{ij}(ry).$$

For $\psi \in C_0^\infty(K_1)$, we have

$$\int_{K_1} \sum_{|i|,|j| \le k} \overline{b}_{ij}(y) D_y^i \psi D_y^j \overline{v}\, dy = 0.$$

The operator

$$B = \sum_{|i|,|j| \le k} (-1)^{|i|} D_y^i (b_{ij} D_y^j) \tag{7.66}$$

is $W_0^{k,2}(\Omega)$-elliptic, cf. Lemma 3.1. Now we use Theorem 4.4.1 for $\Omega = K_1$ and B^* and the Dirichlet problem $B^* M_r = D^\mu \delta(y)$ with homogeneous boundary conditions. The very weak solution of this problem is M_r. According to Theorem 4.1.3 $M_r(y) \in C^\infty(K_1 - \{0\})$. Let $\psi \in C_0^\infty(K_1)$, $\psi(y) = 1$ for $|y| < 1/2$, and let us set $P_r(y) = \psi(y) M_r(y)$. It follows from Theorem 4.4.1 that

$$0 = (-1)^{(k-1)} D_y^{\mu} v(0) + \int_{K_1} v N_r(y)\,\mathrm{d}y,$$

where

$$N_r(y) = B^*(P_r(y)) - D^{\mu}\delta(y).$$

We have $N_r \in C^{\infty}(\overline{K_1})$, and $|N_r(y)|_{C(\overline{K_1})} \le c_1$. Returning to the charts x and setting $N(x,r) = (-1)^k N_r(x/r)$, the result follows. □

Theorem 3.5. *Let $\Omega \in C_{ext}$, A be the operator (7.46), (7.64), the form $A(v,u)$ be $W_0^{k,2}(\Omega)$-elliptic, $N = 2$. Let $u_0 \in W^{k,2}(\Omega) \cap C^{k-1}(\overline{\Omega})$, u the weak solution of the Dirichlet problem $Au = 0$ in $\Omega, u - u_0 \in W_0^{k,2}(\Omega)$. Then $u \in C^{k-1}(\overline{\Omega})$, $D^i u(x) = D^i u_0(x)$, $x \in \partial\Omega$, $|i| \le k-1$, and*

$$|u|_{C^{k-1}(\overline{\Omega})} \le c(|u_0|_{W^{k,2}(\Omega)} + |u_0|_{C^{k-1}(\overline{\Omega})}). \tag{7.67}$$

Proof. Using Theorem 4.1.2, we have $u \in C^{\infty}(\Omega)$ and it suffices to consider a sequence $x_n \in \Omega$, $\lim_{n\to\infty} x_n = x \in \Omega$. Let us set $u = u_1 + u_0$. Let us consider the balls $K(x_n, d_n/2)$ introduced in the proof of Theorem 3.1 and we use the same notation for M_n. From Theorem 2.3.4 we have $W_0^{k,2}(\Omega) \subset C^{k-2}(\overline{\Omega})$ if $k \ge 2$; then it is sufficient to consider $D^{\mu} u_1$, $|\mu| = k-1$. We apply the previous lemma for u in the discs $K(x_n, d_n/2)$. Hence we get

$$D^{\mu} u(x_n) = \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2) u(x)\,\mathrm{d}x.$$

It follows that

$$D^{\mu} u_1(x_n) = \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2) u(x)\,\mathrm{d}x - D^{\mu} u_0(x_n) =$$

$$= \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2) u_1(x)\,\mathrm{d}x + \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2) u_0(x)\,\mathrm{d}x - D^{\mu} u_0(x_n).$$

Denote

$$p_n(x) = \sum_{|i| \le k-1} \frac{1}{i!} (x - x_n)^i D^i u_0(x_n),$$

and let $u_2 \in W_0^{k,2}(\Omega)$ be the solution of $Au_2 = Ap_n$ in Ω. According to Theorem 3.1 we get

$$|u_2|_{C^{k-1}(\overline{\Omega})} \le c_1 |u_0|_{C^{k-1}(\overline{\Omega})}, \tag{7.68}$$

and for $|i| \le k-1, \lim_{n\to\infty} D^i u(x_n) = 0$. Then we obtain

$$\begin{aligned} D^{\mu} u_1(x_n) = &\left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2) u_1(x)\, \mathrm{d}x + \\ &+ \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2)(u_0(x) - p_n(x))\, \mathrm{d}x + \\ &+ \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2)(p_n(x) - u_2(x))\, \mathrm{d}x + \\ &+ \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2) u_2(x)\, \mathrm{d}x - D^{\mu} u_0(x). \end{aligned} \tag{7.69}$$

But we have the estimate

$$d^{-k} |u_1|_{L^2(K_n)} \le c_2 |u_1|_{W^{k,2}(M_n)},$$

and hence

$$\left| \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2) u_1(x)\, \mathrm{d}x \right| \le c_3 |u_1|_{W^{k,2}(M_n)}. \tag{7.70}$$

Obviously, we have

$$\lim_{n\mapsto\infty} \left| \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2)(u_0(x) - p_n(x))\, \mathrm{d}x \right| = 0. \tag{7.71}$$

We obtain

$$\left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2)(p_n(x) - u_2(x)))\, \mathrm{d}x = D^{\mu} u_0(x_n) - D^{\mu} u_2(x_n),$$

and finally

$$\left| \left(\frac{2}{d_n}\right)^{k+1} \int_{K_n} N(x, d_n/2) u_2(x)\, \mathrm{d}x \right| \le c_4 |u_2|_{W^{k,2}(M_n)}. \tag{7.72}$$

It follows from (7.69)–(7.72) that $\lim_{n\to\infty} D^{\mu} u_1(x_n) = 0$, and thus $u \in C^{(k-1)}(\overline{\Omega})$. The inequality (7.67) then follows from (7.69). □

Exercise 3.1. Let A be the operator (7.46), (7.64),and suppose the sesquilinear form corresponding to A is coercive. Assuming the hypotheses given in Theorem 3.5, prove that

$$|u|_{C^{k-1}(\Omega)} \le c\big(|u_0|_{W^{k,2}(\Omega)} + |u_0|_{C^{k-1}(\overline{\Omega})} + |u|_{W^{k,2}(\Omega)}\big).$$

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